The Solutions of Initial Value Problems for Nonlinear Fourth-Order Impulsive Integro-Differential Equations in Banach Spaces

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Abstract: In this paper, we investigate the maximal and minimal solutions for initial value problem of fourth order impulsive differential equations by using cone theory and the monotone iterative method to some existence results of solution are obtained. As an application, we give an example to illustrate our results.

Key-words: Banach space, Cone, Initial value problem, Impulsive integro-differential equations

1. Introduction

Impulsive integro-differential equations have become more important in recent years in some mathematical models in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equations in R_n, see [1]. Impulsive integro-differential equations both for first and second order have been studied by many authors, see [4-13].Only a few papers have implemented the fourth order impulsive equations, see [14-15].In [14],the author use variation methods and a three critical points theorem to investigate impulsive equation without impulsive differential inequalities.

In this paper, by applying a new corresponding result connected with fourth-order impulsive differential inequalities, we apply cone theory and the monotone iterative method to investigate the maximal and minimal solutions.

Consider the following initial value problem of fourth order impulsive differential equations:

$$\begin{cases} x^{(4)}(t) = f(t, x(t), x'(t), x''(t), x'''(t), \\ \qquad (Tx)(t), (Sx)(t)), \forall t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = I_{0k}(x''(t_k)), \\ \Delta x'|_{t=t_k} = I_{1k}(x'(t_k), x''(t_k)), \\ \Delta x''|_{t=t_k} = I_{2k}(x'''(t_k)), \\ \Delta x'''|_{t=t_k} = I_{3k}(x(t_k), x'(t_k), x''(t_k), x'''(t_k)) \\ \qquad (k = 1, 2, \cdots, m), \\ x(0) = x_0^*, x'(0) = x_1^*, x''(0) = x_2^*, x'''(0) = x_3^* \end{cases} \tag{1}$$

where $J = [0, a](a > 0)$, $x_0^*, x_1^*, x_2^*, x_3^* \in E$, θ is the zero element of E,
$f \in C[J \times E \times E \times E \times E \times E \times E, E]$,
$0 < t_1 < \cdots < t_k < \cdots < t_m < a, I_{0k} \in C[E, E]$,
$I_{1k} \in C[E \times E, E], I_{2k} \in C[E, E]$,
$I_{3k} \in C[E \times E \times E \times E, E] \quad (k = 1, 2, \cdots, m)$,

$$(Tx)(t) = \int_0^t k(t,s)x(s)ds,$$

$$(Sx)(t) = \int_0^a h(t,s)x(s)ds,$$

$\forall t \in J,\ k \in C[D, R_+],\ D = \{(t,s) \in J \times J \mid t \geq s\}$, $h \in C[J \times J, R_+],\ R_+ = [0, +\infty)$.

$$\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-),$$

$$\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-),$$

$x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of x at t_k,respectively. Similarly, $x'(t_k^+)$ and $x'(t_k^-)$ denote the right and left limits of x' at t_k, respectively.

$$\Delta x''|_{t=t_k} = x''(t_k^+) - x''(t_k^-),$$

$$\Delta x'''|_{t=t_k} = x'''(t_k^+) - x'''(t_k^-),$$

$x''(t_k^+)$ and $x''(t_k^-)$ denote the right and left limits of x'' at t_k, respectively. Similarly, $x'''(t_k^+)$ and $x'''(t_k^-)$ denote the right and left limits of x''' at t_k, respectively.

Let $PC[J, E] = \{x : J \to E \mid x(t)$ is continuous at $t \neq t_k$, $x(t_k^+)$ exist, $x(t_k^-) = x(t_k)\}$.

Indeed, $PC[J, E]$ is a Banach space with the norm

$$\| x \|_{PC} = \sup_{t \in J} \| x(t) \|$$

Let $PC^3[J, E] = \{x \in PC[J, E] \mid x'''(t)$ is continuous at $t \neq t_k$, $x'''(t_k^+)$ and $x'''(t_k^-)$ exist $\}$

For $x \in PC^3[J, E]$, we have

$$x''(t_k-\varepsilon)=x''(t)+\int_t^{t_k-\varepsilon}x'''(s)ds,$$
$$\forall t_{k-1}<t<t_k-\varepsilon<t_k,\varepsilon>0 \qquad (2)$$

Because $x'''(t_k^-)$ exists, there exists the limit $x''(t_k^-)$ of (2) as $\varepsilon\to 0^+$, and

$$x''(t_k^-)=x''(t)+\int_t^{t_k}x'''(s)ds,\ \forall t_{k-1}<t<t_k.$$

In the same way, we obtain

$$x''(t_k^+),\ x'(t_k^-),\ x'(t_k^+).$$

Let $x'(t_k)=x'(t_k^-),\ x''(t_k)=x''(t_k^-),\ x'''(t_k)=x'''(t_k^-)$. Obviously, $x',x'',x'''\in PC[J,E]$ .Indeed, $PC^3[J,E]$ is a Banach space with the respective norm:
$\|x\|_{PC^3}=\max\{\|x\|_{PC},\|x'\|_{PC},\|x''\|_{PC},\|x'''\|_{PC}\}$.

Let $PC^2[J,E]=\{x\in PC[J,E]\,|\,x''(t)$ is continuous at $t\neq t_k$, $x''(t_k^-)$ and $x''(t_k^+)$ exist $\}$. For $x\in PC^2[J,E]$, similarly, $x'(t_k^-),\ x'(t_k^+)$ exist. Let $x'(t_k)=x'(t_k^-),\ x''(t_k)=x''(t_k^-)$. Obviously, $x',x''\in PC[J,E]$. Indeed, $PC^2[J,E]$ is a Banach space with the respective norm:

$$\|x\|_{PC^2}=\max\{\|x\|_{PC},\|x'\|_{PC},\|x''\|_{PC}\}$$

Let $PC^1[J,E]=\{x\in PC[J,E]\,|\,x'(t)$ is continuous at $t\neq t_k$, $x'(t_k^-)$ and $x'(t_k^+)$ exist$\}$.

For $x\in PC^1[J,E]$, let $x'(t_k)=x'(t_k^-)$. Obviously $x'\in PC[J,E]$. Indeed, $PC^1[J,E]$ is a Banach space with respect to the norm:

$$\|x\|_{PC^1}=\max\{\|x\|_{PC},\|x'\|_{PC}\}$$

Let $J'=J\setminus\{t_1,t_2,\cdots,t_m\}$, $t_0=0$, $t_{m+1}=a$, $J_0=[0,t_1]$, $J_1=(t_1,t_2],\cdots,J_{m-1}=(t_{m-1},t_m],J_m=(t_m,a]$, $\tau=\max\{t_i-t_{i-1}\,|\,i=1,2,\cdots,m+1\}$ .Denote the norm $\|\cdot\|_{C^1[J_0,E]}$ in the space $C^1[J_0,E]$ and denote the norm $\|\cdot\|_{PC^1[\bar J_i,E]}$ and $\|\cdot\|_{PC^1[J_i,E]}$ in the space $PC^1[\bar J_i,E]$ and $PC^1[J_i,E]$ respectively. $\bar J_i$ is the closure of J_i. If there exists x such that $x\in PC^3[J,E]\cap C^4[J',E]$ and IVP (1), then x is called the solution of IVP (1).

2. Preliminaries

Suppose that E is a real Banach space which is partially ordered by a cone $P\subset E$. we say "$x\le y$" if and only if $y-x\in P$ .Moreover P is called normal if there exists a constant $N>0$ such that for all $x,y\in E$, $\theta\le x\le y$ implies $\|x\|\le N\|y\|$. In the case N is called the normality constant of P. P is called regular if there exists $y\in E$ such that $x_1\le x_2\le\cdots\le x_n\le\cdots\le y$ implies $x^*\in E$ such that $\|x_n-x^*\|\to 0$ as $n\to\infty$ .Further information can be found in [2].

Lemma 2.1. Assume that $p\in PC^1[J,E]\cap C^2[J',E]$ satisfies

$$\begin{cases} p''(t)\le -M_1(t)p(t)-M_2(t)p'(t),\forall t\in J,t\neq t_k,\\ \Delta p|_{t=t_k}=C_k p'(t_k),(k=1,2,\cdots,m)\\ \Delta p'|_{t=t_k}\le -L_k p(t_k)-L_k^* p'(t_k),\\ p'(0)\le p(0)\le\theta,\end{cases} \qquad (3)$$

where $M_1(t)$, $M_2(t)$ are bounded with $M_1\ge 0$, $M_2\ge 0$ on J and $M_1,M_2\in L^1[0,a]$. C_k, L_k, L_k^* are all nonnegative constants, and we have

(i) $$a(M_1^*(1+a+\sum_{k=1}^m C_k)+M_2^*)+\sum_{k=1}^m(L_k(1+t_k+\sum_{i=0}^{k-1}C_i)+L_k^*)\le 1 \qquad (4)$$

(ii) $$M_2^*>0,\ aM_1^*((e^{aM_2^*})+\frac{(e^{aM_2^*})-1}{M_2^*}+(e^{aM_2^*})\sum_{k=1}^m(C_k(e^{-M_2^*t_k}))+\sum_{k=1}^m(L_k((e^{M_2^*t_k})+\frac{(e^{aM_2^*t_k})-1}{M_2^*}+\sum_{i=0}^{k-1}C_i(e^{M_2^*(t_k-t_i)}))+L_k^*)\le 1 \qquad (5)$$

where
$M_1^*=\sup\{M_1(t)\,|\,t\in J\},M_2^*=\sup\{M_2(t)\,|\,t\in J\}$, $C_0=0$, then $p(t)\le\theta,\ p'(t)\le\theta,\ \forall t\in J$.

Proof. Let $P^*=\{g\in E^*\,|\,g(x)\ge 0,\forall x\in P\}$ .For any $g\in P^*$ such that $v(t)=g(p(t))$, then $v\in PC^1[J,R]\cap C^2[J',R]$ and $v''(t)=g(p''(t))$, $v'(t)=g(p'(t))$, $\forall t\in J$. By (3) we have

$$\begin{cases} v''(t)\le -M_1(t)v(t)-M_2(t)v'(t),\forall t\in J,t\neq t_k,\\ \Delta v|_{t=t_k}=C_k v'(t_k),\\ \Delta v'|_{t=t_k}\le -L_k v(t_k)-L_k^* v'(t_k),(k=1,2,\cdots m),\\ v'(0)\le v(0)\le 0.\end{cases} \qquad (6)$$

Put

$v_*(t) = v'(t)\ (t \in J)$, then $v_* \in PC[J,R] \cap C^1[J',R]$ and

$$v(t) = v(0) + \int_0^t v_*(s)ds + \sum_{0<t_k<t} \Delta v|_{t=t_k}$$
$$= v(0) + \int_0^t v_*(s)ds + \sum_{0<t_k<t} C_k v_*(t_k), \forall t \in J \quad (7)$$

So we have by (6)

$$\begin{cases} v_*'(t) \le -M_1(t)(v(0) + \int_0^t v_*(s)ds + \sum_{0<t_k<t} C_k v_*(t_k)) \\ \qquad - M_2(t)v_*(t), \forall t \in J, t \ne t_k \\ \Delta v_*|_{t=t_k} \le -L_k(v(0) + \int_0^{t_k} v_*(s)ds \\ \qquad + \sum_{i=0}^{k-1} C_i v_*(t_i)) - L_k^* v_*(t_k) \\ v_*(0) \le v(0) \le 0, (k = 1, 2, \cdots, m), . \end{cases} \quad (8)$$

Next, we show

$$v_*(t) \le 0, \forall t \in J \quad (9)$$

We suppose the inequality $v_*(t) \le 0, t \in J$ is not true. This means that we can find $t^* \in J$ such that $v_*(t^*) > 0$.We have the next two cases:

Case (a): Assume that $t^* \in J_j = (t_j, t_{j+1}]$.Let

$$\inf_{0 \le t \le t^*} v_*(t) = -\lambda.$$

Then $\lambda \ge 0.$

(i) $\lambda = 0.$ By (8), we have

$$v_*'(t) \le 0,\ \Delta v_*|_{t=t_k} \le 0.$$

Then $v_*(t)$ is decreasing on $[0, t^*]$, so

$$v_*(t^*) \le v_*(0) \le 0.$$

This is a contradiction with $v_*(t^*) > 0$.

(ii) $\lambda > 0.$ There exists $t_* \in J_n,\ n \in \{1, 2, \cdots, m\}$ such that $v_*(t_*) = -\lambda$ or $v_*(t_n^+) = -\lambda.$ Below we discuss only the situation when $v_*(t_*) = -\lambda.$ (The proof is similar, when $v_*(t_n^+) = -\lambda$).We obtain by (8)

$$v_*'(t) \le M_1^*(1 + a + \sum_{k=1}^m C_k)\lambda + M_2^*\lambda = M_0\lambda,$$
$$\forall t \in [0, t^*], t \ne t_k, \quad (10)$$

$$\Delta v_*|_{t=t_k} \le L_k(1 + t_k + \sum_{i=0}^{k-1} C_i)\lambda + L_k^*\lambda,\ \forall t_k \le t^* \quad (11)$$

where

$$M_0 = M_1^*(1 + a + \sum_{k=1}^m C_k) + M_2^*. \quad (12)$$

Then we have

$$\begin{cases} v_*(t^*) - v_*(t_j^+) = v_*'(\xi_j)(t^* - t_j), t_j < \xi_j < t^*, \\ v_*(t_j) - v_*(t_{j-1}^+) = v_*'(\xi_{j-1})(t_j - t_{j-1}), t_{j-1} < \xi_{j-1} < t_j, \\ \cdots\cdots \\ v_*(t_{n+2}) - v_*(t_{n+1}^+) = v_*'(\xi_{n+1})(t_{n+2} - t_{n+1}) \\ \qquad t_{n+1} < \xi_{n+1} < t_{n+2}, \\ v_*(t_{n+1}) - v_*(t_*) = v_*'(\xi_n)(t_{n+1} - t_*), t_* < \xi_n < t_{n+1}. \end{cases} \quad (13),$$

By (11) we know

$$v_*(t_k^+) = v_*(t_k) + \Delta v_*|_{t=t_k} \le v_*(t_k) + L_k(1 + t_k + \sum_{i=0}^{k-1} C_i)\lambda$$
$$+ L_k^*\lambda. \quad (14)$$

Combing (10), (11) and (13),(14), this yields

$$\begin{cases} v_*(t^*) - v_*(t_j) \le L_j(1 + t_j + \sum_{i=0}^{j-1} C_i)\lambda + L_j^*\lambda \\ \qquad + \lambda M_0(t^* - t_j) \\ v_*(t_j) - v_*(t_{j-1}) \le L_{j-1}(1 + t_{j-1} + \sum_{i=0}^{j-2} C_i)\lambda + L_{j-1}^*\lambda \\ \qquad + \lambda M_0(t_j - t_{j-1}), \\ \cdots\cdots \\ v_*(t_{n+2}) - v_*(t_{n+1}) \le L_{n+1}(1 + t_{n+1} + \sum_{i=0}^{n} C_i)\lambda \\ \qquad + L_{n+1}^*\lambda + \lambda M_0(t_{n+2} - t_{n+1}), \\ v_*(t_{n+1}) + \lambda \le \lambda M_0(t_{n+1} - t_*). \end{cases} \quad (15)$$

Adding those inequalities,we have

$$\lambda < v_*(t^*) + \lambda$$
$$\le \lambda \sum_{k=n+1}^{j} L_k(1 + t_k + \sum_{i=0}^{k-1} C_i) + \lambda \sum_{k=n+1}^{j} L_k^*$$
$$+ \lambda M_0(t^* - t_*)$$
$$\le \lambda \sum_{k=1}^{m} L_k(1 + t_k + \sum_{i=0}^{k-1} C_i)$$
$$+ \lambda \sum_{k=1}^{m} L_k^* + \lambda M_0 a, \quad (16)$$

This means that

$$1 < \sum_{k=1}^{m} L_k(1 + t_k + \sum_{i=0}^{k-1} C_i) + \sum_{k=1}^{m} L_k^* + M_0 a. \quad (17)$$

This is a contradiction with (4).

Case (b): when (ii) satisfies, putting

$$w(t) = v_*(t) \quad e^{\int_0^t M_2(s)ds},$$

by (8) we have

$$\begin{cases} w'(t) \le -M_1(t)(v(0)(e^{\int_0^t M_2(s)ds} + \int_0^t (e^{\int_s^t M_2(r)dr})w(s)ds \\ \quad + \sum_{0<t_k<t} C_k (e^{\int_{t_k}^t M_2(s)ds})w(t_k), \qquad \forall t \in J, t \ne t_k, \\ \Delta w|_{t=t_k} \le -L_k(v(0)(e^{\int_0^{t_k} M_2(s)ds}) \\ \quad + \int_0^{t_k} (e^{\int_s^{t_k} M_2(r)dr})w(s)ds \\ \quad + \sum_{i=0}^{k-1} C_i (e^{\int_{t_i}^{t_k} M_2(r)dr})w(t_i)) - L_k^* w(t_k), \\ w(0) \le v(0) \le 0, (k = 1, 2, \cdots, m). \end{cases}$$

In the same way, we have

$$w(t) \le 0.$$

Hence, $v_*(t) \le 0$. It means that $v'(t) \le 0,\ \forall t \in J$. This yields

$$v(t) = v(0) + \int_0^t v_*(s)ds + \sum_{0<t_k<t} C_k v_*(t_k) \le 0, \forall t \in J.$$

Moreover, for any $g \in P^*$, we have $p(t) \le \theta$, $p'(t) \le \theta,\ \forall t \in J$. This ends the proof.

Lemma 2.2.[1] Let $m \in PC[J, R_+], k \in C[D, R_+]$, $\beta_i \ge 0 (i = 1, 2, \cdots, m)$ is constant and

$$m(t) \le \int_0^t k(t,s)m(s)ds + \sum_{0<t_i<t} \beta_i m(t_i),\ \forall t \in J.$$

Then $m(t) \le 0$.

Lemma 2.3.[1] If $H \subset PC[J, E]$ is a bounded and countable set, then we have $\alpha(H(t)) \in L[J, R_+]$ and

$$\alpha(\{ \int_0^a x(t)\ dt: x \in H\}) \le 2\int_0^a \alpha(H(t))dt.$$

Lemma 2.4.[1] Assume that $H \subset PC^1[J, E]$ is bounded set, and the functions belonging to H' are equicontinuity on $J_k (k = 1, 2, \cdots m)$

$$\alpha_{PC^1}(H) = \max\{\sup_{t \in J} \alpha(H(t)), \sup_{t \in J} \alpha(H'(t))\}.$$

where α_{PC^1} is a measure of noncompactness in $PC^1[J, E]$.

In order to study the fourth-order impulsive integro-differential equations, we study the second-order impulsive differential equations firstly by method of the reduction of order.

3. Some results of the second order impulsive differential equations

We investigate the following second order impulsive differential equations:

$$\begin{cases} u''(t) = f(t, (Bu)(t), (Fu)(t), u(t), u'(t), \\ \quad (TBu)(t), (SBu)(t)), \forall t \in J, t \ne t_k, \\ \Delta u|_{t=t_k} = I_{2k}(u'(t_k)), \\ \Delta u'|_{t=t_k} = I_{3k}((Bu)(t_k), (Fu)(t_k), u(t_k), u'(t_k)) \\ \quad (k = 1, 2, \cdots m), \\ u(0) = x_2^*, u'(0) = x_3^* \end{cases} \tag{18}$$

where $J = [0, a] (a > 0)$, $f \in C[J \times E \times E \times E \times E \times E \times E,\ E]$, $0 < t_1 < \cdots < t_k < \cdots < t_m < a$, $I_{2k} \in C[E, E]$, $I_{3k} \in C[E \times E \times E \times E, E]$ $(k = 1, 2, \cdots, m)$, $x_2^*, x_3^* \in E$, $(Tu)(t) = \int_0^t k(t,s)u(s)ds$, $(Su)(t) = \int_0^a h(t,s)u(s)ds,\ \forall t \in J,\ k \in C[D, R_+],\ D = \{(t,s) \in J \times J \mid t \ge s\}$, $h \in C[J \times J, R_+], R_+ = [0, +\infty)$. $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $\Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$. $u(t_k^+)$ and $u(t_k^-)$ denote the right and left limits of u at t_k, respectively. Similarly, $u'(t_k^+)$ and $u'(t_k^-)$ denote the right and left limits of u' at t_k, respectively. Define two operators B and F

$B: PC^1[J, E] \cap C^2[J', E] \to PC^3[J, E] \cap C^4[J', E]$

$F: PC^1[J, E] \cap C^2[J', E] \to PC^2[J, E] \cap C^3[J', E]$

They are continuous and increasing operators.

Assume that the following conditions are satisfied:

(H_1) There exist $u_0, v_0 \in PC^1[J, E] \cap C^2[J', E]$ such that $u_0(t) \le v_0(t), u_0'(t) \le v_0'(t),\ \forall t \in J$ and

$$\begin{cases} u_0''(t) \le f(t, (Bu_0)(t), (Fu_0)(t), u_0(t), u_0'(t), \\ \quad (TBu_0)(t), (SBu_0)(t)), \forall t \in J, t \ne t_k, \\ \Delta u_0|_{t=t_k} = I_{2k}(u_0'(t_k)), \\ \Delta u_0'|_{t=t_k} \le I_{3k}((Bu_0)(t_k), (Fu_0)(t_k), u_0(t_k), u_0'(t_k)) \\ \quad (k = 1, 2, \cdots m), \\ u_0(0) \le x_2^*, u_0'(0) - u_0(0) \le x_3^* - x_2^*, \end{cases} \tag{19}$$

$$\begin{cases} v_0''(t) \ge f(t, (Bv_0)(t), (Fv_0)(t), v_0(t), v_0'(t), \\ \quad (TBv_0)(t), (SBv_0)(t)), \forall t \in J, t \ne t_k \\ \Delta v_0|_{t=t_k} = I_{2k}(v_0'(t_k)), \\ \Delta v_0'|_{t=t_k} \ge I_{3k}((Bv_0)(t_k), (Fv_0)(t_k), v_0(t_k), v_0'(t_k)) \\ \quad (k = 1, 2, \cdots m), \\ v_0(0) \ge x_2^*, v_0'(0) - v_0(0) \ge x_3^* - x_2^* \end{cases} \tag{20}$$

(H_2) There exist $M_1(t),\ M_2(t)$ are bounded with $M_1 \ge 0,\ M_2 \ge 0$ on J and $M_1, M_2 \in L^1[0,a]. C_k \ge 0, L_k \ge 0, L_k^* \ge 0,\ (k=1,2,\cdots m)$ such that

$$f(t,x,y,z,u,v,w) - f(t,\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w})$$
$$\ge -M_1(t)(z-\bar{z}) - M_2(t)(u-\bar{u}), \forall t \in J,$$
$$I_{2k}(u) - I_{2k}(\bar{u}) = C_k(u-\bar{u}),$$
$$I_{3k}(x,y,z,u) - I_{3k}(\bar{x},\bar{y},\bar{z},\bar{u})$$
$$\ge -L_k(z-\bar{z}) - L_k^*(u-\bar{u}),$$
$$(Bu_0)(t) \le \bar{x} \le x \le (Bv_0)(t),$$
$$(Fu_0)(t) \le \bar{y} \le y \le (Fv_0)(t),$$
$$u_0(t) \le \bar{z} \le z \le v_0(t),\ u_0'(t) \le \bar{u} \le u \le v_0'(t),$$
$$(TBu_0)(t) \le \bar{v} \le v \le (TBv_0)(t)$$
$$(SBu_0)(t) \le \bar{w} \le w \le (SBv_0)(t).$$

(H_3) For any $r>0$, there exist $d_r \ge 0, d_r^* \ge 0$, and $b_r^{(k)} \ge 0,\ a_r^{(k)} \ge 0,\ (k=1,2,\cdots m)$ such that

$$\alpha(f(J,U_1,U_2,U_3,U_4,U_5,U_6))$$
$$\le d_r\alpha(U_3) + d_r^*\alpha(U_4),$$
$$\forall U_i \subset B_r, (i=1,2,3,4,5,6),$$
$$\alpha(I_{3k}(V_1,V_2,V_3,V_4)) \le b_r^{(k)} \max\{\alpha(V_3),\alpha(V_4)\},$$
$$\forall V_j \subset B_r,\ (j=1,2,3,4),\ (k=1,2,\cdots,m),$$
$$\alpha(I_{2k}(V_4)) \le a_r^{(k)}\alpha(V_4),\ \forall V_4 \subset B_r\ (k=1,2,\cdots,m),$$

where $B_r = \{u \in E \,|\, \|u\| \le r\}$. α is the measure of noncompactness in E with the *Kuratowski* property.

Denote

$$[u_0,v_0] = \{u \in PC^1[J,E] \,|\, u_0(t) \le u(t) \le v_0(t), u_0'(t) \le u'(t) \le v_0'(t), \forall t \in J\}.$$

Theorem 3.1. Suppose E is a real Banach space, P is a normal cone, B and F are bounded operators, and (H_1) - (H_3) hold, assume that (4) or (5) is satisfied. Then there exist monotone sequences

$$\{u_n\},\{v_n\} \subset PC^1[J,E] \cap C^2[J',E],$$

are uniform convergence at

$$u^*, v^* \in PC^1[J,E] \cap C^2[J',E]$$

where u^* is a minimal solution and v^* is a maximal solution of (18) on $[u_0,v_0]$ and $\{u_n'\},\{v_n'\}$ are convergent at $(u^*)',(v^*)'$ respectively, and

$$u_0(t) \le u_1(t) \le \cdots \le u_n(t) \le \cdots \le u^*(t) \le u(t) \quad (21)$$
$$\le v^*(t) \le \cdots \le v_n(t) \le \cdots \le v_1(t) \le v_0(t) \qquad \forall t \in J.$$
$$u_0'(t) \le u_1'(t) \le \cdots \le u_n'(t) \le \cdots \le (u^*)'(t) \le u'(t)$$
$$\le (v^*)'(t) \le \cdots \le v_n'(t) \le \cdots \le v_1'(t) \le v_0'(t) \quad \forall t \in J.$$

Proof. For any $\eta \in [u_0,v_0]$, we consider the solution of linear impulsive differential equation of type

$$\begin{cases} u''(t) = -M_1(t)u(t) - M_2(t)u'(t) + \sigma(t), \forall t \in J, t \ne t_k \\ \Delta u|_{t=t_k} = I_{2k}(\eta'(t_k)) + C_k(u'(t_k) - \eta'(t_k)), \\ \Delta u'|_{t=t_k} = I_{3k}((B\eta)(t_k),(F\eta)(t_k),\eta(t_k),\eta'(t_k)) \\ \qquad - L_k(u(t_k)-\eta(t_k)) - L_k^*(u'(t_k)-\eta'(t_k)) \\ \qquad\qquad (k=1,2,\cdots m) \\ u(0)=x_2^*, u'(0)=x_3^*, \end{cases} \quad (22)$$

where

$$\sigma(t) = f(t,(B\eta)(t),\ (F\eta)(t),\eta(t),\eta'(t_k),(TB\eta)(t),$$
$$(SB\eta)(t)) + M_1(t)\eta(t) + M_2(t)\eta'(t_k)).$$

Obviously, $u \in PC^1[J,E] \cap C^2[J',E]$ is a solution of (22) if and only if $u \in PC^1[J,E]$ and

$$u(t) = x_2^* + tx_3^* + \int_0^t (t-s)(\sigma(s) - M_1(s)u(s)$$
$$-M_2(s)u'(s))ds + \sum_{0<t_k<t} (I_{2k}(\eta'(t_k))$$
$$+C_k(u'(t_k)-\eta'(t_k))) + \sum_{0<t_k<t} (t-t_k)$$
$$(I_{3k}((B\eta)(t_k),(F\eta)(t_k),\eta(t_k),\eta'(t_k))$$
$$-L_k(u(t_k)-\eta(t_k)) - L_k^*(u'(t_k)-\eta'(t_k))). \quad (23)$$

Next, we show that u is a unique solution of IVP (22).Let

$$f_*(t,u,u') = \sigma(t) - M_1(t)u(t) - M_2(t)u'(t),\ t \in J.$$

Firstly, we consider the following linear differential equation:

$$\begin{cases} u''(t) = f_*(t,u,u'), t \in J_0 \\ u(0) = x_2^*, u'(0) = x_3^*. \end{cases} \quad (24)$$

It's easy to prove that $u \in C^2[J_0,E]$ is a solution of (24) if and only if $u \in C^1[J_0,E]$

$$u(t) = x_2^* + tx_3^* + \int_0^t (t-s)(\sigma(s) - M_1(s)u(s)$$
$$-M_2(s)u'(s))ds.$$

Let

$$(A_0u)(t) = x_2^* + tx_3^* + \int_0^t (t-s)(\sigma(s) - M_1(s)u(s)$$
$$-M_2(s)u'(s))ds. \quad (25)$$

Then $(A_0u)'(t) = x_3^* + \int_0^t (\sigma(s) - M_1(s)u(s)$

$$-M_2(s)u'(s))ds. \quad (26)$$

For any $u,v \in C^1[J_0,E]$, by (25) and (26) we have

$$\begin{aligned}&\|(A_0u)(t)-(A_0v)(t)\|\\ &\le\int_0^t(t-s)\ (M_1^*\|u(s)-v(s)\|\\ &\quad+M_2^*\|u'(s)-v'(s)\|)ds\\ &\le\int_0^t\tau(M_1^*\|u(s)-v(s)\|+M_2^*\|u'(s)-v'(s)\|)ds\\ &\le(\tau+1)(M_1^*+M_2^*)t\|u-v\|_{C^1[J_0,E]},t\in J_0.\end{aligned}$$

$$\begin{aligned}&\|(A_0u)'(t)-(A_0v)'(t)\|\\ &\le\int_0^t(M_1^*\|u(s)-v(s)\|+M_2^*\|u'(s)-v'(s)\|)ds\\ &\le(\tau+1)(M_1^*+M_2^*)t\|u-v\|_{C^1[J_0,E]},t\in J_0\end{aligned}$$

$$\begin{aligned}&\|(A_0^2u)(t)-(A_0^2v)(t)\|\\ &\le\int_0^t\tau(M_1^*\|(A_0u)(s)-(A_0v)(s)\|\\ &\quad+M_2^*\|(A_0u)'(s)-(A_0v)'(s)\|)ds\\ &\le(\tau+1)^2(M_1^*+M_2^*)^2(\frac{t^2}{2})\|u-v\|_{C^1[J_0,E]},t\in J_0\end{aligned}$$

$$\begin{aligned}&\|(A_0^2u)'(t)-(A_0^2v)'(t)\|\\ &\le(\tau+1)^2(M_1^*+M_2^*)^2(\frac{t^2}{2})\|u-v\|_{C^1[J_0,E]},t\in J_0.\end{aligned}$$

Hence

$$\begin{aligned}&\|(A_0^nu)(t)-(A_0^nv)(t)\| \qquad (27)\\ &\le(\tau+1)^n\ (M_1^*+M_2^*)^n(\frac{t^n}{n!})\ \|u-v\|_{C^1[J_0,E]}\ ,t\in J_0.\end{aligned}$$

$$\begin{aligned}&\|(A_0^nu)'(t)-(A_0^nv)'(t)\| \qquad (28)\\ &\le(\tau+1)^n\ (M_1^*+M_2^*)^n(\frac{t^n}{n!})\ \|u-v\|_{C^1[J_0,E]}\ ,t\in J_0\end{aligned}$$

and

$$\begin{aligned}&\|(A_0^nu)-(A_0^nv)\|_{C^1[J_0,E]} \qquad (29)\\ &\le(\tau+1)^n\ (M_1^*+M_2^*)^n(\frac{\tau^n}{n!})\ \|u-v\|_{C^1[J_0,E]}\ ,t\in J_0.\end{aligned}$$

There exists $n_0\in N$ such that

$$(\tau+1)^{n_0}(M_1^*+M_2^*)^{n_0}(\frac{\tau^{n_0}}{n_0!})<1. \qquad (30)$$

So by (29), (30) and the Banach fixed point theorem, then $A_0^{n_0}$ has a unique fixed point $w_0\in C^1[J_0,E]$. It means that $w_0\in C^2[J_0,E]$ is a unique solution of the (24) such that

$$\begin{cases}w_0''(t)=f_*(t,w_0,w_0'),t\in J_0,\\ w_0(0)=x_2^*,w_0'(0)=x_3^*.\end{cases} \qquad (31)$$

In the following we consider

$$\begin{cases}u''=f_*(t,u,u'),t\in J_1,\\ u(t_1^+)=I_{21}(\eta'(t_1))+C_1(w_0'(t_1)-\eta'(t_1))+w_0(t_1),\\ u'(t_1^+)=I_{31}((B\eta)(t_1),(F\eta)(t_1),\eta(t_1),\eta'(t_1)) \qquad (32)\\ \qquad -L_1(w_0(t_1)-\eta(t_1))-L_1^*(w_0'(t_1)-\eta'(t_1))\\ \qquad +w_0'(t_1)\end{cases}$$

It is easy to prove that $u\in PC^1[J_1,E]\bigcap C^2[(t_1,t_2),E]$ is a solution of (32) if and only if $u\in PC^1[J_1,E]$ such that

$$\begin{aligned}u(t)=&I_{21}(\eta'(t_1))+C_1(w_0'(t_1)-\eta'(t_1))\\ &+w_0(t_1)+(t-t_1)\ (I_{31}((B\eta)(t_1),\\ &(F\eta)(t_1),\ \eta(t_1),\eta'(t_1))-L_1(w_0(t_1)-\eta(t_1))\\ &-L_1^*(w_0'(t_1)-\eta'(t_1))+w_0'(t_1))\\ &+\int_{t_1}^t(t-s)(\sigma(s)-M_1(s)u(s)\\ &-M_2(s)u'(s))ds\end{aligned}$$

Put

$$\begin{aligned}(A_1u)(t)=&I_{21}(\eta'(t_1))+C_1(w_0'(t_1)-\eta'(t_1))\\ &+w_0(t_1)+(t-t_1)(I_{31}((B\eta)(t_1),\\ &(F\eta)(t_1),\eta(t_1),\eta'(t_1))\\ &-L_1(w_0(t_1)-\eta(t_1))\\ &-L_1^*(w_0'(t_1)-\eta'(t_1))+w_0'(t_1))\\ &+\int_{t_1}^t(t-s)(\sigma(s)-M_1(s)u(s)\\ &-M_2(s)u'(s))ds,\quad t\in\overline{J}_1. \qquad (33)\end{aligned}$$

Then for any $t\in\overline{J}_1$ we have

$$\begin{aligned}(A_1u)'(t)=&(I_{31}((B\eta)(t_1),\ (F\eta)(t_1),\ \eta(t_1),\\ &\eta'(t_1))-L_1(w_0(t_1)-\eta(t_1))\\ &-L_1^*(w_0'(t_1)-\eta'(t_1))+w_0'(t_1))\\ &+\int_{t_1}^t(\sigma(s)-M_1(s)u(s)-M_2(s)u'(s))ds,\end{aligned}$$

Obviously, $A_1:PC^1[\overline{J}_1,E]\to PC^1[\overline{J}_1,E]$.

For any $u,v\in PC^1[\overline{J}_1,E]$, using the similar method used in (29) we obtain

$$\begin{aligned}&\|(A_1^nu)-(A_1^nv)\|_{PC^1[\overline{J}_1,E]}\\ &\le(\tau+1)^n\ (M_1^*+M_2^*)^n(\frac{\tau^n}{n!})\|u-v\|_{PC^1[\overline{J}_1,E]} \qquad (34)\end{aligned}$$

By (30), (34) and the Banach fixed point theorem, $A_1^{n_0}$ has a unique fixed point $w_1\in PC^1[\overline{J}_1,E]$.

It means that $w_1\in PC^1[\overline{J}_1,E]$ is a unique solution to (32) such that

$$\begin{cases} w_1''= f_*(t,w_1,w_1'), t\in J_1, \\ w_1(t_1^+) = I_{21}(\eta'(t_1)) + C_1(w_0'(t_1)-\eta'(t_1)) + w_0(t_1), \\ w_1'(t_1^+) = I_{31}((B\eta)(t_1),(F\eta)(t_1),\eta(t_1),\eta'(t_1)) \\ \qquad - L_1(w_0(t_1)-\eta(t_1)) \\ \qquad - L_1^*(w_0'(t_1)-\eta'(t_1)) + w_0'(t_1) \end{cases} \quad (35)$$

Again, we want to prove that linear differential equation for any $i,(i=1,2,\cdots m)$

$$\begin{cases} u''= f_*(t,u,u'), t\in J_i, \\ u(t_i^+) = I_{2i}(\eta'(t_i)) + C_i(w_{i-1}'(t_i)-\eta'(t_i)) + w_{i-1}(t_i), \\ u'(t_i^+) = I_{3i}((B\eta)(t_i),(F\eta)(t_i),\eta(t_i),\eta'(t_i)) \\ \qquad - L_i(w_{i-1}(t_i)-\eta(t_i)) \\ \qquad - L_i^*(w_{i-1}'(t_i)-\eta'(t_i)) + w_{i-1}'(t_i), \end{cases}$$

has a unique solution $w_i \in PC^1[J_i,E]\cap C^2[(t_i,t_{i+1}),E]$ such that

$$\begin{cases} w_i''= f_*(t,w_i,w_i'), t\in J_i, \\ w_i(t_i^+) = I_{2i}(\eta'(t_i)) + C_i(w_{i-1}'(t_i)-\eta'(t_i)) + w_{i-1}(t_i), \\ w_i'(t_i^+) = I_{3i}((B\eta)(t_i),(F\eta)(t_i),\eta(t_i),\eta'(t_i)) \\ \qquad - L_i(w_{i-1}(t_i)-\eta(t_i)) - L_i^*(w_{i-1}'(t_i)-\eta'(t_i)) \\ \qquad + w_{i-1}'(t_i). \end{cases} \quad (36)$$

Let

$$u_\eta(t) = \begin{cases} w_0(t), t\in J_0, \\ w_1(t), t\in J_1, \\ \cdots\cdots\cdots\cdots \\ w_m(t), t\in J_m. \end{cases} \quad (37)$$

Combing (31) and (35), (36), (37), we have $u_\eta \in PC^1[J,E]\bigcap C^2[J',E]$ is a unique solution of IVP (22).

Putting $u_\eta = A\eta$, Then

$$A:[u_0,v_0] \to PC^1[J,E]\bigcap C^2[J',E].$$

Next we prove two cases:

Case (1): $u_0 \le Au_0, u_0' \le (Au_0)'$,

$$Av_0 \le v_0, (Av_0)' \le v_0'.$$

Case (2): if $\eta_1,\eta_2 \in [u_0,v_0]$ and $\eta_1\le\eta_2, \eta_1'\le\eta_2'$, then $A\eta_1 \le A\eta_2, (A\eta_1)'\le(A\eta_2)'$.

First, consider case (1).

Put $u_1 = Au_0, p = u_0 - u_1$. By (22), we have

$$u_1''(t) = -M_1(t)u_1(t) - M_2(t)u_1'(t) + M_1(t)u_0(t) + M_2(t)u_0'(t) + f(t,(Bu_0)(t),(Fu_0)(t), u_0(t),u_0'(t),(TBu_0)(t),(SBu_0)(t)), \forall t\in J, t\ne t_k,$$

$$\Delta u_1|_{t=t_k} = I_{2k}(u_0'(t_k)) + C_k(u_1'(t_k)-u_0'(t_k)),$$

$$\Delta u_1'|_{t=t_k} = I_{3k}((Bu_0)(t_k),(Fu_0)(t_k), u_0(t_k), u_0(t_k),u_0'(t_k)) - L_k(u_1(t_k)-u_0(t_k)) - L_k^*(u_1'(t_k)-u_0'(t_k))\ (k=1,2,\cdots,m),$$

$$u_1(0)=x_2^*, u_1'(0)=x_3^*.$$

Moreover, by (H_1) we have

$$\begin{cases} p''(t) = u_0''(t) - u_1''(t) \\ \qquad \le -M_1(t)p(t) - M_2(t)p'(t), \forall t\in J, t\ne t_k, \\ \Delta p|_{t=t_k} = \Delta u_0|_{t=t_k} - \Delta u_1|_{t=t_k} = C_k p'(t_k), \\ \Delta p'|_{t=t_k} = \Delta u_0'|_{t=t_k} - \Delta u_1'|_{t=t_k} \\ \qquad \le -L_k p(t_k) - L_k^* p'(t_k)\ (k=1,2,\cdots,m), \\ p'(0) = u_0'(0) - u_1'(0) = u_0'(0) - x_3^* \\ \qquad \le u_0(0) - x_2^* = p(0) \le \theta. \end{cases} \quad (38)$$

Hence, by Lemma 2.1 we obtain

$$p(t)\le\theta, p'(t)\le\theta, \forall t\in J.$$

This means that

$$u_0 \le Au_0, u_0' \le (Au_0)'.$$

In the same way, $Av_0 \le v_0, (Av_0)' \le v_0'$,

Next, consider case (2):

Let $\eta_1,\eta_2\in[u_0,v_0]$ such that $\eta_1\le\eta_2, \eta_1'\le\eta_2'$ and put $p=\lambda_1-\lambda_2$, where $\lambda_1 = A\eta_1, \lambda_2 = A\eta_2$.

Combing (22) and (H_2), we have

$$\begin{cases} p''(t) = \lambda_1''(t) - \lambda_2''(t) = -M_1(t)p(t) - M_2(t)p'(t) \\ \qquad -(f(t,(B\eta_2)(t),(F\eta_2)(t),\eta_2(t),\eta_2'(t), \\ \qquad (TB\eta_2)(t),(SB\eta_2)(t)) - f(t,(B\eta_1)(t), \\ \qquad (F\eta_1)(t),\eta_1(t),\eta_1'(t),(TB\eta_1)(t),(SB\eta_1)(t)) \\ \qquad + M_1(t)(\eta_2(t)-\eta_1(t)) + M_2(t)(\eta_2'(t)-\eta_1'(t))) \\ \qquad \le -M_1(t)p(t) - M_2(t)p'(t), \forall t\in J, t\ne t_k, \\ \Delta p|_{t=t_k} = \Delta\lambda_1|_{t=t_k} - \Delta\lambda_2|_{t=t_k} = I_{2k}(\eta_1'(t_k)) \\ \qquad + C_k(\lambda_1'(t_k)-\eta_1'(t_k)) - I_{2k}(\eta_2'(t_k)) \\ \qquad - C_k(\lambda_2'(t_k)-\eta_2'(t_k)) = C_k p'(t_k), \\ \Delta p'|_{t=t_k} = \Delta\lambda_1'|_{t=t_k} - \Delta\lambda_2'|_{t=t_k} \\ \qquad = I_{3k}((B\eta_1)(t_k),(F\eta_1)(t_k),\eta_1(t_k),\eta_1'(t_k)) \\ \qquad - L_k(\lambda_1(t_k)-\eta_1(t_k)) - L_k^*(\lambda_1'(t_k)-\eta_1'(t_k)) \\ \qquad - I_{3k}((B\eta_2)(t_k),(F\eta_2)(t_k),\eta_2(t_k),\eta_2'(t_k)) \\ \qquad + L_k(\lambda_2(t_k)-\eta_2(t_k)) + L_k^*(\lambda_2'(t_k)-\eta_2'(t_k)) \\ \qquad \le -L_k p(t_k) - L_k^* p'(t_k)(k=1,2\cdots,m), \\ p'(0) = p(0) = \theta. \end{cases}$$

Moreover, by Lemma2.1 we obtain

$$p(t) \le \theta, p'(t) \le \theta, \forall t \in J,$$

this means that

$$(A\eta_1)(t) \le (A\eta_2)(t), (A\eta_1)'(t) \le (A\eta_2)'(t).$$

Let

$$u_n = Au_{n-1}, v_n = Av_{n-1} (n = 1, 2, \cdots). \quad (39)$$

By Case (1) and Case (2), we have

$$u_0(t) \le u_1(t) \le \cdots \le u_n(t) \le \cdots$$
$$\le v_n(t) \le \cdots \le v_1(t) \le v_0(t), \forall t \in J.$$
$$u_0'(t) \le u_1'(t) \le \cdots \le u_n'(t) \le \cdots$$
$$\le v_n'(t) \le \cdots \le v_1'(t) \le v_0'(t), \forall t \in J. \quad (40)$$

Let

$$U = \{u_n \mid n = 1, 2 \cdots\}, U' = \{u_n' \mid n = 1, 2, \cdots\},$$
$$U(t) = \{u_n(t) \mid n = 1, 2, \cdots\},$$
$$U'(t) = \{u_n'(t) \mid n = 1, 2, \cdots\}, t \in J.$$

By normality of P and (40), then U, U' are both bounded sets in $PC[J, E]$. For any $\eta \in [u_0, v_0]$, combing (H_1) and (H_2), we have

$$u_0''(t) + M_1(t)u_0(t) + M_2(t)u_0'(t)$$
$$\le f(t, (Bu_0)(t), (Fu_0)(t), u_0(t), u_0'(t), (TBu_0)(t), (SBu_0)(t)) + M_1(t)u_0(t) + M_2(t)u_0'(t)$$
$$\le f(t, (B\eta)(t), (F\eta)(t), \eta(t), \eta'(t), (TB\eta)(t), (SB\eta)(t)) + M_1(t)\eta(t) + M_2(t)\eta'(t)$$
$$\le f(t, (Bv_0)(t), (Fv_0)(t), v_0(t), v_0'(t), (TBv_0)(t), (SBv_0)(t)) + M_1(t)v_0(t) + M_2(t)v_0'(t)$$
$$\le v_0''(t) + M_1(t)v_0(t) + M_2(t)v_0'(t). \quad (41)$$

Moreover, we obtain

$$\{f(t, B\eta, F\eta, \eta, \eta', TB\eta, SB\eta) + M_1(t)\eta + M_2(t)\eta' \mid \eta \in [u_0, v_0]\}$$

is a bounded set. Hence, there exists a constant $\gamma > 0$ such that

$$\| f(t, (Bu_{n-1})(t), (Fu_{n-1})(t), u_{n-1}(t), u_{n-1}'(t), (TBu_{n-1})(t), (SBu_{n-1})(t)) - M_1(t)(u_n(t) - u_{n-1}(t)) - M_2(t)(u_n'(t) - u_{n-1}'(t)) \| \le \gamma, \forall t \in J (n = 1, 2, \cdots), \quad (42)$$

and $\{\sigma_n \mid n = 1, 2, \cdots\}$ is a bounded set in $PC[J, E]$, where

$$\sigma_n(t) = f(t, (Bu_{n-1})(t), (Fu_{n-1})(t), u_{n-1}(t), u_{n-1}'(t), (TBu_{n-1})(t), (SBu_{n-1})(t)) + M_1(t)u_{n-1}(t) + M_2(t)u_{n-1}'(t).$$

By the definition of $u_n(t)$ and (23), we have

$$u_n(t) = x_2^* + tx_3^* + \int_0^t (t-s)(f(s, (Bu_{n-1})(s), (Fu_{n-1})(s), u_{n-1}(s), u_{n-1}'(s), (TBu_{n-1})(s), (SBu_{n-1})(s)) + M_1(s)u_{n-1}(s) + M_2(s)u_{n-1}'(s) - M_1(s)u_n(s) - M_2(s)u_n'(s))ds + \sum_{0<t_k<t} (I_{2k}(u_{n-1}'(t_k)) + C_k(u_n'(t_k) - u_{n-1}'(t_k))) + \sum_{0<t_k<t} (t - t_k)(I_{3k}((Bu_{n-1})(t_k), (Fu_{n-1})(t_k), u_{n-1}(t_k), u_{n-1}'(t_k)) - L_k(u_n(t_k) - u_{n-1}(t_k)) - L_k^*(u_n'(t_k) - u_{n-1}'(t_k))), \forall t \in J (n = 1, 2, \cdots). \quad (43)$$

Then, we have

$$u_n'(t) = x_3^* + \int_0^t (f(s, (Bu_{n-1})(s), (Fu_{n-1})(s), u_{n-1}(s), u_{n-1}'(s), (TBu_{n-1})(s), (SBu_{n-1})(s)) + M_1(s)u_{n-1}(s) + M_2(s)u_{n-1}'(s) - M_1(s)u_n(s) - M_2(s)u_n'(s))ds + \sum_{0<t_k<t} (I_{3k}((Bu_{n-1})(t_k), (Fu_{n-1})(t_k), u_{n-1}(t_k), u_{n-1}'(t_k)) - L_k(u_n(t_k) - u_{n-1}(t_k)) - L_k^*(u_n'(t_k) - u_{n-1}'(t_k))), \forall t \in J (n = 1, 2, \cdots). \quad (44)$$

By (42), (43) and (44), the functions belonging to U, U' are equiv-continuity on $J_k (k = 0, 1, 2, \cdots m)$. So by Lemma 2.4, we have $\forall t \in J$

$$\alpha_{PC^1}(U) = \max\left\{\sup_{t \in J} \alpha(U(t)), \sup_{t \in J} \in \alpha(U'(t))\right\}.$$

By (H_3), there exist constants $d \ge 0, d^* \ge 0$ and $b^{(k)} \ge 0, a^{(k)} \ge 0$ $(k = 1, 2, \cdots, m)$ such that

$$\alpha(f(t, (BU)(t), (FU)(t), U(t), U'(t), (TBU)(t), (SBU)(t))) \le d\alpha(U(t)) + d^*\alpha(U'(t)), \forall t \in J. \quad (45)$$

$$\alpha(I_{3k}((BU)(t_k), (FU)(t_k), U(t_k), U'(t_k))) \le b^{(k)} \max\{\alpha(U(t_k)), \alpha(U'(t_k))\} \quad (k = 1, 2, \cdots, m). \quad (46)$$

$$\alpha(I_{2k}(U'(t_k))) \le a^{(k)}\alpha(U'(t_k)) \quad (k = 1, 2, \cdots, m) \quad (47)$$

Hence, for any $t \in J$, combing (43), (45), (46), (47) and Lemma 2.3, we have

$$\alpha(U(t)) \le 2a\int_0^t (\alpha(f(s, (BU)(s), (FU)(s), U(s), U'(s), (TBU)(s), (SBU)(s))) + 2M_1^*\alpha(U(s)) + 2M_2^*\alpha(U'(s)))ds$$

$$+\sum_{0<t_k<t}(a^{(k)}\alpha(U'(t_k))+2C_k\alpha(U'(t_k)))$$
$$+\sum_{0<t_k<t}(ab^{(k)}\max\{\alpha(U(t_k)),\ \alpha(U'(t_k))\}$$
$$+2aL_k\alpha(U(t_k))\ +2aL_k^*\alpha(U'(t_k)))$$
$$\le 2a\int_0^t(d\alpha(U(s))+d^*\alpha(U'(s))$$
$$+2M_1^*\alpha(U(s))+2M_2^*\alpha(U'(s)))ds$$
$$+\sum_{0<t_k<t}(a^{(k)}\alpha(U'(t_k))+2C_k\alpha(U'(t_k)))$$
$$+\sum_{0<t_k<t}(ab^{(k)}\max\{\alpha(U(t_k)),\alpha(U'(t_k))\}$$
$$+2aL_k\alpha(U(t_k))+2aL_k^*\alpha(U'(t_k))) \qquad (48)$$

$$\alpha(U'(t))\le 2\int_0^t(d\alpha(U(s))+d^*\alpha(U'(s))$$
$$+2M_1^*\alpha(U(s))+2M_2^*\alpha(U'(s)))ds$$
$$+\sum_{0<t_k<t}(b^{(k)}\max\{\alpha(U(t_k)),\alpha(U'(t_k))\}$$
$$+2L_k\alpha(U(t_k))+2L_k^*\alpha(U'(t_k))) \qquad (49)$$

Let

$$m(t)=\max\{\alpha(U(t)),\alpha(U'(t))\}.$$

Because the functions belonging to U,U' are equiv-continuity on $J_k(k=1,2,\cdots m)$ and U,U' are bounded, we have $m(t)\in PC[J,E]$, $m(t)\ge 0$. Combing (48) and (49), we obtain

$$m(t)\le 2(a+1)(d+d^*+2M_1^*+2M_2^*)\int_0^t m(s)ds \qquad (50)$$
$$+\sum_{0<t_k<t}(a^{(k)}+2C_k+(a+1)(b^{(k)}+2L_k+2L_k^*))m(t_k).$$

Moreover, by Lemma 2.2, we have $m(t)\le 0,$ this means $m(t)\equiv 0,\ t\in J,$ moreover,

$$\alpha(U(t))\equiv 0,\alpha(U'(t))\equiv 0,t\in J.$$

This yields that U possesses the relatively compactness in $PC^1[J,E]$, U' possesses the relatively compactness in $PC[J,E]$ .Hence, by (40) and the normality of P, $\{u_n\}$ is convergent at $u^*\in PC^1[J,E],\{u_n'\}$ is convergent at $(u^*)'$ and

$$\|u_n-u^*\|_{PC^1}\to 0,\|u_n'-(u^*)'\|_{PC}\to 0 \qquad (51)$$

Because f is continuous, by the definition of σ_n and (51), we have

$$\|\sigma_n-\sigma^*\|_{PC}\to 0,(n\to\infty) \qquad (52)$$

where

$$\sigma^*(t)=f(t,(Bu^*)(t),(Fu^*)(t),u^*(t),$$
$$(u^*)'(t),(TBu^*)(t),(SBu^*)(t))$$
$$+M_1(t)u^*(t)+M_2(t)(u^*)'(t).$$

By (42), (51), (52) and Lebesgue control convergent theorem, we have

$$\lim_{n\to\infty}u_n=u^*(t),\ \lim_{n\to\infty}u_n'=(u^*)'(t)$$

Moreover,

$$u^*(t)=x_2^*+tx_3^*+\int_0^t(t-s)f(s,(Bu^*)(s),$$
$$(Fu^*)(s),u^*(s),(u^*)'(s),(TBu^*)(s),$$
$$(SBu^*)(s))ds+\sum_{0<t_k<t}I_{2k}((u^*)'(t_k))$$
$$+\sum_{0<t_k<t}(t-t_k)I_{3k}((Bu^*)(t_k),(Fu^*)(t_k),$$
$$u^*(t_k),(u^*)'(t_k)),\forall t\in J,$$

$$(u^*)'(t)=x_3^*+\int_0^t f(s,(Bu^*)(s),(Fu^*)(s),u^*(s),$$
$$(u^*)'(s),(TBu^*)(s),(SBu^*)(s))ds$$
$$+\sum_{0<t_k<t}I_{3k}((Bu^*)(t_k),(Fu^*)(t_k),$$
$$u^*(t_k),(u^*)'(t_k)),\forall t\in J,$$

It is easy to prove that $u^*\in PC^1[J,E]\cap C^2[J',E]$ is a solution of IVP (18).In the same way, there exists $v^*\in PC^1[J,E]\cap C^2[J',E]$ such that

$$\|v_n-v^*\|_{PC^1}\to 0,\|v_n'-(v^*)'\|_{PC}\to 0.$$

v^* is a solution of IVP (18),and by (40),we have

$$u_0(t)\le u_1(t)\le\cdots\le u_n(t)\le\cdots\le u^*(t)\le v^*(t)$$
$$\le\cdots\le v_n(t)\le\cdots\le v_1(t)\le v_0(t),\forall t\in J. \qquad (53)$$
$$u_0'(t)\le u_1'(t)\le\cdots\le u_n'(t)\le\cdots\le(u^*)'(t)\le(v^*)'(t)$$
$$\le\cdots\le v_n'(t)\le\cdots\le v_1'(t)\le v_0'(t)\forall t\in J.$$

For $u\in PC^1[J,E]\cap C^2[J',E]$ is any solution of IVP (18) on $[u_0,v_0]$, then

$$u_0(t)\le u(t)\le v_0(t),\ u_0'(t)\le u'(t)\le v_0'(t),\forall t\in J.$$

Assume that $u_{n-1}(t)\le u(t)\le v_{n-1}(t),\ u_{n-1}'(t)\le u'(t)\le v_{n-1}'(t),\forall t\in J.$ Let $p(t)=u_n(t)-u(t).$ By (22), (39) and (H_2), we have

$$p''(t)=-M_1(t)p(t)-M_2(t)p'(t)$$
$$-(f(t,(Bu)(t),(Fu)(t),u(t),u'(t),$$
$$(TBu)(t),(SBu)(t))$$
$$-f(t,(Bu_{n-1})(t),(Fu_{n-1})(t),u_{n-1}(t),$$
$$u_{n-1}'(t),(TBu_{n-1})(t),(SBu_{n-1})(t))$$
$$+M_1(t)(u(t)-u_{n-1}(t))$$
$$+\mathrm{M}_2(t)(u'(t)-u_{n-1}'(t)))$$

$$\le -M_1(t)p(t) - M_2(t)p'(t),\ \forall t \in J, t \ne t_k,$$
$$\Delta p|_{t=t_k} = I_{2k}(u'_{n-1}(t_k)) + C_k(u'_n(t_k) - u'_{n-1}(t_k))$$
$$-I_{2k}(u'(t_k)) \quad = C_k p'(t_k),$$
$$\Delta p'|_{t=t_k} = I_{3k}((Bu_{n-1})(t_k), (Fu_{n-1})(t_k), u_{n-1}(t_k),$$
$$u'_{n-1}(t_k)) - L_k(u_n(t_k) - u_{n-1}(t_k))$$
$$-L_k^*(u'_n(t_k) - u'_{n-1}(t_k))$$
$$-I_{3k}((Bu)(t_k), (Fu)(t_k), u(t_k), u'(t_k))$$
$$\le -L_k p(t_k) - L_k^* p'(t_k)\ (k = 1, 2, \cdots m),$$
$$p'(0) = p(0) = \theta.$$

By Lemma 2.1, we have $p(t) \le \theta, p'(t) \le \theta, \forall t \in J$. Moreover, $u_n(t) \le u(t), u'_n(t) \le u'(t), \forall t \in J$ .In the same way, we can show that

$$u(t) \le v_n(t), u'(t) \le v'_n(t), \forall t \in J,$$

Hence, we obtain

$$u_n(t) \le u(t) \le v_n(t), u'_n(t) \le u'(t) \le v'_n(t),$$
$$\forall t \in J (n = 0, 1, 2, \cdots). \qquad (54)$$

Now if $n \to \infty$, for any $t \in J$,

$$u^*(t) \le u(t) \le v^*(t), (u^*)'(t) \le u'(t) \le (v^*)'(t).$$

By (53), then (22) holds. This ends the proof.

Theorem 3.2. Suppose E is a real Banach space, P is a regular cone, and $(H_1), (H_2)$ hold. Assume (4) or (5) is satisfied, then (21) holds.

Proof. The proof is similar to the proof of Theorem 3.1, the only difference is that we verify relative compactness of U, U' and the regularity of P by (40) instead of H_3.This ends the proof.

Corollary 3.1. If E is a weak sequentially complete Banach space, P is a normal cone, H_1, H_2 hold, and (4) or (5) is satisfied, then (21) holds.

Proof. If E is a weak sequentially complete Banach space, the normality of P is equivalent to the regularity. Hence, (21) holds by Theorem 3.2. This ends the proof.

Remark 3.1. f is relative to operators B, F. To my knowledge, in all papers connected with the second order impulsive integro-differential equation has been not investigated this situation, so IVP (18) is a new problem.

Remark 3.2. B, F relative to Theorem 3.1 are bounded and continuous operators, however, B, F relative to Theorem 3.2 are continuous and increasing.

4. Some results of the four order impulsive differential equations

Let us list the following assumptions for convenience:

(G_1) There exist

$$y_0, z_0 \in PC^3[J, E] \cap C^4[J', E]$$

such that $y_0(t) \le z_0(t), y'_0(t) \le z'_0(t), y''_0(t) \le z''_0(t)$, $y'''_0(t) \le z'''_0(t), t \in J$,

$$\begin{cases} y_0^{(4)}(t) \le f(t, y_0(t), y'_0(t), y''_0(t), y'''(t), \\ \qquad (Ty_0)(t), (Sy_0)(t)), \forall t \in J, t \ne t_k, \\ \Delta y_0|_{t=t_k} = I_{0k}(y''_0(t_k)), \\ \Delta y'_0|_{t=t_k} = I_{1k}(y'_0(t_k), y''_0(t_k)), \\ \Delta y''_0|_{t=t_k} = I_{2k}(y'''_0(t_k)), (k = 1, 2, \cdots, m), \\ \Delta y'''_0|_{t=t_k} \le I_{3k}(y_0(t_k), y'_0(t_k), y''_0(t_k) y'''_0(t_k)) \\ y_0(0) = x_0^*, y'_0(0) = x_1^*, y''_0(0) \le x_2^*, \\ \qquad y'''_0(0) - y''_0(0) \le x_3^* - x_2^*. \end{cases} \qquad (55)$$

$$\begin{cases} z_0^{(4)}(t) \ge f(t, z_0(t), z'_0(t), z''_0(t), z'''_0(t), (Tz_0)(t), \\ \qquad (Sz_0)(t)), \forall t \in J, t \ne t_k, \\ \Delta z_0|_{t=t_k} = I_{0k}(z''_0(t_k)), \\ \Delta z'_0|_{t=t_k} = I_{1k}(z'_0(t_k), z''_0(t_k)), \\ \Delta z''_0|_{t=t_k} = I_{2k}(z'''_0(t_k)), (k = 1, 2, \cdots, m), \\ \Delta z'''_0|_{t=t_k} \ge I_{3k}(z_0(t_k), z'_0(t_k), z''_0(t_k), z'''_0(t_k)), \\ z_0(0) = x_0^*, z'_0(0) = x_1^*, z''_0(0) \ge x_2^*, \\ \qquad z'''_0(0) - z''_0(0) \ge x_3^* - x_2^* \end{cases} \qquad (56)$$

(G'_1) There exist

$$y_0, z_0 \in PC^3[J, E] \cap C^4[J', E]$$

such that $y_0(t) \le z_0(t), y'_0(t) \le z'_0(t), y''_0(t) \le z''_0(t)$, $y'''_0(t) \le z'''_0(t), t \in J$,

$$\begin{cases} y_0^{(4)}(t) \le f(t, y_0(t), y'_0(t), y''_0(t), y'''_0(t), \\ \qquad (Ty_0)(t), (Sy_0)(t)), \forall t \in J, t \ne t_k, \\ \Delta y_0|_{t=t_k} \le I_{0k}(y''_0(t_k)), \\ \Delta y'_0|_{t=t_k} \le I_{1k}(y'_0(t_k), y''_0(t_k)), \\ \Delta y''_0|_{t=t_k} = I_{2k}(y'''(t_k)), \\ \Delta y'''_0|_{t=t_k} \le I_{3k}(y_0(t_k), y'_0(t_k), y''_0(t_k), y'''_0(t_k)), \\ y_0(0) \le x_0^*, y'_0(0) \le x_1^*, y''_0(0) \le x_2^*, \\ \qquad y'''_0(0) - y''_0(0) \le x_3^* - x_2^*, \end{cases} \qquad (57)$$

$$\begin{cases} z_0^{(4)}(t) \geq f(t, z_0(t), z_0'(t), z_0''(t), z_0'''(t), (Tz_0)(t), \\ \qquad (Sz_0)(t)), \forall t \in J, t \neq t_k, \\ \Delta z_0 |_{t=t_k} \geq I_{0k}(z_0''(t_k)), \\ \Delta z_0' |_{t=t_k} \geq I_{1k}(z_0'(t_k), z_0''(t_k)), \\ \Delta z_0'' |_{t=t_k} = I_{2k}(z_0'''(t_k)), \\ \Delta z''' |_{t=t_k} \geq I_{3k}(z_0(t_k), z_0'(t_k), z_0''(t_k), z_0'''(t_k)), \\ \qquad (k = 1, 2, \cdots, m), \\ z_0(0) \geq x_0^*, z_0'(0) \geq x_1^*, z_0''(0) \geq x_2^*, \\ \qquad z_0'''(0) - z_0''(0) \geq x_3^* - x_2^* \end{cases} \tag{58}$$

(G_2) There exist $M_1(t), M_2(t)$ are bounded with $M_1 \geq 0, M_2 \geq 0$ on J and M_1, M_2 $\in L^1[0,a]$. C_k, L_k, L_k^* $(k = 1, 2, \cdots m)$ are all nonnegative constants such that

$$f(t,x,y,z,u,v,w) - f(t,\overline{x},\overline{y},\overline{z},\overline{u},\overline{v},\overline{w})$$
$$\geq -M_1(t)(z-\overline{z}) - M_2(t)(u-\overline{u}), \forall t \in J,$$
$$I_{0k}(z) \geq I_{0k}(\overline{z}), I_{1k}(y,z) \geq I_{1k}(\overline{y},\overline{z}),$$
$$I_{2k}(u) - I_{2k}(\overline{u}) = C_k(u-\overline{u})$$
$$I_{3k}(x,y,z,u) - I_{3k}(\overline{x},\overline{y},\overline{z},\overline{u})$$
$$\geq -L_k(z-\overline{z}) - L_k^*(u-\overline{u}) \ (k = 1, 2, \cdots, m)$$

where

$$y_0(t) \leq \overline{x} \leq x \leq z_0(t),\ y_0'(t) \leq \overline{y} \leq y \leq z_0'(t),$$
$$y_0''(t) \leq \overline{z} \leq z \leq z_0''(t),\ y_0'''(t) \leq \overline{u} \leq u \leq z_0'''(t),$$
$$(Ty_0)(t) \leq \overline{v} \leq v \leq (Tz_0)(t),$$
$$(Sy_0)(t) \leq \overline{w} \leq w \leq (Sz_0)(t), \forall t \in J$$

(G_3) There exist

$$b_{0k} \geq 0, a_{1k} \geq 0, b_{1k} \geq 0, (k = 1, 2, \cdots, m)$$

such that

$$\| I_{0k}(z) - I_{0k}(\overline{z}) \| \leq b_{0k} \| z - \overline{z} \|,$$
$$\| I_{1k}(y,z) - I_{1k}(\overline{y},\overline{z}) \| \leq a_{1k} \| y - \overline{y} \| + b_{1k} \| z - \overline{z} \|,$$
$$y, z, \overline{y}, \overline{z} \in E (k = 1, 2, \cdots, m).$$

Denote

$$[y_0, z_0] = \{ y \in PC^3[J,E] \mid y_0(t) \leq y(t) \leq z_0(t),$$
$$y_0'(t) \leq y'(t) \leq z_0'(t),\ y_0''(t) \leq y''(t) \leq z_0''(t),$$
$$y_0'''(t) \leq y'''(t) \leq z_0'''(t), \forall t \in J \}.$$

Theorem 4.1 Suppose E is a real Banach space, P is normal cone, and $(G_1), (G_2), (G_3), (H_3)$ hold. Assume (4) or (5) is satisfied, then IVP (1) has the maximal and minimal solutions

$$y^*, z^* \in PC^3[J,E] \cap C^4[J',E]$$

on $[y_0, z_0]$.

Proof. Consider IVP (1). Let $x''(t) = u(t), t \in J$. Then we have

$$\begin{cases} x''(t) = u(t), \forall t \in J, t \neq t_k, \\ u''(t) = f(t, x(t), x'(t), u(t), u'(t), (Tx)(t), (Sx)(t)), \\ \qquad \forall t \in J, t \neq t_k, \\ \Delta x |_{t=t_k} = I_{0k}(u(t_k)), \\ \Delta x' |_{t=t_k} = I_{1k}(x'(t_k), u(t_k)), \\ \Delta u |_{t=t_k} = I_{2k}(u'(t_k)), \\ \Delta u' |_{t=t_k} = I_{3k}(x(t_k), x'(t_k), u(t_k), u'(t_k)) \\ \qquad (k = 1, 2, \cdots, m), \\ x(0) = x_0^*, x'(0) = x_1^*, u(0) = x_2^*, u'(0) = x_3^* \end{cases} \tag{59}$$

For any $u \in PC[J,E]$, we have

$$\begin{cases} x''(t) = u(t), \forall t \in J, t \neq t_k, \\ \Delta x |_{t=t_k} = I_{0k}(u(t_k)), \\ \Delta x' |_{t=t_k} = I_{1k}(x'(t_k), u(t_k)) (k = 1, 2, \cdots, m), \\ x(0) = x_0^*, x'(0) = x_1^*. \end{cases} \tag{60}$$

Obviously, if $x \in PC^1[J,E] \cap C^2[J',E]$ is a solution of (60) if and only if

$$x(t) = x_0^* + x_1^* t + \int_0^t (t-s) u(s) ds + \sum_{0<t_k<t} I_{0k}(u(t_k))$$
$$+ \sum_{0<t_k<t} (t-t_k) I_{1k}(x'(t_k), u(t_k)), \tag{61}$$

and

$$x'(t) = x_1^* + \int_0^t u(s) ds + \sum_{0<t_k<t} I_{1k}(x'(t_k), u(t_k)). \tag{62}$$

Let

$$x(t) = (Bu)(t), t \in J. \tag{63}$$
$$x'(t) = (Fu)(t), t \in J. \tag{64}$$

Then define two operators B, F,

$$B: PC[J,E] \to PC^1[J,E] \cap C^2[J',E],$$
$$F: PC[J,E] \to PC[J,E] \cap C^1[J',E].$$

Next, we show that

(i) B is bounded and continuous.

When $m = 3, 4, \cdots$, for any $y_1, y_2 \in PC[J,E]$, by (62),(63), we have

$$\| (By_1)(t) - (By_2)(t) \|$$
$$\leq \int_0^t (t-s) \| y_1(s) - y_2(s) \| ds$$
$$+ \sum_{0<t<t} \| I_{0k}(y_1(t_k)) - I_{0k}(y_2(t_k)) \|$$
$$+ \sum_{0<t_k<t} (t-t_k) \| I_{1k}((Fy_1)(t_k), y_1(t_k))$$

$$-I_{1k}((Fy_2)(t_k), y_2(t_k))\|$$

$$\leq \frac{a^2}{2}\| y_1 - y_2 \|_{PC} + \sum_{k=1}^{m} b_{0k} \| y_1 - y_2 \|_{PC} \quad (65)$$

$$+ a\sum_{k=1}^{m} a_{1k} \| (Fy_1)(t_k) - (Fy_2)(t_k) \|$$

$$+ a\sum_{k=1}^{m} b_{1k} \| y_1 - y_2 \|_{PC}$$

$$= \frac{a^2}{2}\| y_1 - y_2 \|_{PC} + \sum_{k=1}^{m} b_{0k} \| y_1 - y_2 \|_{PC}$$

$$+ a\sum_{k=1}^{m} b_{1k} \| y_1 - y_2 \|_{PC}$$

$$+ a(a_{1m} \| (Fy_1)(t_m) - (Fy_2)(t_m) \|$$

$$+ \sum_{k=1}^{m-1} a_{1k} \| (Fy_1)(t_k) - (Fy_2)(t_k) \|)$$

$$\leq \frac{a^2}{2}\| y_1 - y_2 \|_{PC} + \sum_{k=1}^{m} b_{0k} \| y_1 - y_2 \|_{PC}$$

$$+ a\sum_{k=1}^{m} b_{1k} \| y_1 - y_2 \|_{PC}$$

$$+ a((a_{1m}+1)\sum_{k=1}^{m-1} a_{1k} \| (Fy_1)(t_k) - (Fy_2)(t_k) \|$$

$$+ a_{1m}t_m \| y_1 - y_2 \|_{PC} + a_{1m}\sum_{k=1}^{m-1} b_{1k} \| y_1 - y_2 \|_{PC})$$

$$\leq \frac{a^2}{2}\| y_1 - y_2 \|_{PC} + \sum_{k=1}^{m} b_{0k} \| y_1 - y_2 \|_{PC}$$

$$+ a\sum_{k=1}^{m} b_{1k} \| y_1 - y_2 \|_{PC} \quad (66)$$

$$+ a(a_{1m}\sum_{k=1}^{m-1} b_{1k} + \sum_{k=2}^{m-1}(a_{1k}(\prod_{j=k+1}^{m}(a_{1j}+1))\sum_{i=1}^{k-1} b_{1i})$$

$$+ a_{1m}t_m + \sum_{k=1}^{m-1} a_{1k}t_k \prod_{j=k+1}^{m}(a_{1j}+1)) \| y_1 - y_2 \|_{PC}.$$

Hence,

$$\| By_1 - By_2 \|_{PC} \leq N_1^* \| y_1 - y_2 \|_{PC},$$

where

$$N_1^* = \frac{a^2}{2} + \sum_{k=1}^{m} b_{0k} + a\sum_{k=1}^{m} b_{1k} + a(a_{1m}\sum_{k=1}^{m-1} b_{1k}$$

$$+ \sum_{k=2}^{m-1}(a_{1k}(\prod_{j=k+1}^{m}(a_{1j}+1))\sum_{i=1}^{k-1} b_{1i})$$

$$+ a_{1m}t_m + \sum_{k=1}^{m-1} a_{1k}t_k \prod_{j=k+1}^{m}(a_{1j}+1)) \quad (67)$$

In the same way,

$$\| (By_1)' - (By_2)' \|_{PC} \leq N_2^* \| y_1 - y_2 \|_{PC},$$

where

$$N_2^* = a + \sum_{k=1}^{m} b_{1k} + (a_{1m}\sum_{k=1}^{m-1} b_{1k} \quad (68)$$

$$+ \sum_{k=2}^{m-1}(a_{1k}(\prod_{j=k+1}^{m}(a_{1j}+1))\sum_{i=2}^{k-1} b_{1i}$$

$$+ a_{1m}t_m + \sum_{k=1}^{m-1} a_{1k}t_k \prod_{j=k+1}^{m}(a_{1j}+1)).$$

So

$$\| By_1 - By_2 \|_{PC^1} \leq N^* \| y_1 - y_2 \|_{PC},$$

where $N^* = \max\{N_1^*, N_2^*\}$. Hence, B is bounded and continuous. When $m = 1, 2$, the proof is similar.

(ii) B is increasing.

For any $y_1, y_2 \in PC[J, E], y_1 \leq y_2$, by (G_2) and (61), we have

$$(By_1)(t) - (By_2)(t)$$

$$= \int_0^t (t-s)(y_1(s) - y_2(s))ds \leq \theta,\ t \in J_0.$$

Then

$$(By_1)(t) \leq (By_2)(t), \forall t \in J_0.$$

In particular, $(By_1)(t_1) \leq (By_2)(t_1)$. Moreover, for any $t \in J_1$, we have

$$\sum_{0<t_k<t}(I_{0k}(y_1(t_k)) - I_{0k}(y_2(t_k)))$$

$$+ \sum_{0<t_k<t}(t-t_k)(I_{1k}((By_1)(t_k), y_1(t_k))$$

$$-I_{1k}((By_2)(t_k), y_2(t_k))) \quad (69)$$

$$= I_{01}(y_1(t_1)) - I_{01}(y_2(t_1))$$

$$+(t-t_1)(I_{11}((By_1)(t_1), y_1(t_1))$$

$$-I_{11}((By_2)(t_1), y_2(t_1))) \leq \theta.$$

Then

$$(By_1)(t) \leq (By_2)(t), \forall t \in J_1.$$

In particular, $(By_1)(t_2) \leq (By_2)(t_2)$. In the same way, we have

$$(By_1)(t) \leq (By_2)(t), \forall t \in J_k,$$

$$(By_1)(t_{k+1}) \leq (By_2)(t_{k+1}) (k = 1, 2, \cdots, m).$$

Hence,

$$(By_1)(t) \leq (By_2)(t), \forall t \in J,$$

then $By_1 \leq By_2$. In the same way, F is a bounded continuous operator with increasing. Combing (61) and (62), it is easy to show if

$$y \in PC^1[J, E] \cap C^2[J', E],$$

then $By \in PC^3[J,E] \cap C^4[J',E]$,
and if

$$y \in PC^1[J,E] \cap C^2[J',E],$$

then $Fy \in PC^2[J,E] \cap C^3[J',E]$.
In the same way, we can show

B:$PC^1[J,E] \cap C^2[J',E] \to PC^3[J,E] \cap C^4[J',E]$

F:$PC^1[J,E] \cap C^2[J',E] \to PC^2[J,E] \cap C^3[J',E]$

They are all bounded continuous operators with increasing. Hence, by (59)-(64), IVP (1) is equivalent to IVP (18).

Obviously, if $u \in PC^1[J,E] \cap C^2[J',E]$ is a solution of IVP (18), then

$$x(t) \in PC^3[J,E] \cap C^4[J',E]$$

is a solution of IVP (1) by(63).

Putting $u_0(t) = y_0''(t), v_0(t) = z_0''(t), t \in J$, we have $u_0 \le v_0$. By (G_1), we obtain

$$y_0(t) = x_0^* + x_1^* t + \int_0^t (t-s)u_0(s)ds + \sum_{0<t_k<t} I_{0k}(u_0(t_k)) + \sum_{0<t_k<t} (t-t_k)I_{1k}(y_0'(t_k), u_0(t_k)), \forall t \in J, \tag{70}$$

$$z_0(t) = x_0^* + x_1^* t + \int_0^t (t-s)v_0(s)ds + \sum_{0<t_k<t} I_{0k}(v_0(t_k)) + \sum_{0<t_k<t} (t-t_k)I_{1k}(z_0'(t_k), v_0(t_k)), \forall t \in J, \tag{71}$$

$$y_0'(t) = x_1^* + \int_0^t u_0(s)ds + \sum_{0<t_k<t} I_{1k}(y_0'(t_k), u_0(t_k)), \forall t \in J, \tag{72}$$

$$z_0'(t) = x_1^* + \int_0^t v_0(s)ds + \sum_{0<t_k<t} I_{1k}(z_0'(t_k), v_0(t_k)), \forall t \in J, \tag{73}$$

then

$$y_0(t) = (Bu_0)(t), z_0(t) = (Bv_0)(t),$$
$$y_0'(t) = (Fu_0)(t), z_0'(t) = (Fv_0)(t), \forall t \in J,$$

where u_0, v_0 satisfy (H_1).

By (G_2), it is easy to know that (H_2) holds. Hence, applying Theorem3.1, there exist the maximal and minimal solutions $u^*, v^* \in PC^1[J,E] \cap C^2[J',E]$ of IVP (18) on $[u_0, v_0]$.

Let $y^* = Bu^*, z^* = Bv^*$. Then

$$y^*, z^* \in PC^3[J,E] \cap C^4[J',E]$$

and

$$y^*(t) = x_0^* + x_1^* t + \int_0^t (t-s)u^*(s)ds + \sum_{0<t_k<t} I_{0k}(u^*(t_k)) + \sum_{0<t_k<t} (t-t_k)I_{1k}((y^*)'(t_k), u^*(t_k)), \ \forall t \in J. \tag{74}$$

By (74), we have

$$\begin{cases} (y^*)''(t) = u^*(t), \forall t \in J, t \ne t_k, \\ \Delta y^* |_{t=t_k} = I_{0k}(u^*(t_k)), \\ \Delta (y^*)' |_{t=t_k} = I_{1k}((y^*)'(t_k), u^*(t_k))(k = 1, 2, \cdots, m), \\ y^*(0) = x_0^*, (y^*)'(0) = x_1^*, \end{cases} \tag{75}$$

If there exist u^* such that (18) and y^* such that (75), then y^* is a solution of IVP (1).In the same way, z^* is a solution of IVP (1).It is easy to verify $y^*, z^* \in PC^3[J,E] \cap C^4[J',E]$ are the maximal and minimal solutions of IVP of (1) on $[y_0, z_0]$, respectively. This ends the proof.

Theorem 4.2. Suppose E is a real Banach space, P is a regular cone, and $(G_1),(G_2),(G_3)$ hold. Assume (4) or (5) is satisfied, then there exist the maximal and minimal solutions $y^*, z^* \in PC^3[J,E] \cap C^4[J',E]$ of IVP (1) on $[y_0, z_0]$, respectively.

Proof. The proof is similar to the proof of Theorem 4.1.If Theorem 3.2 satisfies, then there exist $u^*, v^* \in PC^1[J,E] \cap C^2[J',E]$ the maximal and minimal solutions of IVP (1) respectively. This ends the proof.

Corollary 4.1. If E is a weak sequentially complete Banach space, P is a normal cone, $(G_1),(G_2),(G_3)$ hold, and (4) or (5) is satisfied, then IVP (1) has the maximal and minimal solutions $y^*, z^* \in PC^3[J,E] \cap C^4[J',E]$ on $[y_0, z_0]$.

Proof. If E is a weak sequentially complete Banach space, the normality of P is equivalent to the regularity of P .Hence the conclusion of Corollary4.1 holds by Theorem 4.2.This ends the proof.

Theorem 4.3. Suppose E is a real Banach space. P is regular cone, and $(G_1'),(G_2),(G_3),(H_3)$ hold. Assume (4) or (5) is satisfied. If for any $z, u \in E$,

$$f(t,x,y,z,u,v,w) \ge f(t,\bar{x},\bar{y},z,u,\bar{v},\bar{w}) \quad \forall x \ge \bar{x}, y \ge \bar{y}, v \ge \bar{v}, w \ge \bar{w}, \tag{76}$$

then IVP (1) has the maximal and minimal solutions

$y^*, z^* \in PC^3[J,E] \cap C^4[J',E]$ on $[y_0, z_0]$.

Proof. Similar to the proof of Theorem 4.1,we consider IVP (1).Let $x''(t)=u(t), t\in J$, then

$$x(t)=(Bu)(t), x'(t)=(Fu)(t), t\in J.$$

Hence, IVP (1) is equivalent to IVP (18).Let

$$u_0(t)=y_0''(t), v_0(t)=z_0''(t), t\in J. \tag{77}$$

Then $u_0 \le v_0$ .Combing (77) and (G_1') , for any $t\in J$, we have

$$y_0(t)=y_0(0)+y_0'(0)t+\int_0^t (t-s)u_0(s)ds + \sum_{0<t_k<t}\Delta y_0(t_k)+\sum_{0<t_k<t}(t-t_k)\Delta y_0'(t_k)\cdot$$

$$y_0'(t)=y_0(0)+\int_0^t u_0(s)ds+\sum_{0<t_k<t}\Delta y_0'(t_k), \forall t\in J,$$

$$z_0(t)=z_0(0)+z_0'(0)t+\int_0^t (t-s)v_0(s)ds + \sum_{0<t_k<t}\Delta z_0(t_k)+\sum_{0<t_k<t}(t-t_k)\Delta z_0'(t_k),$$

$$z_0'(t)=z_0(0)+\int_0^t v_0(s)ds+\sum_{0<t_k<t}\Delta z_0'(t_k).$$

It is easy to verify

$$y_0(t)\le (Bu_0)(t), y_0'(t)\le (Fu_0)(t),$$
$$(Bv_0)(t)\le z_0(t), (Fv_0)(t)\le z_0'(t), \quad t\in J_0.$$

In particular,

$$y_0(t_1)\le (Bu_0)(t_1), y_0'(t_1)\le (Fu_0)(t_1),$$
$$(Bv_0)(t_1)\le z_0(t_1), (Fv_0)(t_1)\le z_0'(t_1).$$

Moreover, we have for any $k, (k=1,2,\cdots,m)$

$$y_0(t)\le (Bu_0)(t), y_0'(t)\le (Fu_0)(t),$$
$$(Bv_0)(t)\le z_0(t), (Fv_0)(t)\le z_0'(t), t\in J_k,$$
$$y_0(t_{k+1})\le (Bu_0)(t_{k+1}), y_0'(t_{k+1})\le (Fu_0)(t_{k+1}),$$
$$(Bv_0)(t_{k+1})\le z_0(t_{k+1}), (Fv_0)(t_{k+1})\le z_0'(t_{k+1}).$$

So we have

$$y_0\le Bu_0, y_0'\le Fu_0, Bv_0\le z_0, Fv_0\le z_0'.$$

Hence, by (G_1') ,we know (H_1) holds. Similar to the proof Theorem 4.1,we obtain the conclusion. This ends the proof.

Theorem 4.4. Suppose E is a real Banach space. P is regular cone, $(G_1'), (G_2), (G_3)$ hold. Assume (4) or (5) is satisfied. If for any $z,u\in E$, (76) holds, then IVP (1) has the maximal and minimal solutions

$$y^*, z^* \in PC^3[J,E]\cap C^4[J',E]$$

on $[y_0, z_0]$.

Proof. Similar to Theorem 4.3, it is easy to know (H_1) holds. Then the rest of the proof is similar to the proof of Theorem 4.3.This ends of the proof.

Corollary 4.2. If E is a weak sequentially complete Banach space, P is a normal cone, (G_1'), (G_2), (G_3) hold. Assume (4) or (5) is satisfied. If for any $z,u\in E$, (76) holds, then IVP (1) has the maximal and minimal solutions

$$y^*, z^* \in PC^3[J,E]\cap C^4[J',E]$$

on $[y_0, z_0]$.

Proof. If E is a weak sequentially complete Banach space, the normality of P is equivalent to the regularity of P. Hence, the conclusion of Corollary4.2 holds by Theorem 4.4.This ends the proof.

Remark 4.1. In Theorem 3.2 and Theorem 4.2, Theorem 4.4, the condition (H_3) is more easy to use and verify.

5 Application

Example 5.1. Consider the following initial value problem for fourth-order impulsive integro-differential equations:

$$\begin{cases} x_n^{(4)}(t)=\frac{1}{3n}(t^2+x_{2n}(t))+\frac{1}{4n}(\frac{t}{n}+x_n'(t)) \\ \quad +\frac{t}{18n}(\frac{t^2}{2n}-x_n''(t))+\frac{t^2}{9}(\frac{t}{n}-x_n'''(t)) \\ \quad +\frac{1}{6n}(t+\int_0^t e^{-ts}x_n(s)ds) \\ \quad +\frac{1}{2n}\int_0^1\frac{1}{1+t+s}x_{2n}(s)ds, \forall 0\le t\le 1, t\ne\frac{1}{2}, \\ \Delta x_n|_{t=\frac{1}{2}}=\frac{1}{25(n+1)^2}x_n''(\frac{1}{2}), \\ \Delta x_n'|_{t=\frac{1}{2}}=\frac{1}{12}x_n'(\frac{1}{2})+\frac{1}{10n^2}x_n''(\frac{1}{2}) \\ \Delta x_n''|_{t=\frac{1}{2}}=\frac{1}{4}x_n'''(\frac{1}{2}), \\ \Delta x_n'''|_{t=\frac{1}{2}}=\frac{1}{2n}x_n(\frac{1}{2})+\frac{1}{4n}x_n'(\frac{1}{2})-\frac{1}{15}x_n''(\frac{1}{2}) \\ \quad -\frac{1}{8n}x_n'''(\frac{1}{2}), (n=1,2,3,\cdots) \\ x_n(0)=0, x_n'(0)=0, x_n''(0)=0, x_n'''(0)=0 \end{cases} \tag{78}$$

Conclusion IVP (78) has the maximal and minimal solutions belonging to C^4 on $[0,\frac{1}{2})\cup(\frac{1}{2},1)]$ such that

$$0 \le x_n(t) \le \begin{cases} \frac{t^4}{24n}, t \in [0, \frac{1}{2}], \\ \frac{t^4}{12n}, t \in (\frac{1}{2}, 1], \end{cases} \quad (n = 1, 2, \cdots)$$

$$0 \le x_n'(t) \le \begin{cases} \frac{t^3}{6n}, t \in [0, \frac{1}{2}], \\ \frac{t^3}{3n}, t \in (\frac{1}{2}, 1], \end{cases} \quad (n = 1, 2, \cdots)$$

$$0 \le x_n''(t) \le \begin{cases} \frac{t^2}{2n}, t \in [0, \frac{1}{2}], \\ \frac{t^2}{n}, t \in (\frac{1}{2}, 1], \end{cases} \quad (n = 1, 2, \cdots)$$

$$0 \le x_n'''(t) \le \begin{cases} \frac{t}{n}, t \in [0, \frac{1}{2}], \\ \frac{2t}{n}, t \in (\frac{1}{2}, 1], \end{cases} \quad (n = 1, 2, \cdots).$$

Proof. Let

$$E = c_0 = \{x = (x_1, x_2, \cdots, x_n, \cdots): x_n \to 0\}$$

with the norm $\| x \| = \sup_n | x_n |$,

$$P = \{x = (x_1, x_2, \cdots, x_n, \cdots) \in c_0 : x_n \ge 0, n = 1, 2, 3, \cdots\}$$

Then P is a normal cone in E, and (78) is a initial value problem in E, where

$$a = 1, k(t, s) = e^{-ts}, h(t, s) = \frac{1}{1 + t + s},$$

$$x_0^* = x_1^* = x_2^* = x_3^* = (0, \cdots, 0, \cdots),$$

$$x = (x_1, x_2, \cdots, x_n, \cdots), y = (y_1, y_2, \cdots, y_n, \cdots),$$

$$z = (z_1, z_2, \cdots, z_n, \cdots), u = (u_1, u_2, \cdots, u_n, \cdots),$$

$$v = (v_1, v_2, \cdots, v_n, \cdots), w = (w_1, w_2, \cdots, w_n, \cdots),$$

$$f = (f_1, f_2, \cdots, f_n, \cdots)$$

and

$$f_n(t, x, y, z, u, v, w) = \frac{1}{3n}(t^2 + x_{2n}) + \frac{1}{4n}(\frac{t}{n} + y_n) + \frac{t}{18n}(\frac{t^2}{2n} - z_n) + \frac{t^2}{9}(\frac{t}{n} - u_n) + \frac{1}{6n}(t + v_n) + \frac{1}{2n} w_{2n}, \quad (79)$$

$$m = 1, t_1 = \frac{1}{2}, I_{01} = (I_{011}, I_{012}, \cdots, I_{01n}, \cdots),$$

$$I_{11} = (I_{111}, I_{112}, \cdots, I_{11n}, \cdots), I_{21} = (I_{211}, I_{212}, \cdots, I_{21n}, \cdots),$$

$$I_{31} = (I_{311}, I_{312}, \cdots, I_{31n}, \cdots),$$

where

$$I_{01n}(z) = \frac{1}{25(n+1)^2} z_n,$$

$$I_{11n}(y, z) = \frac{1}{12} y_n + \frac{1}{10n^2} z_n, I_{21n}(u) = \frac{1}{4} u_n,$$

$$I_{31n}(x, y, z, u) = \frac{1}{2n} x_n + \frac{1}{4n} y_n - \frac{1}{15} z_n - \frac{1}{8n} u_n.$$

Let $J = [0, 1]$, obviously, $f \in C[J \times E \times E \times E \times E \times E \times E, E] \cdot$

Let $y_0(t) = (0, 0, \cdots 0, \cdots), t \in [0, 1]$,

$$z_0(t) = \begin{cases} (\frac{t^4}{24}, \frac{t^4}{48}, \cdots, \frac{t^4}{24n}, \cdots), t \in [0, \frac{1}{2}], \\ (\frac{t^4}{12}, \frac{t^4}{24}, \cdots, \frac{t^4}{12n}, \cdots), t \in (\frac{1}{2}, 1]. \end{cases}$$

We have

$$y_0'(t) = (0, 0, \cdots, 0, \cdots), t \in [0, 1],$$

$$y_0''(t) = (0, 0, \cdots, 0, \cdots), t \in [0, 1],$$

$$y'''(t) = (0, 0, \cdots, 0, \cdots), t \in [0, 1],$$

$$y_0^{(4)}(t) = (0, 0, \cdots, 0, \cdots), t \in [0, 1],$$

$$z_0'(t) = \begin{cases} (\frac{t^3}{6}, \frac{t^3}{12}, \cdots, \frac{t^3}{6n}, \cdots), t \in [0, \frac{1}{2}], \\ (\frac{t^3}{3}, \frac{t^3}{6}, \cdots, \frac{t^3}{3n}, \cdots), t \in (\frac{1}{2}, 1], \end{cases}$$

$$z_0''(t) = \begin{cases} (\frac{t^2}{2}, \frac{t^2}{4}, \cdots, \frac{t^2}{2n}, \cdots), t \in [0, \frac{1}{2}], \\ (t^2, \frac{t^2}{2}, \cdots, \frac{t^2}{n}, \cdots), t \in (\frac{1}{2}, 1], \end{cases}$$

$$z_0'''(t) = \begin{cases} (t, \frac{t}{2}, \cdots, \frac{t}{n}, \cdots), t \in [0, \frac{1}{2}], \\ (2t, t, \cdots, \frac{2t}{n}, \cdots), t \in (\frac{1}{2}, 1], \end{cases}$$

$$z_0^{(4)}(t) = \begin{cases} (1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots), t \in [0, \frac{1}{2}], \\ (2, 1, \cdots, \frac{2}{n}, \cdots), t \in (\frac{1}{2}, 1]. \end{cases}$$

Hence, we have $y_0, z_0 \in PC^3[J, E] \cap C^4[J', E]$,

$$y_0'(t) \le z_0'(t), y_0''(t) \le z_0''(t), \ y_0'''(t) \le z_0'''(t), t \in J$$

and

$$y_0(0) = z_0(0) = (0, 0, \cdots, 0, \cdots) = x_0^*,$$

$$y_0'(0) = z_0'(0) = (0, 0, \cdots, 0, \cdots) = x_1^*,$$

$$y_0''(0) = z_0''(0) = (0, 0, \cdots, 0, \cdots) = x_2^*,$$

$$y_0'''(0) = z_0'''(0) = (0, 0, \cdots, 0, \cdots) = x_3^*,$$

$$f_n(t, y_0(t), y_0'(t), y_0''(t), y_0'''(t), (Ty_0)(t), (Sy_0)(t))$$
$$=\frac{t^2}{3n}+\frac{t}{4n^2}+\frac{t^3}{36n^2}+\frac{t^3}{9n}+\frac{t}{6n}\ge 0, \forall t\in[0,1]$$

when $0\le t\le \frac{1}{2}$,

$$f_n(t, z_0(t), z_0'(t), z_0''(t), z'''(t), (Tz_0)(t), (Sz_0)(t))$$
$$\le \frac{1}{3n}(t^2+\frac{t^4}{48n})+\frac{1}{4n}(\frac{t}{n}+\frac{t^3}{6n})$$
$$+\frac{1}{6n}(t+\int_0^t \frac{s^4}{24n}ds)+\frac{1}{2n}\int_0^1 \frac{s^4}{48n}ds\le \frac{1}{n}.$$

when $\frac{1}{2}<t\le 1$,

$$f_n(t, z_0(t), z_0'(t), z_0''(t), z_0'''(t), (Tz_0)(t), (Sz_0)(t))$$
$$\le \frac{1}{3n}(t^2+\frac{t^4}{24n})+\frac{1}{4n}(\frac{t}{n}+\frac{t^3}{3n})+\frac{t}{18n}(\frac{t^2}{2n}-\frac{t^2}{n})$$
$$+\frac{t^2}{9}(\frac{t}{n}-\frac{2t}{n})+\frac{1}{6n}(t+\int_0^t \frac{s^4}{12n}ds)+\frac{1}{2n}\int_0^1 \frac{s^4}{24n}ds$$
$$\le \frac{2}{n},$$

$$\Delta y_0|_{t=\frac{1}{2}}=(0,0,\cdots,0,\cdots)=I_{01}(y_0''(\frac{1}{2})),$$

$$\Delta y_0'|_{t=\frac{1}{2}}=(0,0,\cdots,0,\cdots)=I_{11}(y_0'(\frac{1}{2}), y_0''(\frac{1}{2})),$$

$$\Delta y_0''|_{t=\frac{1}{2}}=(0,0,\cdots,0,\cdots)=I_{21}(y_0'''(\frac{1}{2})),$$

$$\Delta y_0'''|_{t=\frac{1}{2}}=(0,0,\cdots,0,\cdots)$$
$$=I_{31}(y_0(\frac{1}{2}), y_0'(\frac{1}{2}), y_0''(\frac{1}{2}), y_0'''(\frac{1}{2})),$$

$$\Delta z_0|_{t=\frac{1}{2}}=(\frac{1}{384},\frac{1}{768},\cdots,\frac{1}{384n},\cdots)\ge I_{01}(z_0''(\frac{1}{2})),$$

$$\Delta z_0'|_{t=\frac{1}{2}}=(\frac{1}{48},\frac{1}{96},\cdots,\frac{1}{48n},\cdots)\ge I_{11}(z_0'(\frac{1}{2}), z_0''(\frac{1}{2})),$$

$$\Delta z_0''|_{t=\frac{1}{2}}=(\frac{1}{8},\frac{1}{16},\cdots,\frac{1}{8n},\cdots)=I_{21}(z_0'''(\frac{1}{2})),$$

$$\Delta z_0'''|_{t=\frac{1}{2}}=(\frac{1}{2},\frac{1}{4},\cdots,\frac{1}{2n},\cdots)$$
$$\ge I_{31}(z_0(\frac{1}{2}), z_0'(\frac{1}{2}), z_0''(\frac{1}{2}), z_0'''(\frac{1}{2})),$$

so (G_1') is satisfied. On the other hand, for any $t\in J$,

$y_0(t)\le \overline{x}\le x\le z_0(t),\ y_0'(t)\le \overline{y}\le y\le z_0'(t),$

$y_0''(t)\le \overline{z}\le z\le z_0''(t),\ y_0'''(t)\le \overline{u}\le u\le z_0'''(t),$

$Ty_0(t)\le \overline{v}\le v\le Tz_0(t),\ Sy_0(t)\le \overline{w}\le w\le Sz_0(t)$

we have

$$f(t,x,y,z,u,v,w)-f(t,\overline{x},\overline{y},\overline{z},\overline{u},\overline{v},\overline{w})$$
$$=\frac{1}{3n}(x_{2n}-\overline{x}_{2n})+\frac{1}{4n}(y_n-\overline{y}_n)+\frac{t}{18n}(\overline{z}_n-z_n)$$
$$+\frac{t^2}{9}(\overline{u}_n-u_n)+\frac{1}{6n}(v_n-\overline{v}_n)+\frac{1}{2n}(w_{2n}-\overline{w}_{2n})$$
$$\ge -\frac{t}{18}(z_n-\overline{z}_n)-\frac{t^2}{9}(u_n-\overline{u}_n)(n=1,2,3,\cdots)$$

$$I_{01}(z)\ge I_{01}(\overline{z}), I_{11}(y,z)\ge I_{11}(\overline{y},\overline{z}),$$

$$I_{21}(u)-I_{21}(\overline{u})=\frac{1}{4}(u-\overline{u}),$$

$$I_{31}(x,y,z,u)-I_{31}(\overline{x},\overline{y},\overline{z},\overline{u})$$
$$\ge -\frac{1}{15}(z-\overline{z})-\frac{1}{8}(u-\overline{u}),$$

so (G_2) is satisfied, where

$$M_1(t)=\frac{t}{18}, M_2(t)=\frac{t^2}{9}, C_1=\frac{1}{4}, L_1=\frac{1}{15}, L_1^*=\frac{1}{8},$$

It is easy to verify (4) holds.

Obviously, for any $y, z, \overline{y}, \overline{z}\in E$, we have

$$\| I_{01}(z)-I_{01}(\overline{z})\|\le \frac{1}{100}\| z-\overline{z}\|,$$

$$\| I_{11}(y,z)-I_{11}(\overline{y},\overline{z})\|\le \frac{1}{12}\| y-\overline{y}\|+\frac{1}{10}\| z-\overline{z}\|,$$

so (G_3) is satisfied. By (79), then

$$f=f^{(1)}+f^{(2)}, f^{(1)}=(f_1^{(1)}, f_2^{(1)},\cdots, f_n^{(1)},\cdots),$$
$$f^{(2)}=(f_1^{(2)}, f_2^{(2)},\cdots, f_n^{(2)},\cdots),$$

where

$$f_n^{(1)}(t,x,y,z,u,v,w)$$
$$=\frac{1}{3n}(t^2+x_{2n})+\frac{1}{4n}(\frac{t}{n}+y_n)$$
$$+\frac{t}{18n}(\frac{t^2}{2n}-z_n)+\frac{1}{6n}(t+v_n)+\frac{1}{2n}w_{2n}, \tag{80}$$

$$f_n^{(2)}(t,x,y,z,u,v,w)=\frac{t^2}{9}(\frac{t}{n}-u_n). \tag{81}$$

For any $r>0$, assume $\{t^{(b)}\}_{b=1}^{\infty}\subset J$, $\{x^{(b)}\}_{b=1}^{\infty}, \{y^{\{b\}}\}_{b=1}^{\infty}, \{z^{(b)}\}_{b=1}^{\infty}, \{u^{(b)}\}_{b=1}^{\infty}, \{v^{(b)}\}_{b=1}^{\infty}$, $\{w^{(b)}\}_{b=1}^{\infty}\subset B_r$,

where

$$x^{(b)}=(x_1^{\{b\}}, x_2^{(b)},\cdots, x_n^{(b)},\cdots),$$
$$y^{(b)}=(y_1^{(b)}, y_2^{(b)},\cdots, y_n^{(b)},\cdots),$$

$$z^{(b)} = (z_1^{(b)}, z_2^{(b)}, \cdots, z_n^{(b)}, \cdots),$$
$$u^{(b)} = (u_1^{(b)}, u_2^{(b)}, \cdots, u_n^{(b)}, \cdots),$$
$$v^{(b)} = (v_1^{(b)}, v_2^{(b)}, \cdots, v_n^{(b)}, \cdots),$$
$$w^{(b)} = (w_1^{(b)}, w_2^{(b)}, \cdots, w_n^{(b)}, \cdots)$$

By (80), we have

$$f_n^{(1)}(t^{(b)}, x^{(b)}, y^{(b)}, z^{(b)}, u^{(b)}, v^{(b)}, w^{(b)})$$
$$\leq \frac{1}{3n}(1+\| x^{(b)} \|) + \frac{1}{4n}(\frac{1}{n}+\| y^{(b)} \|)$$
$$+\frac{1}{18n}(\frac{1}{2n}+\| z^{(b)} \|) + \frac{1}{6n}(1+\| v^{(b)} \|) + \frac{1}{2n}\| w^{(b)} \|$$
$$\leq \frac{1}{3n}(1+r) + \frac{1}{4n}(\frac{1}{n}+r) + \frac{1}{18n}(\frac{1}{2n}+r)$$
$$+\frac{1}{6n}(1+r) + \frac{1}{2n}r(b, n = 1, 2, 3, \cdots). \quad (82)$$

So

$$\{ f_n^{(1)}(t^{(b)}, x^{(b)}, y^{(b)}, z^{(b)}, u^{(b)}, v^{(b)}, w^{(b)}) \}$$

is bounded, moreover, we choose subsequence $\{b_i\} \subset \{b\}$ such that

$$f_n^{(1)}(t^{(b_i)}, x^{(b_i)}, y^{(b_i)}, z^{(b_i)}, u^{(b_i)}, v^{(b_i)}, w^{(b_i)}) \to \zeta_n,$$
$$i \to \infty (n = 1, 2, 3, \cdots). \quad (83)$$

Combing (82) and (83), we have

$$|\zeta_n| \leq \frac{1}{3n}(1+r) + \frac{1}{4n}(\frac{1}{n}+r) + \frac{1}{18n}(\frac{1}{2n}+r)$$
$$+\frac{1}{6n}(1+r) + \frac{1}{2n}r(n = 1, 2, 3, \cdots), \quad (84)$$

so $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n, \cdots) \in c_0 = E.$ For any $\varepsilon > 0,$ by (82) and (84),there exists a positive integer n_0 such that

$$| f_n^{(1)}(t^{(b_i)}, x^{(b_i)}, y^{(b_i)}, z^{(b_i)}, u^{(b_i)}, v^{(b_i)}, w^{(b_i)}) | < \varepsilon,$$
$$|\zeta_n| < \varepsilon, \forall n > n_0, \quad (i = 1, 2, 3, \cdots). \quad (85)$$

By (83), there exists a positive integer i_0 such that

$$| f_n^{(1)}(t^{(b_i)}, x^{(b_i)}, y^{(b_i)}, z^{(b_i)}, u^{(b_i)}, v^{(b_i)}, w^{(b_i)}) - \zeta_n | < \varepsilon,$$
$$\forall i > i_0, (n = 1, 2, \cdots, n_0). \quad (86)$$

Then, combing (85) and (86),we have

$$\| f^{(1)}(t^{(b_i)}, x^{(b_i)}, y^{(b_i)}, z^{(b_i)}, u^{(b_i)}, v^{(b_i)}, w^{(b_i)}) - \zeta \|$$
$$= \sup_n | f_n^{(1)}(t^{(b_i)}, x^{(b_i)}, y^{(b_i)}, z^{(b_i)}, u^{(b_j)}, v^{(b_i)}, w^{(b_i)}) - \zeta_n |$$
$$\leq 2\varepsilon, \forall i > i_0.$$

Hence,

$$\| f^{(1)}(t^{(b_i)}, x^{(b_i)}, y^{(b_i)}, z^{(b_i)}, u^{(b_i)}, v^{(b_i)}, w^{(b_i)}) - \zeta \|$$
$$\to 0, \quad i \to \infty.$$

Thus,

$$\alpha(f^{(1)}(J, U_1, U_2, U_3, U_4, U_5, U_6)) = 0,$$
$$\forall U_i \subset B_r (i = 1, 2, 3, 4, 5, 6). \quad (87)$$

On the other hand, applying (81),

$$\alpha(f^{(2)}(J, U_1, U_2, U_3, U_4, U_5, U_6)) \leq \frac{1}{9}\alpha(U_4),$$
$$\forall U_i \subset B_r (i = 1, 2, 3, 4, 5, 6) \quad (88)$$

By (87) and (88), we have

$$\alpha(f(J, U_1, U_2, U_3, U_4, U_5, U_6)) \leq \frac{1}{9}\alpha(U_4),$$
$$\forall U_i \subset B_r (i = 1, 2, 3, 4.5, 6). \quad (89)$$

In the same way,

$$\alpha(I_{31}(V_1, V_2, V_3, V_4)) \leq \frac{1}{15}\alpha(V_3),$$
$$\forall V_j \subset B_r (j = 1, 2, 3, 4), \quad (90)$$
$$\alpha(I_{21}(V_4)) \leq \frac{1}{4}\alpha(V_4), \forall V_4 \subset B_r. \quad (91)$$

Hence, (H_3) holds, where

$$d_r = 0, d_r^* = \frac{1}{9}, b_r^{(1)} = \frac{1}{15}, a_r^{(1)} = \frac{1}{4}.$$

Finally, it is easy to prove (76) holds. Then, we have the conclusion by Theorem 4.3.This ends the proof.

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