New Families of Eighth-Order Methods With High Efficiency Index For Solving Nonlinear Equations

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Abstract: In this paper, we construct two new families of eighth-order methods for solving simple roots of nonlinear equations by using weight function and interpolation methods. Per iteration in the present methods require three evaluations of the function and one evaluation of its first derivative, which implies that the efficiency indexes are 1.682. Kung and Traub conjectured that an iteration method without memory based on n evaluations could achieve optimal convergence order 2^{n-1}. The new families of eighth-order methods agree with the conjecture of Kung-Traub for the case $n = 4$. Numerical comparisons are made with several other existing methods to show the performance of the presented methods, as shown in the illustration examples.

Key–Words: Eighth-order convergence; Nonlinear equations; Weight function methods; Convergence order; Efficiency index

1 Introduction

In this paper, we construct iterative methods to find a simple root of a nonlinear equation $f(x) = 0$, where $f : D \subset R \to R$ is a scalar function on an open interval D.

The classical Newton's method for a single nonlinear equation is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

This is an important and basic method [1], which converges quadratically.

In the literature there are some classical iterative methods, such as Newton's method, Ostrowski's method [1], which is defined by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}; \end{aligned} \tag{2}$$

Chebyshev-Halley method [2],which is defined by

$$x_{n+1} = x_n - (1 + \frac{1}{2} \frac{L_f(x)}{1 - \alpha L_f(x)}) \frac{f(x_n)}{f'(x_n)}, \tag{3}$$

where

$$L_f(x) = \frac{f''(x_n) f(x_n)}{f'^2(x_n)};$$

and Jarratt's method [3], which is defined by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - (1 - \frac{3}{2} \frac{f'(y_n) - f'(x_n)}{3f'(y_n) - f'(x_n)}) \frac{f(x_n)}{f'(x_n)}. \end{aligned} \tag{4}$$

Recently, many new modified methods have been proposed to improve the convergence order and efficiency index of the classical iterative methods, see [4]-[28]. Chun and Ham developed a family of sixth-order methods by weight function methods in [4] (see formula (10), (11), (12) therein), which is written as:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - H(\mu_n) \frac{f(z_n)}{f'(x_n)}, \end{aligned} \tag{5}$$

where $\mu_n = \frac{f(y_n)}{f(x_n)}$ and $H(t)$ represents a real-valued function with $H(0) = 1,\ H'(0) = 2, H''(0) < \infty$. Wang and Liu in [5] developed a family of sixth-order methods as follows:

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} ,\\
z_n &= x_n - \frac{9f'(x_n) - 5f'(y_n)}{10f'(x_n) - 6f'(y_n)}\frac{f(x_n)}{f'(y_n)} ,\\
x_{n+1} &= z_n - \frac{f(z_n)}{\frac{3}{2}W_f(x_n)f'(y_n) + (1 - \frac{3}{2}W_f(x_n))f'(x_n)} ,
\end{aligned} \tag{6}
$$

where

$$W_f(x_n) = \frac{af'(x_n) + bf'(y_n)}{cf'(x_n) + df'(y_n)}\frac{f'(x_n)}{f'(y_n)}$$

and $a = -b+c+d$ (b, c and d are constants). Kou et al. in [6] constructed a family of variants of Ostrowski's method (see formula (8) therein) with seventh-order convergence, which is given by

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} ,\\
z_n &= y_n - \frac{f(y_n)}{f'(x_n)}\frac{f(x_n)}{f(x_n) - 2f(y_n)} ,\\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}[(\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)})^2\\
&\quad + \frac{f(z_n)}{f(y_n) - \alpha f(z_n)}] ,
\end{aligned} \tag{7}
$$

where α is a constant. Kung and Traub [7] conjectured that a multipoint iteration without memory based on n evaluations could achieve optimal convergence order 2^{n-1}. Kung and Traub [7] also provided two families of multipoints iterations based on n evaluations. For the case $n = 4$, the methods can be written as follows:

$$
\begin{aligned}
y_n &= x_n + \beta f(x_n) ,\\
z_n &= y_n - \beta\frac{f(x_n)f(y_n)}{f(y_n) - f(x_n)} ,\\
w_n &= z_n - \frac{f(x_n)f(y_n)}{f(z_n) - f(x_n)}\\
&\quad [\frac{y_n - x_n}{f(y_n) - f(x_n)} - \frac{z_n - y_n}{f(z_n) - f(y_n)}] ,\\
x_{n+1} &= w_n - \frac{f(x_n)f(y_n)f(z_n)}{f(w_n) - f(x_n)}\\
&\quad \{\frac{1}{f(w_n) - f(y_n)}[\frac{w_n - z_n}{f(w_n) - f(z_n)}\\
&\quad - \frac{z_n - y_n}{f(z_n) - f(y_n)}] - \frac{1}{f(z_n) - f(x_n)}\\
&\quad [\frac{z_n - y_n}{f(z_n) - f(y_n)} - \frac{y_n - x_n}{f(y_n) - f(x_n)}]\},
\end{aligned} \tag{8}
$$

where β is a constant, and

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} ,\\
z_n &= y_n - \frac{f(x_n)f(y_n)}{[(f(x_n) - f(y_n)]^2}\frac{f(x_n)}{f'(x_n)} ,\\
x_{n+1} &= z_n - \frac{P(x_n, y_n, z_n)}{Q(x_n, y_n, z_n)}\frac{f(x_n)}{f'(x_n)},
\end{aligned} \tag{9}
$$

where

$$
\begin{aligned}
P(x_n, y_n, z_n) &= f(x_n)f(y_n)f(z_n)\{f(x_n)^2\\
&\quad + f(y_n)[f(y_n) - f(z_n)]\},\\
Q(x_n, y_n, z_n) &= [f(x_n) - f(y_n)]^2[f(x_n)\\
&\quad - f(z_n)]^2[f(y_n) - f(z_n)].
\end{aligned}
$$

Bi et al. in [8] developed a family of eighth-order convergence methods (see formula (14) therein), which is given by

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} ,\\
z_n &= y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)}\frac{f(y_n)}{f'(x_n)} ,\\
x_{n+1} &= z_n - H(\mu_n)\\
&\quad \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)},
\end{aligned} \tag{10}
$$

where $\mu_n = \frac{f(z_n)}{f(x_n)}$ and $H(t)$ represents a real-valued function with $H(0) = 1,\ H'(0) = 2$ and $|H''(0)| < \infty$. Using the function difference, Bi's group constructed another family of eighth-order iterative methods (see [9] formula (13) therein):

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} ,\\
z_n &= y_n - h(\mu_n)\frac{f(y_n)}{f'(x_n)} ,\\
x_{n+1} &= z_n - \frac{f(x_n) + (\gamma + 2)f(z_n)}{f(x_n) + \gamma f(z_n)}\\
&\quad \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)},
\end{aligned} \tag{11}
$$

where $\gamma \in R$ is a constant, $\mu_n = \frac{f(y_n)}{f(x_n)}$ and $h(t)$ represents a real-valued function with $h(0) = 1,\ h'(0) = 2,\ h''(0) = 10$ and $|h'''(0)| < \infty$.

In 2010, Wang and Liu [10] (see formula (16) therein) proposed a robust optimal eighth-order method by using weight functions. The formula for the method is as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} , \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)} , \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \left[\frac{1}{2} + u\left(\frac{1}{2} + \frac{f(z_n)}{f(y_n)}\right)\right] , \end{aligned} \tag{12}$$

where

$$u = \frac{5f(x_n)^2 + 8f(x_n)f(y_n) + 2f(y_n)^2}{5f(x_n)^2 - 12f(x_n)f(y_n)}.$$

Also in 2010, Thukral and Petković [11] (see (12) therein) proposed a family of three-point methods of optimal order for solving nonlinear equations as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} , \\ z_n &= y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n)+bf(y_n)}{f(x_n)+(b-2)f(y_n)} , \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} [\varphi(\frac{f(y_n)}{f(x_n)}) + v(x_n, y_n, z_n)] , \end{aligned} \tag{13}$$

where

$$\varphi(\frac{f(y_n)}{f(x_n)}) = \frac{f(x_n)^2}{f(x_n)^2 - 2f(x_n)f(y_n) - f(y_n)^2},$$

$$v(x_n, y_n, z_n) = \frac{f(z_n)}{f(y_n) - af(z_n)} + 4\frac{f(z_n)}{f(x_n)},$$

and

$$\begin{aligned} &\varphi(0) = 1, \varphi'(0) = 2, \varphi''(0) = 10 - 4b, \\ &\varphi'''(0) = 12b^2 - 72b + 72. \end{aligned}$$

In this paper, based on Newton's method, Lagrange interpolation and Hermite interpolation, we derive two new families of eighth-order methods. In terms of computational cost, they require the evaluations of only three functions and one first-order derivative per iteration. This gives 1.682 as efficiency index of the derived methods. The new methods are comparable with Newton's method and other known methods. The efficacy of the methods is tested on a number of numerical examples.

2 The methods and analysis of convergence

Inspired by scheme (2)-(13), we consider the following three-step iteration scheme by Newton's method and the weight function method:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} , \\ z_n &= y_n - G(\mu_n)\frac{f(y_n)}{f'(x_n)} , \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)} , \end{aligned} \tag{14}$$

where $\mu_n = \frac{f(y_n)}{f(x_n)}$ and $G(t)$ represents a real-valued function. We can prove that scheme (14) is eighth-order convergence under some conditions and it requires five evaluations of the function and its first derivative. Scheme (14) has efficiency index [12] $8^{\frac{1}{5}} = 1.516$. To derive higher efficiency index, we approximate $f'(z_n)$ using other known information by interpolation methods. We first construct Lagrange interpolation polynomial $L_2(x)$ to approximate $f'(z_n)$ so as to meet the interpolation conditions:

$$L_2(x_n) = f(x_n),\ L_2(y_n) = f(y_n),\ L_2(z_n) = f(z_n).$$

Then we get

$$\begin{aligned} L_2(x) = &\frac{(x-y_n)(x-z_n)}{(x_n-y_n)(x_n-z_n)} f(x_n) \\ &+ \frac{(x-x_n)(x-z_n)}{(y_n-x_n)(y_n-z_n)} f(y_n) \\ &+ \frac{(x-x_n)(x-y_n)}{(z_n-x_n)(z_n-y_n)} f(z_n), \end{aligned} \tag{15}$$

and

$$\begin{aligned} L_2'(x) = &\frac{2x-(y_n+z_n)}{(x_n-y_n)(x_n-z_n)} f(x_n) \\ &+ \frac{2x-(x_n+z_n)}{(y_n-x_n)(y_n-z_n)} f(y_n) \\ &+ \frac{2x-(x_n+y_n)}{(z_n-x_n)(z_n-y_n)} f(z_n). \end{aligned} \tag{16}$$

We can obtain an approximation of $f'(z_n)$ by

$$\begin{aligned} f'(z_n) \approx L_2'(z_n) = &\frac{f(z_n)-f(x_n)}{z_n-x_n} \\ &+ \frac{f(z_n)-f(y_n)}{z_n-y_n} - \frac{f(y_n)-f(x_n)}{y_n-x_n}, \\ = &f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n], \end{aligned} \tag{17}$$

where

$$f[x_n, z_n] = \frac{f(z_n) - f(x_n)}{z_n - x_n},$$
$$f[y_n, z_n] = \frac{f(z_n) - f(y_n)}{z_n - y_n},$$
$$f[x_n, y_n] = \frac{f(y_n) - f(x_n)}{y_n - x_n}.$$

Next, we present the first new family of methods as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - G(\mu_n)\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - H(\nu_n)\times \\ &\frac{f(z_n)}{f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n]}. \end{aligned} \quad (18)$$

where

$$\mu_n = \frac{f(y_n)}{f(x_n)}, \quad \nu_n = \frac{f(z_n)}{f(x_n)},$$

$G(t)$ and $H(t)$ represent real-valued functions.

The order of convergence of the preceding methods (18) is analyzed in the following Theorem 1.

Theorem 1 *Assume that functions $G,\ H,\ f$ are sufficiently differentiable functions and f has a simple zero $x^* \in D$. If the initial point x_0 is sufficiently close to x^*, then the methods defined by (18) converge to x^* with eighth-order under the conditions $G(0) = 1,\ G'(0) = 2,\ G''(0) = 10,\ H(0) = 1,\ H'(0) = 1$.*

Proof: Let

$$\begin{aligned} e_n &= x_n - x^*, \\ s_n &= y_n - x^*, \\ p_n &= \frac{f(y_n)}{f'(x_n)}, \\ d_n &= z_n - x^*, \\ c_k &= \frac{f^{(k)}(x^*)}{k!f'(x^*)},\ k = 2, 3, \cdots. \end{aligned}$$

Using Taylor expansion about x^* and taking into account $f(x^*) = 0$, we have

$$\begin{aligned} f(x_n) &= f'(x^*)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 \\ &\quad + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + O(e_n^9)], \\ f'(x_n) &= f'(x^*)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 \\ &\quad + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7 \\ &\quad + O(e_n^8), \\ \frac{f(x_n)}{f'(x_n)} &= e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 \\ &\quad - 4c_2^3 - 3c_4)e_n^4 + 2(4c_2^4 - 10c_2^2c_3 + 3c_3^2 \\ &\quad + 5c_2c_4 - 2c_5)e_n^5 - s_6e_n^6 - s_7e_n^7 - s_8e_n^8 \\ &\quad + O(e_n^9), \end{aligned}$$

and hence

$$\begin{aligned} s_n &= c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 4c_2^3 \\ &\quad - 3c_4)e_n^4 - 2(4c_2^4 - 10c_2^2c_3 + 3c_3^2 \\ &\quad + 5c_2c_4 - 2c_5)e_n^5 + s_6e_n^6 + s_7e_n^7 \\ &\quad + s_8e_n^8 + O(e_n^9), \\ f(y_n) &= f'(x^*)[s_n + c_2s_n^2 + c_3s_n^3 + c_4s_n^4 \\ &\quad + O(e_n^9)], \\ f[x_n, y_n] &= f'(x^*)[1 + c_2e_n + (c_2^2 + c_3)e_n^2 \\ &\quad + (-2c_2^3 + 3c_2c_3 + c_4)e_n^3 + (4c_2^4 \\ &\quad - 8c_2^2c_3 + 2c_3^2 + 4c_2c_4 + c_5)e_n^4 \\ &\quad + O(e_n^5)], \end{aligned} \quad (19)$$

where $s_i\ (i = 6, 7, 8)$ are expressions about coefficients $c_i\ (i = 2, \cdots, 8)$, we omit their specific forms for the sake of brevity. In the following we will use other notations represent the same meanings.

With (19), we obtain that

$$\begin{aligned} \mu_n &= c_2e_n + (-3c_2^2 + 2c_3)e_n^2 + (8c_2^3 \\ &\quad - 10c_2c_3 + 3c_4)e_n^3 + \mu_4e_n^4 + \mu_5e_n^5 \\ &\quad + \mu_6e_n^6 + O(e_n^7), \\ p_n &= c_2e_n^2 + (-4c_2^2 + 2c_3)e_n^3 + (13c_2^3 \\ &\quad - 14c_2c_3 + 3c_4)e_n^4 + (-38c_2^4 + \\ &\quad 64c_2^2c_3 - 12c_3^2 - 20c_2c_4 + 4c_5)e_n^5 \\ &\quad + p_6e_n^6 + p_7e_n^7 + p_8e_n^8 + O(e_n^9), \end{aligned} \quad (20)$$

where $\mu_i\ (i = 4, 5, 6),\ p_i\ (i = 6, 7, 8)$ are expressions about $c_i\ (i = 2, \cdots, 8)$.

Expanding $G(\mu_n)$ at point 0 yields

$$\begin{aligned} G(\mu_n) &= G(0) + G'(0)\mu_n + \frac{G''(0)}{2!}\mu_n^2 \\ &\quad + \frac{G'''(0)}{3!}\mu_n^3 + O(e_n^4). \end{aligned} \quad (21)$$

Using formula (19)-(21) together with conditions $G(0) = 1$ and $G'(0) = 2$, we obtain that

$$\begin{aligned} d_n = s_n - G(\mu_n)p_n = d_4e_n^4 + d_5e_n^5 \\ + d_6e_n^6 + d_7e_n^7 + d_8e_n^8 + O(e_n^9), \end{aligned} \tag{22}$$

where

$$\begin{aligned} d_4 &= -c_2c_3 + c_2^3(5 - \frac{G''(0)}{2}), \\ d_5 &= -2c_3^2 - 2c_2c_4 + c_2^2c_3(32 - 3G''(0)) \\ &\quad + c_2^4(-36 + 5G''(0) - \frac{1}{6}G'''(0)), \end{aligned} \tag{23}$$

and d_i $(i = 6, 7, 8)$ are expressions about c_i $(i = 2, \cdots, 8)$.

Further from (19)-(22) we obtain that

$$\begin{aligned} f(z_n) &= f'(x^*)\left[d_n + c_2d_n^2 + O(e_n^9)\right], \\ \nu_n &= d_4e_n^3 + (d_5 - c_2d_4)e_n^4 + O(e_n^5), \end{aligned} \tag{24}$$

and

$$\begin{aligned} f[x_n, z_n] &= f'(x^*)[1 + c_2e_n + c_3e_n^2 + \\ &\quad c_4e_n^3 + (c_5 + c_2d_4)e_n^4 + O(e_n^5)], \\ f[y_n, z_n] &= f'(x^*)[1 + c_2^2e_n^2 - 2(c_2^3 \\ &\quad - c_2c_3)e_n^3 + (4c_2^4 - 6c_2^2c_3 \\ &\quad + c_2(3c_4 + d_4))e_n^4 + O(e_n^5)]. \end{aligned} \tag{25}$$

From (19), (23) and (24) we get

$$\begin{aligned} & f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n] \\ & = f'(x^*)[1 - c_2c_3e_n^3 + (2c_2^2c_3 - 2c_3^2 - c_2(c_4 \\ & \quad - 2d_4))e_n^4 + O(e_n^5)], \\ & \frac{f(z_n)}{f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n]} \\ & = d_4e_n^4 + d_5e_n^5 + d_6e_n^6 + (d_7 + c_2c_3d_4)e_n^7 \\ & \quad + (d_8 - 2c_2^2c_3d_4 + 2c_3^2d_4 + c_2(c_4d_4 \\ & \quad - d_4^2 + c_3d_5))e_n^8 + O(e_n^9). \end{aligned} \tag{26}$$

Expanding $H(\nu_n)$ at point 0 yields

$$H(\nu_n) = H(0) + H'(0)\nu_n + O(e_n^5). \tag{27}$$

Using (22),(26) and (27), we have

$$\begin{aligned} x_{n+1} - x^* &= d_n - H(\nu_n) \\ & \frac{f(z_n)}{f[x_n, z_n] + f(y_n, z_n] - f[x_n, y_n]} \\ & = R_4e_n^4 + R^5e_n^5 + R_6e_n^6 + R_7e_n^7 \\ & \quad + R_8e_n^8 + O(e_n^9). \end{aligned} \tag{28}$$

where R_i $(i = 4, \cdots, 8)$ are expressions about c_i $(i = 2, \cdots, 8)$.

The conditions

$$H(0) = 1, H'(0) = 1, G''(0) = 10$$

will lead to

$$\begin{aligned} R_4 &= (1 - H(0))d_4 = 0, \\ R_5 &= 0, \\ R_6 &= 0, \\ R_7 &= \frac{1}{2}c_2d_4(2c_3(H'(0) - 1) \\ &\quad + c_2^2H'(0)(G''(0) - 10)) \\ &= 0, \\ R_8 &= c_2c_3(2c_3^2 + c_2c_4 + d_5) \\ &= -\frac{1}{6}c_2^2c_3(-12c_2c_3 + 6c_4 \\ &\quad + c_2^3(G'''(0) - 84)). \end{aligned} \tag{29}$$

It is clear that $R_8 \neq 0$. Thus (18) converges to x^* with eighth-order, in this case, the error equation becomes

$$\begin{aligned} e_{n+1} = -\frac{1}{6}c_2^2c_3(-12c_2c_3 + 6c_4 \\ + c_2^3(G'''(0) - 84))e_n^8 + O(e_n^9). \end{aligned} \tag{30}$$

This finishes the proof of Theorem 1. □

According to scheme (14), we now consider another approximation of $f'(z_n)$. We construct Hermite interpolation polynomial $H_3(x)$ so as to meet the interpolation conditions

$$\begin{aligned} H_3(x_n) = f(x_n),\ H_3(y_n) = f(y_n), \\ H_3(z_n) = f(z_n),\ H_3'(x_n) = f'(x_n). \end{aligned} \tag{31}$$

Clearly, Hermite interpolation polynomial $H_3(x)$ is of the form

$$\begin{aligned} H_3(x) = & \frac{(x - y_n)(x - z_n)}{(x_n - y_n)(x_n - z_n)}[1 - \\ & \frac{(x - x_n)(2x_n - y_n - z_n)}{(x_n - y_n)(x_n - z_n)}]f(x_n) \\ & + \frac{(x - x_n)^2(x - z_n)}{(y_n - x_n)^2(y_n - z_n)}f(y_n) \\ & + \frac{(x - x_n)^2(x - y_n)}{(z_n - x_n)^2(z_n - y_n)}f(z_n) \\ & + \frac{(x - x_n)(x - y_n)(x - z_n)}{(x_n - y_n)(x_n - z_n)}f'(x_n), \end{aligned}$$

$$\begin{aligned} H_3'(z_n) &= -\frac{(3x_n - 2y_n - z_n)(y_n - z_n)}{(x_n - y_n)^2(x_n - z_n)} f(x_n) \\ &+ \frac{(x_n - z_n)^2}{(y_n - x_n)^2(y_n - z_n)} f(y_n) \\ &- \frac{x_n + 2y_n - 3z_n}{(z_n - x_n)(z_n - y_n)} f(z_n) \\ &- \frac{y_n - z_n}{y_n - x_n} f'(x_n). \end{aligned} \tag{32}$$

Simplifying $H_3'(z_n)$ yields

$$\begin{aligned} H_3'(z_n) &= 2\frac{f(z_n) - f(x_n)}{z_n - x_n} + \frac{f(z_n) - f(y_n)}{z_n - y_n} \\ &- 2\frac{f(y_n) - f(x_n)}{y_n - x_n} + \frac{y_n - z_n}{y_n - x_n} \\ &\frac{f(y_n) - f(x_n)}{y_n - x_n} - \frac{y_n - z_n}{y_n - x_n} f'(x_n), \\ &= 2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] \\ &+ (y_n - z_n) f[y_n, x_n, x_n], \end{aligned} \tag{33}$$

where the differences are

$$\begin{aligned} f[x_n, z_n] &= \frac{f(z_n) - f(x_n)}{z_n - x_n}, \\ f[x_n, y_n] &= f[y_n, x_n] = \frac{f(y_n) - f(x_n)}{y_n - x_n}, \\ f[y_n, z_n] &= \frac{f(z_n) - f(y_n)}{z_n - y_n}, \\ f[y_n, x_n, x_n] &= \frac{f[y_n, x_n] - f'(x_n)}{y_n - x_n}. \end{aligned}$$

Thus we can obtain an approximation of $f'(z_n)$ given by

$$\begin{aligned} f'(z_n) \approx H_3'(z_n) &= 2f[x_n, z_n] \\ &+ f[y_n, z_n] - 2f[x_n, y_n] \\ &+ (y_n - z_n) f[y_n, x_n, x_n]. \end{aligned} \tag{34}$$

Therefore, we obtain a new scheme as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - W(\mu_n)\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{H(x_n, y_n, z_n)}, \end{aligned} \tag{35}$$

where

$$\begin{aligned} \mu_n &= \frac{f(y_n)}{f(x_n)}, \\ H(x_n, y_n, z_n) &= 2f[x_n, z_n] + f[y_n, z_n] \\ &- 2f[x_n, y_n] + (y_n - z_n) f[y_n, x_n, x_n] \end{aligned}$$

and $W(t)$ represents real-valued function.

Modified the proof of Theorem 1, we can make the following conclusion.

Theorem 2 *Assume that W and f are sufficiently smooth functions and f has a simple zero $x^* \in D$. If the initial point x_0 is sufficiently close to x^*, then the methods defined by (35) converge to x^* with eighth-order under the conditions $W(0) = 1,\ W'(0) = 2$. The error equation of (35) is*

$$\begin{aligned} e_{n+1} &= \frac{1}{4}c_2^2(2c_3 + c_2^2(W''(0) - 10)) \\ &(2c_2c_3 - 2c_4 + c_2^3(W''(0) \\ &- 10))e_n^8 + O(e_n^9). \end{aligned} \tag{36}$$

In what follows, we give some concrete iterative forms of schemes (18) and (35).

Example 1.1 The functions $G(t)$ and $H(t)$ defined by

$$G(t) = \frac{2 - t}{2 - 5t},\ H(t) = \frac{1 + (1 + a)t}{1 + at}$$

satisfy the conditions of Theorem 1. A new family of one-parameter eighth-order methods is obtained as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(x_n) + (1 + a)f(z_n)}{f(x_n) + af(z_n)} \\ &\frac{f(z_n)}{f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n]}, \end{aligned} \tag{37}$$

where a is a constant. The error equation of (37) is

$$e_{n+1} = \frac{1}{2}c_2^2c_3(3c_2^3 + 4c_2c_3 - 2c_4)e_n^8.$$

Example 1.2 Let $G(t)$ and $H(t)$ be defined by

$$G(t) = \frac{1 + 2t + bt^2}{1 + (b - 5)t^2},\ H(t) = 1 + t.$$

Then they satisfy all conditions in Theorem 1. A new family of one-parameter eighth-order methods is obtained as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)^2 + 2f(x_n)f(y_n) + bf(y_n)^2}{f(x_n)^2 + (b-5)f(y_n)^2} \\ &\quad \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \left[1 + \frac{f(z_n)}{f(x_n)}\right] \\ &\quad \frac{f(z_n)}{f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n]}, \end{aligned} \tag{38}$$

where b is a constant. The error equation of (38) is

$$e_{n+1} = c_2^2 c_3 (2(2+b)c_2^3 + 2c_2c_3 - c_4)e_n^8.$$

Example 1.3 Consider functions $G(t)$ and $H(t)$ defined by

$$G(t) = 1 + 2t + 5t^2, \ H(t) = 1 + \frac{t}{1+ct}.$$

It is easy to check that they satisfy all conditions in Theorem 1. Thus a new eighth-order methods with one-parameter c is obtained as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - (1 + 2\frac{f(y_n)}{f(x_n)} + 5\frac{f(y_n)}{f(x_n)}^2)\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \left[1 + \frac{f(z_n)}{f(x_n) + cf(z_n)}\right] \\ &\quad \frac{f(z_n)}{f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n]}, \end{aligned} \tag{39}$$

where $c \in R$ is a constant. The corresponding error equation is

$$e_{n+1} = c_2^2 c_3 (14c_2^3 + 2c_2c_3 - c_4)e_n^8.$$

Example 1.4 Let $G(t)$ and $H(t)$ be defined by

$$G(t) = \frac{1 + 3t + 6t^2}{1 + t - t^2}, \ H(t) = \exp^t.$$

Then G and H are the functions required in Theorem 1. A new eighth-order methods is obtained as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)^2 + 3f(x_n)f(y_n) + 6f(y_n)^2}{f(x_n)^2 + f(x_n)f(y_n) - f(y_n)^2}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \exp^{\frac{f(z_n)}{f(x_n)}} \\ &\quad \frac{f(z_n)}{f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n]}, \end{aligned} \tag{40}$$

The error equation of (40) is

$$e_{n+1} = c_2^2 c_3 (17c_2^3 + 2c_2c_3 - c_4)e_n^8.$$

In the following examples, we give some functions and eight-order methods bases on Theorem 2.

Example 2.1 Let $W(t)$ be

$$W(t) = \frac{2-t}{2-5t}.$$

Then it holds that $W(0) = 1$, $W'(0) = 2$. So all conditions in Theorem 2 are satisfied. So a new eighth-order method is given by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{H(x_n, y_n, z_n)}. \end{aligned} \tag{41}$$

The error equation of (41) is

$$e_{n+1} = c_2^2 c_3 (c_2c_3 - c_4)e_n^8.$$

Example 2.2 The function $W(t)$ defined by

$$W(t) = \frac{1 + 2t + dt^2}{1 + (d-5)t^2}$$

satisfies the conditions of Theorem 2. A new family of one-parameter eighth-order methods is obtained as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)^2 + 2f(x_n)f(y_n) + df(y_n)^2}{f(x_n)^2 + (d-5)f(y_n)^2}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{H(x_n, y_n, z_n)}, \end{aligned} \tag{42}$$

where d is a constant. The error equation of (42) is

$$\begin{aligned} e_{n+1} &= c_2^2((30 - 11d + d^2)c_2^2 + c_3) \\ &\quad ((30 - 11d + d^2)c_2^3 + c_2c_3 - c_4)e_n^8. \end{aligned}$$

Example 2.3 Consider function $W(t)$ defined by

$$W(t) = \frac{1 + (2+\alpha_1)t}{1 + \alpha_1 t + \alpha_2 t^2}.$$

Clearly, it satisfies the conditions of Theorem 2. A new family of two-parameter eighth-order methods is obtained as follows:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - \frac{f(x_n)^2 + (2+\alpha_1)f(x_n)f(y_n)}{f(x_n)^2 + \alpha_1 f(x_n)f(y_n) + \alpha_2 f(y_n)^2} \\
&\quad \frac{f(y_n)}{f'(x_n)}, \\
x_{n+1} &= z_n - \frac{f(z_n)}{H(x_n, y_n, z_n)},
\end{aligned} \tag{43}
$$

where α_1, α_2 are constants. The error equation of (43) is

$$
\begin{aligned}
e_{n+1} = c_2^2((5 + 2\alpha_1 + \alpha_2)c_2^2 - c_3) \\
((5 + 2\alpha_1 + \alpha_2)c_2^3 - c_2c_3 + c_4)e_n^8.
\end{aligned}
$$

Example 2.4 Take function $W(t)$ as

$$W(t) = 1 + 2t + \alpha_3 t^2,$$

which satisfies the conditions of Theorem 2. A new family of one-parameter eighth-order methods is obtained as follows:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - (1 + 2\frac{f(y_n)}{f(x_n)} + \alpha_3 \frac{f(y_n)}{f(x_n)}^2) \\
&\quad \frac{f(y_n)}{f'(x_n)}, \\
x_{n+1} &= z_n - \frac{f(z_n)}{H(x_n, y_n, z_n)},
\end{aligned} \tag{44}
$$

where α_3 is a constant. The error equation of (44) is

$$
\begin{aligned}
e_{n+1} = c_2^2((-5 + \alpha_3)c_2^2 + c_3) \\
((-5 + \alpha_3)c_2^3 + c_2c_3 - c_4)e_n^8.
\end{aligned}
$$

In terms of computational cost, the developed methods require evaluations of only three functions and one first derivative per iteration. With the definition of efficiency index [12], the new methods have the efficiency indexes $8^{\frac{1}{4}} = 1.682$, which are better than $3^{\frac{1}{3}} = 1.442$ in [13]-[15] and [25], $4^{\frac{1}{3}} = 1.587$ in [26] and [27], $5^{\frac{1}{4}} = 1.495$ in [16] and [17], $6^{\frac{1}{4}} = 1.565$ in [4], [18]-[24] and [28], $7^{\frac{1}{4}} = 1.627$ in [6] and Newton's method $2^{\frac{1}{2}} = 1.414$ in [1].

3 Numerical results and conclusions

In this section, we present some results of the numerical simulations to compare the efficiencies of the present methods with the others. The old methods considered are NM method (1), CM6 method (5) with

$$H(\mu_n) = 1 + 2\mu_n + \mu_n^2 + \mu_n^3,$$

KM7 method (7) ($\alpha = 1$), BM8 method (10) with

$$H(\mu_n) = \frac{1 + 3\mu_n}{1 + \mu_n},$$

BM8-2 method (11) with

$$h(\mu_n) = (\frac{1}{1 - 3\mu_n})^{\frac{2}{3}}, \gamma = 1,$$

WLM method (12), TPM method (13) with

$$
\begin{aligned}
a = 1, b = 0, \\
\varphi(\frac{f(y_n)}{f(x_n)}) = 1 + 2\frac{f(y_n)}{f(x_n)} \\
+ 5(\frac{f(y_n)}{f(x_n)})^2 + 12(\frac{f(y_n)}{f(x_n)})^3
\end{aligned}
$$

and new methods (37) ($a = 3$), (38) ($b = 3$), (39) ($c = 5$),(41), (42) ($d = 3$)and (43) ($\alpha_1 = -1, \alpha_2 = -3$).

Numerical results reported here have been carried out in a Mathematica 4.0 environment. Table 1 shows the difference of the root x^* and the approximation x_n, where x^* is the exact root computed with 800 significant digits and x_n is calculated by the same total number of function evaluations (TNFE) for all methods. The absolute values of the function ($|f(x_n)|$) and the computational order of convergence (COC) are also shown in Table 1. Here, the COC is defined by [15]

$$\rho \approx \frac{\ln |(x_{n+1} - x^*)/(x_n - x^*)|}{\ln |(x_n - x^*)/(x_{n-1} - x^*)|}.$$

The test functions are listed as follows

$$
\begin{aligned}
&f_1(x) = e^{x^2+7x-30} - 1, \quad x^* = 3 \\
&f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \\
&\quad x^* \approx -1.2076478271309189270 \\
&f_3(x) = \cos x - x, \\
&\quad x^* \approx 0.7390851332151606417 \\
&f_4(x) = \sin x - \frac{x}{3}, \\
&\quad x^* \approx 2.2788626600758283127 \\
&f_5(x) = e^{-x^2+x+2} - 1, \quad x^* = -1 \\
&f_6(x) = x^3 + 4x^2 - 10, \\
&\quad x^* \approx 1.3652300134140968458
\end{aligned}
$$

Table 1 Comparison of various iterative methods under the same total number of function evaluations (TNFE=12)

	$\|x_n - x^*\|$	$\|f(x_n)\|$	COC	$\|x_n - x^*\|$	$\|f(x_n)\|$	COC
	$f_1(x), x_0 = 3.1$			$f_2(x), x_0 = -1.3$		
NM	5.32643e-20	6.92436e-19	1.99999851	2.46839e-56	5.01266e-55	2.00000000
CM6	9.49963e-62	1.23495e-60	5.99705711	2.88306e-200	5.85476e-199	6.00000000
KM7	3.52649e-112	4.58443e-111	6.99969160	1.23352e-360	2.50496e-359	7.00000000
BM8	2.02449e-87	2.63183e-86	7.99417911	2.32630e-399	4.72411e-398	8.00000000
BM8-2	3.78409e-26	4.91931e-25	7.99846931	5.93625e-409	1.20550e-407	8.00000000
WLM	2.01292e-197	2.61679e-196	7.99982698	1.80610e-515	3.66773e-514	7.99417911
TPM	3.26251e-117	4.24126e-116	7.99748167	1.30343e-412	2.64692e-411	7.99999999
(37)	7.40302e-112	9.62393e-111	7.99796240	3.25588e-414	6.61184e-413	8.00000000
(38)	6.63474e-123	8.62516e-122	8.00612815	1.28170e-453	2.60280e-452	7.99999999
(39)	7.26578e-123	9.44552e-122	8.00751084	6.29989e-408	1.27935e-406	8.00000006
(41)	5.88114e-169	7.64548e-168	8.00041192	1.99255e-535	4.04636e-534	8.00000000
(42)	1.97332e-150	2.56532e-149	7.99251954	4.39175e-531	8.91851e-530	8.00000000
(43)	6.63474e-123	8.62516e-122	8.00612815	2.22954e-519	4.52761e-518	8.00000000
	$f_3(x), x_0 = 1.2$			$f_4(x), x_0 = 2.8$		
NM	2.83086e-71	4.73777e-71	2.00000000	4.23677e-55	4.16771e-55	2.00000000
CM6	2.48233e-184	4.15445e-184	5.99999900	2.58239e-140	2.54029e-140	5.99997882
KM7	2.01626e-293	3.37443e-293	6.99999990	1.91715e-224	1.88590e-224	6.99999681
BM8	3.33577e-510	5.58278e-510	8.00000000	1.96953e-394	1.93743e-394	7.99999988
BM8-2	1.27423e-512	2.13257e-512	8.00000000	1.91393e-397	1.88273e-397	7.99999971
WLM	2.91269e-439	4.87472e-439	7.99999999	2.26910e-335	2.23211e-335	7.99999960
TPM	4.02340e-417	6.73362e-417	7.99999997	1.95884e-331	1.92691e-331	7.99999815
(37)	1.27802e-516	2.13890e-516	8.00000000	3.83017e-400	3.76773e-400	7.99999979
(38)	1.67161e-557	2.79762e-557	8.00000000	1.32907e-405	1.30740e-405	7.99999997
(39)	1.60906e-548	2.69294e-548	8.00000000	1.17286e-410	1.15374e-410	7.99999999
(41)	2.19998e-557	3.68191e-557	8.00000000	8.96834e-424	8.82215e-424	8.00000004
(42)	2.43944e-545	4.08268e-545	8.00000000	5.65974e-413	5.56748e-413	8.00000033
(43)	7.91592e-555	1.32482e-554	8.00000000	1.76740e-421	1.73859e-421	8.00000008
	$f_5(x), x_0 = 0$			$f_6(x), x_0 = 1.5$		
NM	6.85471e-28	2.05640e-27	1.99999996	2.74724e-77	4.53662e-76	2.00000000
CM6	2.05678e-50	6.17035e-50	5.98965740	6.64702e-238	1.09765e-236	5.99999998
KM7	1.13818e-91	3.41453e-91	6.99817441	3.69560e-393	6.10270e-392	7.00000000
BM8	2.47753e-147	7.43258e-147	7.99869054	4.60627e-631	7.60652e-630	8.00000000
BM8-2	1.73101e-149	5.19303e-149	7.99914092	7.38085e-643	1.21883e-641	8.00000000
WLM	4.09038e-119	1.22711e-118	8.00483114	4.44490e-617	7.34004e-616	8.00000000
TPM	8.56807e-119	2.57042e-118	7.99702662	1.76368e-519	2.91243e-518	8.00000000
(37)	2.56600e-159	7.69799e-159	7.99915981	6.85805e-649	1.13250e-647	8.00000000
(38)	8.21683e-155	2.46505e-155	7.99676171	3.26690e-599	5.39477e-598	8.00000000
(39)	1.32676e-151	3.98027e-151	7.99526033	1.19364e-572	1.97111e-571	8.00000000
(41)	1.08571e-194	3.25714e-194	7.99975290	3.41497e-720	5.63927e-719	8.00000000
(42)	6.45930e-194	1.93779e-193	7.99871853	1.94284e-710	3.20829e-709	8.00000000
(43)	1.54948e-194	4.64844e-194	7.99957730	1.13849e-749	1.88004e-748	8.00000000

In Table 1, $f(x)$ is the test function, x_0 is the original iteration value, COC is the computational order of convergence. Here NM method is the second-order, CM6 is the sixth-order, KM7 is the seventh-order and other methods are the eighth-order. The results presented in Table 1 show that the proposed families have higher convergence order and higher efficiency index compared with the other methods.

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