Global solutions for second order impulsive integro-differential equations in Banach spaces

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Abstract: This paper regards initial value problem for second order impulsive integro-differential equations as some nonlinear vector system. By means of the Mönch′s fixed point theorem, some existence theorems of solutions of the initial value problem are established. The results are newer than all of the previous ones because of the more general form compactness-type condition and the weaker restriction of its coefficients. An example is given to demonstrate our results. Annotation shows that our method can be used to solve the impulsive boundary value problems.

Key–Words: Impulsive integro-differential equations, initial value problem, Boundary value problem, Compactness-type condition, Banach space, Fixed point, Operator norm of the matrix.

1 Introduction

Around the last fifteen years, a lot of works [1-10,13,14] have been done for the following initial value problem for nonlinear second order impulsive integro-differential equations of mixed type in a real Banach space E

$$\begin{cases} u'' = f(t,u,u',Tu,Su), \forall t \in J=[0,a], t \neq t_k \\ \triangle u|_{t=t_k} = I(u(t_k),u'(t_k)), \\ \triangle u'|_{t=t_k} = \bar{I}_k(u(t_k),u'(t_k)), \quad k=1,2,\cdots,m \\ u(0)=x_0, u'(0)=x_1 \end{cases} \tag{1}$$

where $Tu = \int_0^t q(t,s)u(s)ds$, $Su = \int_0^a h(t,s)u(s)ds$, $h(t,s) \in C(J\times J,R)$, $q(t,s)\in C(D,R)$, $D=\{(t,s)\in J\times J: t\geq s\}$. $\triangle u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $(k=1,2,\cdots,m)$ denote the jump of $u(t)$ at $t=t_k$, $u(t_k^-)$ and $u(t_k^+)$ represent the left and right limits of $u(t)$ at $t=t_k$ respectively, and $\triangle u'|_{t=t_k}$ has a similar meaning for $u'(t)$.

In many investigations, for examples [1-4,9, 10, 15], non-compactness type conditions, combined with fixed point theorem, play an important role in the proof of those results. In 1996, Guo[4] studied the unique solution of system IVP(1) employing Banach's fixed point theorem. Zhang [15] studied IVP(1) for the case in which f does not include derivative x' and obtained a global solution by Schauder's fixed point theorem. Zhang et al.[10] improved the results of Zhang[15] by Mönch's fixed point theorem with a new established comparison result. Recently, Liu et al.[9] and Zhang et al.[2] generalized the results of Guo[4] by using Banach's fixed point theorem. Almost at the same time, Guo et al.[3] established the existence of global solutions of IVP(1) by Schauder's fixed point theorem. And then Zhang et al.[1], based on the generalization of Darbo's fixed point theorem, extended the the results of Guo et al.[3] step by step through extending integro-differential equation without impulses on subinterval $\bar{J}_k$ to one with impulses on global interval J. Zhang et al. used the following compactness-type condition:

(H0). For any $r>0$, f is bounded and uniformly continuous on $J\times B_r\times B_r\times B_r\times B_r$, and there exist non-negative Lebesgue integrable functions $L_k \in L(J,R^+)(k=1,2,3)$ such that for any bounded sets $B_i \in E(i=1,2,3,4)$ and $t\in J$,

$$\alpha(f(t,B_1,B_2,B_3,B_4)) \leq L_1(t)\alpha(B_1)+L_2(t)\alpha(B_2)+L_3(t)\alpha(B_3). \tag{2}$$

Apparently, the effect of operator Su in f of IVP (1) is overlooked.

Compactness type condition with both u' and Su is very difficult to deal with in proof. By introducing an operator and transforming IVP(1) into first order IVP without u', Wang et al.[8] obtained some results by using the monotone iterative technique. In this paper, the novelty of our approach is to introduce a vector with components being $u(t)$ defined on each subinterval $[t_k,t_{k+1}]$ (where $t_0=0, t_{m+1}=$

a, $u(t_k) = u(t_k^+)$ at the left point of subinterval and $k = 0, 1, \cdots, m$), then a corresponding integro-differential equation is derived for such an unknown vector system. Further, by means of the Mönch fixed point theorem, we establish the existence of solutions of IVP(1). Under more general form with the item $L_4(t)\alpha(B_4)$ than the condition (2), we obtain some new results.

2 Some Lemmas

Let $PC[J,E] = \{u | u : J \to E$ is continuous at $t \neq t_i$, left continuous at $t = t_i$, and its right limit $u(t_i^+)$ at t_i exists, $i = 1, 2, \cdots, m\}$. Evidently, $PC[J,E]$ is a Banach space with the norm $\|u\|_{PC} = \sup_{t\in J} \|u(t)\|$. Let $PC^1[J,E] = \{u \in PC[J,E] | u'(t)$ is continuous at $t \neq t_i$, and $u'(t_i^-), u'(t_i^+)$ exist, $i = 1, 2, \cdots, m\}$. We can obtain that $u'(t)$ is continuous at the left of t_i by the mean value theorem, and then $PC^1[J,E]$ is a Banach space with the norm

$$\|u\|_{PC^1} = \max\{\|u\|_{PC}, \|u'\|_{PC}\}$$

Let $J' = J\backslash\{t_1, t_2, \cdots, t_m\}$, $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, $\cdots$, $J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, a]$, $t_0 = 0$, $t_{m+1} = a$, $d_i = t_{i+1} - t_i$, $\bar{J}_i$ is the closure of J_i and $B_r = \{x \in E : \|x\| \leq r\}$ for any $r > 0$. For $H \subset PC^1[J,E]$, let $H' = \{x' : x \in H\} \subset PC[J,E]$ and

$$\begin{gathered} H_i = \{x|_{\bar{J}_i} : x \in H\} \subset C^1[\bar{J}_i, E], \\ H_i' = \{x'|_{\bar{J}_i} : x \in H\} \subset C[\bar{J}_i, E], \\ A_iH = \{(Ax)|_{\bar{J}_i} : x \in H\} \subset C^1[\bar{J}_i, E], \\ (A_iH)' = \{(Ax)'|_{\bar{J}_i} : x \in H\} \subset C[\bar{J}_i, E], \end{gathered}$$

where $x(t_i) = x(t_i^+)$, $x'(t_i) = x'(t_i^+)$, $(A_ix)(t_i) = (Ax)(t_i^+)$, $(Ax)'(t_i) = (Ax)'(t_i^+)$, $(i = 1, 2, \cdots, m)$. For any $t \in J$, set

$$\begin{gathered} H(t) = \{x(t) : x \in H\} \subset E, \\ H'(t) = \{x'(t) : x \in H\} \subset E, \\ (TH)(t) = \{(Tx)(t) : x \in H\} \subset E, \\ (SH)(t) = \{(Sx)(t) : x \in H\} \subset E. \end{gathered}$$

For any $t \in J_i$ $(i = 0, 1, \cdots, m)$, set

$$\begin{gathered} H_i(t) = \{x(t) : x \in H, t \in J_i\} \subset E, \\ H_i'(t) = \{x'(t) : x \in H, t \in J_i\} \subset E, \\ (A_iH)(t) = \{(Ax)(t) : x \in H, t \in J_i\} \subset E, \\ (A_iH)'(t) = \{(Ax)'(t) : x \in H, t \in J_i\} \subset E. \end{gathered}$$

Let $\alpha(.), \alpha_1(.)$ and $\alpha_2(.)$ denote the Kuratowski measure of non-compactness in E, $C^1(I,E)$ and $PC^1(J,E)$ respectively. For the details please to refer the references [11][12].

Lemma 1 *[3]. If $H \subset PC^1(J,E)$ is bounded and the elements of H are equicontinuous on each $J_k (k = 0, 1, \ldots, m)$, then $\bar{co}(H) \subset PC^1(J,E)$ is bounded and equicontinuous an each $J_k (k = 0, 1, \ldots, m)$. (Here $\bar{co}(H)$ denotes the closed convex hull of H.*

Lemma 2 *[3]. If for any $r > 0$, f is bounded and uniformly continuous on $J \times B_r \times B_r \times B_r \times B_r$ and $H \subset PC^1(J,E)$ is bounded and equicontinuous on each $J_k (k = 0, 1, \ldots, m)$, then*

$$\{f(t, x(t), x'(t), (Tx)(t), (Sx)(t)) : x \in H\} \subset PC(J,E)$$

is bounded and equicontinuous on each $J_k (k = 0, 1, \ldots, m)$.

Lemma 3 *[11] If $H \subset PC^1[J,E]$ is bounded and the elements of H' are equicontinuous on each $J_k (k = 0, 1, \ldots, m)$, then*

$$\alpha_2(H) = \max\{\sup_{t\in J} \alpha(H(t)), \sup_{t\in J} \alpha(H'(t))\}.$$

Lemma 4 *[15] If $H \subset PC^1[J,E]$ is bounded and equicontinuous on each J_k $(k = 0, 1, 2, \ldots, m)$, then $\alpha(\{u(t) | u \in H\})$ is continuous on $t \in J_k$ $(k = 0, 1, 2, \ldots, m)$ and*

$$\alpha(\{\int_0^a u(t)dt | u \in H\}) \leq \int_0^a \alpha(\{u(t) | u \in H\})dt.$$

Lemma 5 *[12] Let E be a Banach space, $\Omega \subset E$ be a bounded open set, and $\theta \in \Omega, A : E \to E$ be continuous such that, (i) $x \neq \lambda Ax$ for $\forall\lambda \in [0,1]$ and $x \in \partial\Omega$; (ii) that $H \subset \overline{\Omega}$ is countable and $H \subset \overline{co}(\{\theta\} \cup (AH))$ imply that H is relative compact. Then A has at least one fixed point in Ω.*

Lemma 6 *[15] The problem IVP(1) is equivalent to the first-order nonlinear impulsive integro-differential equation*

$$u(t) = (Au)(t) \tag{3}$$

where

$$\begin{aligned} &(Au)(t) = x_0 + tx_1 + \\ &\int_0^t (t-s)f(s, u(s), u'(s), Tu(s), (Su)(s))ds \\ &+ \sum_{0<t_k<t} I_k(u(t_k), u'(t_k)) \\ &+ \sum_{0<t_k<t} (t - t_k)\bar{I}_k(u(t_k), u'(t_k)). \end{aligned} \tag{4}$$

Lemma 7 *Let $V_1, V_2 \subset PC^1[J,E]$ be two countable subset satisfying $V_1 \subset \overline{co}(u_0 \cup V_2)$ for some $u_0 \in PC^1[J,E]$. Then*

$$\begin{gathered} V_{1i} \subset \overline{co}(\{u_{0i}\} \cup V_{2i}), i = 0, 1, 2, \cdots, m \\ V_{1i}' \subset \overline{co}(\{u_{0i}'\} \cup V_{2i}'), i = 0, 1, 2, \cdots, m \end{gathered}$$

and for any $t \in J_i$ $(i = 0, 1, 2, \cdots, m)$,

$$V_{1i}(t) \subset \overline{co}(\{u_{0i}(t)\} \cup V_{2i}(t)),$$
$$V'_{1i}(t) \subset \overline{co}(\{u'_{0i}(t)\} \cup V'_{2i}(t)).$$

Proof: $V_1, V_2 \subset PC^1[J, E]$ are countable imply that $V'_1, V'_2 \subset PC[J, E]$ are countable and $u_0 \in PC^1[J, E]$ imply that $u'_0 \in PC[J, E]$.

For any $x \in V'_{1i}$, there exists $u \in V_1$ such that $u'|_{J_i} = x$. From $u \in V_1 \subset \overline{co}(u_0 \cup V_2)$, there exist

$$u_n = \lambda_0^{(n)} u_0 + \sum_{k=1}^{m_n} \lambda_k^{(n)} v_k^{(n)} \in \overline{co}(\{u_0\} \cup V_2),$$
$$n = 1, 2, \cdots,$$

such that $\|u_n - u\|_{PC^1} \to 0 (n \to \infty)$, where

$$v_k^{(n)} \in V_2, k = 1, 2, \cdots, m_n,$$
$$\lambda_k^{(n)} \geq 0, k = 0, 1, \cdots, m_n,$$
$$\sum_{k=0}^{m_n} \lambda_k^{(n)} = 1.$$

Hence $\|u'_n|_{\bar{J}_k} - u'|_{\bar{J}_k}\|_C \to 0 (n \to \infty)$ and

$$u'_n|_{\bar{J}_k} = \lambda_0^{(n)} u'_0|_{\bar{J}_k} + \sum_{k=1}^{m_n} \lambda_k^{(n)} (v_k^{(n)})'|_{\bar{J}_k}$$
$$\in \overline{co}(\{u'_{0i}\} \cup V'_{2i}),\ n = 1, 2, \cdots,$$

so $x = u'|_{J_k} \in \overline{co}(\{u'_{0i}\} \cup V'_{2i}$, which imply $V'_{1i} \subset \overline{co}(\{u'_{0i}\} \cup V'_{2i})$ and $V'_{1i}(t) \subset \overline{co}(\{u'_{0i}(t)\} \cup V'_{2i}(t))$ for any $t \in J_i (i = 0, 1, 2, \cdots, m)$.

For the same reasons, we have $V_{1i} \subset \overline{co}(\{u_{0i}\} \cup V_{2i})$ and $V_{1i}(t) \subset \overline{co}(\{u_{0i}(t)\} \cup V_{2i}(t))$ for any $t \in J_i (i = 0, 1, 2, \cdots, m)$.

Lemma 8 *Let* $X \in R^{n \times n}$ *be a matrix with following form*

$$X = \begin{pmatrix} t_{11} & 0 & \ldots & 0 \\ t_{12} & t_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ t_{1n} & t_{2n} & \ldots & t_{nn} \end{pmatrix}.$$

Then for any $\varepsilon > 0$ *there exists a norm* $||\cdot||_{mon}$ *on* $R^{n \times n}$*, which is reduced by monotone vector norm, such that*

$$\|X\|_{mon} \leq \rho(X) + \varepsilon.$$

Proof: It is from the proof of theorem 3.7 of [16]. For any $\delta > 0$, let

$$D_\delta = \mathrm{diag}(1, \delta, \delta^2, \cdots, \delta^{n-1}),$$

then

$$D_\delta^{-1} X D_\delta = \begin{pmatrix} t_{11} & 0 & \ldots & 0 & 0 \\ \delta t_{12} & t_{22} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \delta^{n-1} t_{1n} & \delta^{n-2} t_{2n} & \ldots & \delta t_{n-1n} & t_{nn} \end{pmatrix}.$$

For any $\varepsilon > 0$, let $\delta > 0$ such that

$$\sum_{i=1}^{j-1} |\delta^{j-i} t_{ij}| < \varepsilon, \quad j = 2, 3, \cdots, n,$$

and define

$$\|G\|_{mon} = \|D_\delta^{-1} G D_\delta\|_\infty, \quad \forall G \in C^{n \times n}$$

then we can prove the function $\|\cdot\|_{mon}$ is an operator norm reduced by following vector norm

$$\|x\|_{D_\delta} = \|D_\delta^{-1} x\|_\infty, \quad x \in C^n \tag{5}$$

and

$$\|X\|_{mon} = \|D_\delta^{-1} X D_\delta\|_\infty \leq \rho(X) + \varepsilon.$$

It is easily to see that $\|\cdot\|_{D_\delta}$ is a monotone vector norm. Lemma 8 holds.

In what follows, set $u_k(t) = u(t)$ as $t \in \bar{J}_k$ for $u \in PC[J, E]$, i.e. $u_k = u|_{\bar{J}_k}$ (where $u_k(t_k) = u(t_k^+)$ at the left point of interval $\bar{J}_k$ and $u|_{\bar{J}_k}$ denote the section of u restricted on $\bar{J}_k$), then (3) can be recast into the following form

$$u_k(t) = (A_k u)(t),\ t \in \bar{J}_k,\ k = 1, 2, \ldots, m \tag{6}$$

where

$$(A_k u)(t) \triangleq x_0 + t x_1 +$$
$$\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} (t - s) \Gamma(i, s, u(s)) ds$$
$$+ \int_{t_k}^{t} (t - s) \Gamma(k, s, u(s)) ds$$
$$+ \sum_{i=1}^{k} I_i(u_{i-1}(t_i), u'_{i-1}(t_i))$$
$$+ \sum_{i=1}^{k} (t - t_i) \bar{I}_i(u_{i-1}(t_i), u'_{i-1}(t_i))$$

and

$$\Gamma(i, s, u(s)) = f(s, u_i(s), u'_i(s), (T_i u)(s), (Su)(s)),$$
$$(T_k u)(t) = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} K(t, r) u_i(r) dr + \int_{t_k}^{t} q(t, r) u_k(r) dr,\ \ t \in \bar{J}_k,$$
$$(Su)(s) = \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} h(s, r) u_i(r) dr.$$

3 Main Results

For convenience, we give the assumptions as follows.

(**H1**) For any $r > 0$, f is bounded and uniformly continuous on $J \times B_r \times B_r \times B_r \times B_r$, I_k and $\bar{I}_k$ are bounded on $B_r \times B_r$.

(**H2**) For any $r > 0$, there exist non-negative Lebesgue integrable functions $L_k \in L(J, R^+)(k = 1, 2, 3)$ such that for any bounded sets $B_i \subset E(i = 1, 2, 3, 4)$ and $t \in J$,

$$\begin{array}{c} \alpha(f(t, B_1, B_2, B_3, B_4)) \leq \sum_{i=1}^{4} L_i(t)\alpha(B_i), \\ \alpha(I_k(B_1, B_2)) \leq a_k(t)\alpha(B_1) + b_k\alpha(B_2), \\ \alpha(\bar{I}_k(B_1, B_2)) \leq \bar{a}_k(t)\alpha(B_1) + \bar{b}_k\alpha(B_2), \\ k = 1, 2, \cdots, m. \end{array} \tag{7}$$

(**H3**) $\beta = \limsup_{\|x\|+\|y\| \to +\infty} (\sup_{t \in J} \frac{f(t,x,y,Tx,Sx)}{\|x\|+\|y\|})$ is finite.

Let $q_0 = \max\{|q(t, s)| : (t, s) \in D\}, h_0 = \max\{|h(t, s)| : (t, s) \in J \times J\}$.

Theorem 9 *If the assumptions (H1)-(H3) are satisfied and the spectral radius $\rho(M_0^T M_0)$ of matrix $M_0^T M_0$ satisfies*

$$\rho(M_0^T M_0) < 1, \tag{8}$$

where

$$M_0 = \begin{pmatrix} \Delta_0 & d_1\mu_0 & \dots & d_m\mu_0 \\ \Delta_{012} & \Delta_1 & \dots & d_m\mu_1 \\ \dots & \dots & \dots & \dots \\ \Delta_{01m} & \Delta_{02m} & \dots & \Delta_m \end{pmatrix} \tag{9}$$

and

$$\begin{array}{l} \Delta_{ijk} = \delta_i + d_i\sigma_{ik} + \lambda_{jk}, \\ \Delta_k = \delta_k + d_k\sigma_{kk} \\ \delta_i = \max\{t_{i+1}, 1\} \int_{t_i}^{t_{i+1}} [L_1(s) + L_2(s)]ds, \\ \mu_k = h_0 \max\{t_{k+1}, 1\} \int_0^{t_{k+1}} L_4(s)ds, \\ \sigma_{ik} = \mu_k + q_0 \max\{t_{k+1}, 1\} \int_{t_i}^{t_{k+1}} L_3(s)ds, \\ \lambda_{ik} = \max\{(a_i + b_i) + (t_k - t_i)(\bar{a}_i + \bar{b}_i), \bar{a}_i + \bar{b}_i\}, \\ d_i = t_{i+1} - t_i, i = 0, 1, \dots, k, \\ j = i+1, i+2, \cdots, m, k = i+1, i+2, \cdots, m. \end{array} \tag{10}$$

Then IVP(1) has at least one solution $u \in PC^1(J, E) \cap C^2(J', E)$.

Proof: We divide the proof into two steps.

(i) Firstly, let

$$\Omega_0 = \left\{ x \in PC^1(J, E) : \begin{array}{c} \exists 0 \leq \lambda \leq 1 \text{ such that} \\ x = \lambda Ax \end{array} \right\} \tag{11}$$

We will prove that Ω_0 is bounded set in $PC^1(J, E)$.

From the hypothesis (H3), there exists the constant $\beta' > \beta$ and $d > 0$ such that

$$\begin{array}{c} \|f(t, u, v, Tu, (Su))\| \leq \beta'(\|u\| + \|v\|), \\ t \in J, \|u\| + \|v\| > d. \end{array}$$

Since f is bounded and continuous, we get

$$\begin{array}{r} \|f(t, u, v, Tu, (Su))\| \leq \beta'(\|u\| + \|v\|) + G, \\ t \in J, u, v \in E, \end{array} \tag{12}$$

where $G = \sup\{\|f(t, u, v, Tu, Su)\| : t \in J, \|u\| + \|v\| \leq d\} < \infty$.

On the other hand, $\forall u \in \Omega_0$, from (11) there exists $0 \leq \lambda \leq 1$ such that

$$u(t) = \lambda Au(t), \quad t \in J. \tag{13}$$

If $t \in J_0$, from (4), (12) and (13), we have

$$\begin{array}{l} \|u(t)\| \leq \|x_0\| + t_1\|x_1\| + \\ \beta' t_1 \int_0^t (\|u(s)\| + \|u'(s)\| + G)ds \end{array}$$

$$\|u'(t)\| \leq \|x_1\| + \beta' \int_0^t (\|u(s)\| + \|u'(s)\| + G)ds.$$

Let $m_0(t) = \max_{t \in \bar{J}_0}\{\|u(t)\|, \|u'(t)\|\}$, then we have

$$m_0(t) \leq C_0 + \gamma_0 \int_0^t m_0(s)ds$$

where $C_0 = \max\{\|x_0\| + t_1\|x_1\| + \beta' t_1^2 G, \|x_1\| + \beta' t_1 G\}$ and $\gamma_0 = 2\beta' \max\{t_1, 1\}$. From the Gronwall lemma, we get

$$\max_{t \in \bar{J}_0}\{\|u(t)\|, \|u'(t)\|\} = m_0(t) \leq C_0 e^{\gamma_0 t_1} = K_0, \quad t \in J_0.$$

And then $\|u\|_{C^1} \leq K_0$ for any $t \in J_0$. From the hypothesis (H1) there exists the constant $\beta_0 > 0$

$$\begin{array}{l} \|f(t, u, u', Tu, (Su))\| \leq \beta_0, \\ \|I_1(u, u')\| \leq \beta_0, \|\bar{I}_1(u, u')\| \leq \beta_0. \end{array} \tag{14}$$

If $t \in J_1 = (t_1, t_2]$, then (13) change into

$$\begin{array}{l} u(t) = \lambda(x_0 + tx_1) + \\ \lambda \int_0^t (t - s)f(s, u(s), u'(s), Tu(s), (Su)(s))ds + \\ \lambda[I_1(u(t_1), u'(t_1)) + (t - t_1)\bar{I}_1(u(t_1), u'(t_1))]. \end{array} \tag{15}$$

(12)(14)(15) imply that

$$\|u(t)\| \leq \|x_0\| + t_2\|x_1\| + t_1^2\beta_0 + \frac{(t_2 - t_1)^2}{2}\beta' G +$$

$$\beta_0 + (t_2 - t_1)\beta_0 + \beta' t \int_{t_1}^{t} (\|u(s)\| + \|u'(s)\|)ds$$

$$\|u'(t)\| \le \|x_1\| + t_1\beta_0 + (t_2 - t_1)\beta' G + \beta_0 + \beta' \int_{t_1}^{t} (\|u(s)\| + \|u'(s)\|)ds.$$

Let $C_1 = \max\{\|x_0\| + t_2\|x_1\| + (t_2 - t_1 + t_1^2 + 1)\beta_0 + \frac{(t_2-t_1)^2}{2}\beta' G, \|x_1\| + t_1\beta_0 + (t_2 - t_1)\beta' G + \beta_0\}, \gamma_1 = 2\beta' \max\{t_2, 1\}$, and therefore

$$m_1(t) \le C_1 + \gamma_1 \int_{t_1}^{t} m_1(s)ds$$

where $m_1(t) = \max_{t\in \bar{J}_1}\{\|u(t)\|, \|u'(t)\|\}$. And then

$$m_1(t) \le C_1 e^{\gamma_1(t_2-t_1)} = K_1, \quad t \in J_1.$$

Analogously, there exist $K_i > 0$ such that

$$m_i(t) \le K_i, \quad t \in J_i, (i = 2, 3, \cdots, m)$$

where $m_i(t) = \max_{t\in \bar{J}_i}\{\|u(t)\|, \|u'(t)\|\}$. Let $m(t) = \max_{t\in J}\{\|u(t)\|, \|u'(t)\|\}$ and $K = \max_{0\le i\le m} K_i$, then

$$m(t) \le \max_{0\le i\le m} m_i(t)) \le K, t \in J, \text{ i.e. } \|u\|_{PC^1} \le K$$

So Ω_0 is a bounded set on $PC^1[J, E]$.

ii) Let $R_0 > K$ and $\Omega = \{u \in PC^1(J, E) : \|u\| < R_0\}$, then Ω is open bounded set which satisfy that $x \ne \lambda Ax$ for $\forall\lambda \in [0, 1]$ and $x \in \partial\Omega$. As follows, we prove that $H \subset \bar{\Omega}$ is relative compact for any countable set $H \subset \overline{co}(\{\theta\} \bigcup (AH))$.

From (4) and (H1), we have that the operator $A : PC^1[J, E] \to PC^1[J, E]$ is bounded and continuous. And then $(AH) \subset PC^1[J, E]$ is bounded and $(AH), (AH)'$ are equicontinuous on J_k $(k = 0, 1, \cdot, m)$.

Since $H \subset \bar{\Omega}$ is countable, $H \subset \overline{co}(\{\theta\} \cup (AH))$ and $(AH), (AH)'$ are bounded and equicontinuous on J_k, then H, H' are bounded and equicontinuous. Thus all of H_i, H_i', A_iH and $(A_iH)'$ $(i = 0, 1, \cdots, m)$ are countable, bounded and equicontinuous on $\bar{J}_i$ and $H_i \subset \overline{co}(\{\theta_i\} \bigcup (A_iH))$ from lemma 7. From lemma 1, lemma 2 and (H1), we have $f(t, H, H', (TH), (SH)) \subset PC[J, E]$ is bounded and equicontinuous on each $J_k (k = 0, 1, \cdots, m)$. Hence from Lemma 7, Lemma 4 and (H2), we have

$$\begin{aligned}
&\alpha(H_0(t)) \le \alpha((A_0H)(t)) \\
&\le t \textstyle\int_0^t [L_1(s)\alpha(H_0(s)) + L_2(s)\alpha(H_0'(s))]ds+ \\
&t \textstyle\int_0^t [L_3(s)\alpha(T_0H(s)) + L_4(s)\alpha(SH(s))]ds \\
&\le t \textstyle\int_0^t ([L_1(s) + L_2(s) + t_1L_3(s)q_0]\alpha_1(H_0)]ds+ \\
&t \textstyle\int_0^t [L_4(s)h_0 \int_0^a \alpha(H(r))dr]ds \\
&\le t \textstyle\int_0^t \{[L_1(s) + L_2(s) + t_1L_3(s)q_0]\alpha_1(H_0)]ds+ \\
&t \textstyle\int_0^t [h_0L_4(s) \sum_{i=0}^{m} (t_{i+1} - t_i)\alpha_1(H_i)\}ds,
\end{aligned}$$

and

$$\begin{aligned}
&\alpha(H_0'(t)) \le \alpha((A_0H)'(t)) \\
&\le \textstyle\int_0^t [L_1(s)\alpha(H_0(s)) + L_2(s)\alpha(H_0'(s))]ds+ \\
&\textstyle\int_0^t [L_3(s)\alpha(T_0H(s)) + L_4(s)\alpha(SH(s))]ds \\
&\le \textstyle\int_0^t \{[L_1(s) + L_2(s) + t_1L_3(s)q_0]\alpha_1(H_0)]ds+ \\
&\textstyle\int_0^t [h_0L_4(s) \sum_{i=0}^{m} (t_{i+1} - t_i)\alpha_1(H_i)\}ds.
\end{aligned}$$

So from lemma 3 and (10),

$$\begin{aligned}
\alpha_1(H_0) &= \max\{\sup_{t\in J_0} \alpha(H(t)), \sup_{t\in J_0} \alpha(H'(t))\} \\
&\le (\delta_0 + d_0\sigma_{00})\alpha_1(H_0) + \sum_{i=1}^{m} d_i\mu_0\alpha_1(H_i).
\end{aligned} \tag{16}$$

For $t \in \bar{J}_1$, we have

$$\begin{aligned}
&\alpha(H_1(t)) \le \alpha((A_1H)(t)) \\
&\le t \textstyle\int_0^t [L_1(s)\alpha(H(s)) + L_2(s)\alpha(H'(s))]ds+ \\
&t \textstyle\int_0^t [L_3(s)\alpha(T_1H(s)) + L_4(s)\alpha(SH(s))]ds \\
&+a_1\alpha(H_0(t_1)) + b_1\alpha(H_0'(t_1))+ \\
&(a - t_1)[\bar{a}_1\alpha(H_0(t_1)) + \bar{b}_1\alpha(H_0'(t_1))]
\end{aligned}$$

$$\begin{aligned}
&\le t \textstyle\int_0^{t_1} [L_1(s) + L_2(s) + t_1L_3(s)q_0]\alpha_1(H_0)ds+ \\
&t \textstyle\int_0^{t_1} [h_0L_4(s) \sum_{i=0}^{m} (t_{i+1} - t_i)\alpha_1(H_i)]ds \\
&+t \textstyle\int_{t_1}^{t} [L_1(s) + L_2(s)]\alpha_1(H_1)ds+ \\
&t \textstyle\int_{t_1}^{t} (t_2 - t_1)L_3(s)q_0\alpha_1(H_1)ds+ \\
&t \textstyle\int_{t_1}^{t} [t_1q_0L_3(s)\alpha_1(H_0)]ds+ \\
&t \textstyle\int_{t_1}^{t} h_0L_4(s) \sum_{i=0}^{m} (t_{i+1} - t_i)\alpha_1(H_i)ds+ \\
&((a_1 + b_1) + (t - t_1)(\bar{a}_1 + \bar{b}_1))\alpha_1(H_0)
\end{aligned}$$

and

$$\begin{aligned}
&\alpha(H_1'(t)) \le \alpha((A_1H)'(t)) \\
&\le \textstyle\int_0^{t_1} [L_1(s) + L_2(s) + t_1L_3(s)q_0]\alpha_1(H_0)ds+ \\
&\textstyle\int_0^{t_1} [h_0L_4(s) \sum_{i=0}^{m} (t_{i+1} - t_i)\alpha_1(H_i)]ds+ \\
&\textstyle\int_{t_1}^{t} [(L_1(s) + L_2(s) + (t_2 - t_1)L_3(s)q_0)\alpha_1(H_1)]ds+ \\
&\textstyle\int_{t_1}^{t} [t_1q_0L_3(s)\alpha_1(H_0)]ds+ \\
&\textstyle\int_{t_1}^{t} h_0L_4(s) \sum_{i=0}^{m} (t_{i+1} - t_i)\alpha_1(H_i)ds+ \\
&(\bar{a}_1 + \bar{b}_1)\alpha_1(H_0).
\end{aligned}$$

then from lemma 3 and (10), we get

$$\begin{aligned}
\alpha_1(H_1) &= \max\{\sup_{t\in J_1} \alpha(H(t)), \sup_{t\in J_1} \alpha(H'(t))\} \\
&\le \sum_{i=0}^{1} (\delta_i + d_i\sigma_{i1})\alpha_1(H_i)+ \\
&\quad \lambda_{12}\alpha_1(H_0) + \sum_{i=2}^{m} d_i\mu_1\alpha_1(H_i).
\end{aligned} \tag{17}$$

In general, for $t \in \bar{J}_k (k = 2, 3, \dots, m)$, we have

$$\begin{aligned}
&\alpha(H_k(t)) \le \alpha((A_k H)(t)) \\
&\le t \int_0^t [L_1(s)\alpha(H(s)) + L_2(s)\alpha(H'(s))]ds + \\
&t \int_0^t [L_3(s)\alpha(T_k H(s)) + L_4(s)\alpha(SH(s))]ds + \\
&\sum_{i=1}^{k} [(a_i + b_i) + (t - t_i)(\bar{a}_i + \bar{b}_i)]\alpha_1(H_i) \\
&\le t \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} [(L_1(s) + L_2(s))\alpha_1(H_i)]ds + \\
&t \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} [k_0 \sum_{j=0}^{i} (t_{j+1} - t_j) L_3(s)\alpha_1(H_j)]ds \\
&+ h_0 t \int_0^{t_{k+1}} L_4(s) \sum_{j=0}^{m} (t_{j+1} - t_j)\alpha_1(H_j)ds \\
&+ \sum_{i=1}^{k} [(a_i + b_i) + (t_k - t_i)(\bar{a}_i + \bar{b}_i)]\alpha_1(H_{i-1}),
\end{aligned}$$

and

$$\begin{aligned}
&\alpha(H_k'(t)) \le \alpha((A_k H)'(t)) \\
&\le \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} [L_1(s) + L_2(s)]ds\alpha_1(H_i) + \\
&\sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} [k_0 \sum_{i=0}^{k} \int_{t_i}^{t_{k+1}} (t_{i+1} - t_i) L_3(s)\alpha_1(H_i)ds + \\
&\quad h_0 \int_0^{t_{k+1}} L_4(s) \sum_{j=0}^{m} (t_{j+1} - t_j)\alpha_1(H_j)ds + \\
&\quad \sum_{i=1}^{k} (\bar{a}_i + \bar{b}_i)\alpha_1(H_{i-1}).
\end{aligned}$$

Thus for $k = 2, 3, \cdots, m$,

$$\begin{aligned}
\alpha_1(H_k) &= \max\{\sup_{t\in J_k} \alpha(H(t)), \sup_{t\in J_k} \alpha(H'(t))\} \\
&\le \sum_{i=0}^{k} (\delta_i + d_i\sigma_{ik})\alpha_1(H_i) + \\
&\quad \sum_{i=k+1}^{m} d_i\mu_k\alpha_1(H_i) + \sum_{i=1}^{k} \lambda_i\alpha_1(H_{i-1}),
\end{aligned} \tag{18}$$

where $\delta_i, \sigma_{ik}, \mu_i, \lambda_i$ and d_i are defined by (10). Hence from (16) (17) (18), we obtain

$$\begin{pmatrix} \alpha_1(H_0) \\ \alpha_1(H_1) \\ \cdots \\ \alpha_1(H_m) \end{pmatrix} \le M_0 \begin{pmatrix} \alpha_1(H_0) \\ \alpha_1(H_1) \\ \cdots \\ \alpha_1(H_m) \end{pmatrix} \tag{19}$$

where M_0 is defined by (9). Let

$$\begin{pmatrix} y_0 \\ y_1 \\ \cdots \\ y_m \end{pmatrix} = M_0 \begin{pmatrix} \alpha_1(H_0) \\ \alpha_1(H_1) \\ \cdots \\ \alpha_1(H_m) \end{pmatrix}. \tag{20}$$

From (19) and (20), we have

$$\begin{aligned}
&[\alpha_1(H_0)]^2 + [\alpha_1(H_1)]^2 + \cdots + [\alpha_1(H_m)]^2 \\
&\le y_0^2 + y_1^2 + \cdots + y_m^2.
\end{aligned} \tag{21}$$

From the definition and the properties of the 2-norm $\|\cdot\|_2$, we have

$$\begin{aligned}
\left\| \begin{matrix} \alpha_1(H_0) \\ \alpha_1(H_1) \\ \cdots \\ \alpha_1(H_m) \end{matrix} \right\|_2 &\le \left\| \begin{matrix} y_0 \\ y_1 \\ \cdots \\ y_m \end{matrix} \right\|_2 \le \|M_0\|_2 \left\| \begin{matrix} \alpha_1(H_0) \\ \alpha_1(H_1) \\ \cdots \\ \alpha_1(H_m) \end{matrix} \right\|_2 \\
&= \sqrt{\rho(M_0^T M_0)} \left\| \begin{matrix} \alpha_1(H_0) \\ \alpha_1(H_1) \\ \cdots \\ \alpha_1(H_m) \end{matrix} \right\|_2
\end{aligned} \tag{22}$$

where $\|M_0\|_2$ denote 2-norm of the matrix M_0. (8) and (22) imply $\alpha_1(H_k) = 0$ $(k = 0, 1, 2, \dots, m)$, and then from lemma 3,

$$\begin{aligned}
&\alpha_2(H) = \max\{\sup_{t\in J} \alpha(H(t)), \sup_{t\in J} \alpha(H'(t))\} \\
\le\ &\max\{\max_{0\le k\le m} \sup_{t\in J_k} \alpha(H_i(t)), \max_{0\le k\le m} \sup_{t\in J_k} \alpha(H_i'(t))\} \\
=\ &0,
\end{aligned}$$

i.e. H is relative compact. So the operator A defined by (3) has at least one fixed point in Ω from lemma 5. Thus IVP(1) has at least one solution $u(t) \in PC^1(J, E) \cap C^2(J', E)$ from lemma 6.

If we replace the norm in (22) into anyone of the others, which is reduced by monotone vector norm, we obtain the following conclusions.

Theorem 10 *Let the matrix M_0 be defined by (9) and the matrix norm $\|\cdot\|_{mon}$ be an operator norm reduced by some monotone vector norm. If the assumptions (H1)-(H3) hold and*

$$\|M_0\|_{mon} < 1, \tag{23}$$

then IVP(1) has at least one solution $u \in PC^1(J, E) \cap C^2(J', E)$.

Proof: For the proof of theorem 9, we replace the (21) into that

$$0 \le \begin{pmatrix} \alpha(H_0) \\ \alpha(H_1) \\ \cdots \\ \alpha(H_m) \end{pmatrix} \le \begin{pmatrix} y_0 \\ y_1 \\ \cdots \\ y_m \end{pmatrix}$$

imply

$$\begin{aligned}
&\|(\alpha(H_0), \alpha(H_1), \dots, \alpha(H_m))^T\|_{mon} \\
&\le \|(y_0, y_1, \dots, y_m)^T\|_{mon},
\end{aligned}$$

then we get the conclusion of theorem 10.

Remark 1. Since $\delta_k, \mu_k, \sigma_{ik}, \lambda_{ik}, d_k$ $(i = 0, 1, \dots, k, k = 0, 1, \dots, m)$ are nonnegative and $t_k \le a$, $\sigma_{ik} \le \sigma_{im}, \lambda_{ik} \le \lambda_{im}$ $(i < k, k = 1, 2, \dots, m)$, then

$$\delta_0+\delta_1+\cdots+\delta_m \le \max\{a,1\} \int_0^a (L_1(s)+L_2(s))ds,$$

$$d_0\sigma_{0m} + d_1\sigma_{1m} + \cdots + d_1\sigma_{mm} \le a\max\{a,1\} \int_0^a [q_0L_3(s) + h_0L_4(s)]ds.$$

Thus

$$\|A\|_\infty \le \max\{a,1\} \int_0^a L(s)ds + \sum_{i=0}^{m} \lambda_{im},$$

where $L(s) = L_1(s)+L_2(s)+aq_0L_3(s)+ah_0L_4(s)$. We can see that our conclusions imply those of [4] since the vector norm $\|\cdot\|_\infty$ reducing the matrix norm $\|\cdot\|_\infty$ is monotone.

Remark 2. Most of those conclusions in theorem 10 are new, since the compactness-type conditions (H2) involve both the derivative x' and the linear integral operator Su. Usually, it will be convenient that we verify (23) by using the operator norms $\|\cdot\|_1, \|\cdot\|_2$ and $\|\cdot\|_\infty$. Only if one of the three norms satisfy (23), we can obtain the conclusion of theorem 10.

Specially, let $L_4(t) = 0$, then we get the following conclusion.

Theorem 11 *Let $L_4(t) = 0$. If the assumptions (H1)-(H3) are satisfied and*

$$\max_{0\le k\le m} \{\delta_k + d_k\sigma_{kk}\} < 1. \quad (24)$$

Then IVP(1) has at least one solution $u \in PC^1(J,E) \cap C^2(J',E)$.

Proof: $L_4(t) = 0$, then we get $\mu_k = 0(k = 0, 1, 2, \dots, m)$. then the matrix M_0 defined by (9) changes into

$$M_0 = \begin{pmatrix} \Delta_0 & 0 & \dots & 0 \\ \Delta_{012} & \Delta_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \Delta_{01m} & \Delta_{12m} & \dots & \Delta_m \end{pmatrix} \quad (25)$$

where Δ_{ijk} and Δ_k are defined by (10) and σ_{ik} $(i = 0, 1, 2, \cdots, m, k > i)$change into

$$\sigma_{ik} = q_0 \max\{t_{k+1}, 1\} \int_{t_i}^{t_{k+1}} L_3(s)ds. \quad (26)$$

Apparently, the eigenvalues of M_0 are $\delta_k+d_k\sigma_{kk}(k = 0, 1, 2, \dots, m)$, and then

$$\rho(M_0) = \max_{0\le k\le m} \{\delta_k + d_k\sigma_{kk}\} < 1. \quad (27)$$

Let $\varepsilon < 1 - \rho(M_0)$, then from Lemma 8, there exist a operator norm $\|\cdot\|$, which is reduced by monotone vector norm, such that

$$\|M_0\| \le \rho(M_0) + \varepsilon < 1. \quad (28)$$

Thus theorem 11 is valid from the theorem 10.

Remark 3. In some sense, theorem 11 indicate that the result with impulses t_i is equivalent to one without impulse defined in every subinterval $[t_i, t_{i+1}](i = 0, 1, \dots, m)$ only if we ignore the influence of the operator S. This is fair and reasonable. Thus our results improve and generalize ones of the paper [1].

4 An Example

Consider the IVP of infinity systems for nonlinear impulsive integro-differential equation

$$\begin{cases} u''_n = \frac{t}{2}(t + u_n) + \frac{3t}{5}u'_n + t^2 \int_0^t e^{-ts}u_n(s)ds + \\ \qquad \frac{t^3}{18}\sin^3 \int_0^1 \frac{u_n(s)}{1+t+s}ds, \ \ t \in [0,1], t \ne \frac{1}{2} \\ \triangle u_n|_{t=\frac{1}{2}} = \frac{1}{10}\cos^2 u_n(\frac{1}{2}) + \frac{1}{5}u'_n(\frac{1}{2}), \\ \triangle u'_n|_{t=\frac{1}{2}} = \frac{1}{4}u_n(\frac{1}{2}) + \frac{1}{4}u'_n(\frac{1}{2}) \\ u_n(0) = 0, u'_n(0) = \frac{1}{n}, \quad n = 1, 2, 3, \cdots. \end{cases} \quad (29)$$

Then IVP(29) has at least one solution $u^*(t) = (u_1^*(t), u_2^*(t), \cdots, u_n^*(t), \cdots)$ which is continuously differentiable twice on $[0, \frac{1}{2}] \cup (\frac{1}{2}, 1]$ and $u_n^*(t) \to 0(n \to \infty)$ for any $t \in [0,1]$.

Proof. By all appearances, $u_n(t) \equiv 0$ is not a solution of IVP(29). Let $\|u\| = \sup_n |u_n|$ is a norm of $E = \{u = (u_1, u_2, \cdots, u_n, \cdots)|u_n \to 0\}$, then we know that IVP(29) can be regard as a form of IVP(1) in E. In this situation, $k(t,s) = e^{-ts}$, $h(t,s) = (1 + t + s)^{-1}$, $x = (x_1, x_2, \cdots, x_n, \cdots)$, $y = (y_1, y_2, \cdots, y_n, \cdots)$, $z = (z_1, z_2, \cdots, z_n, \cdots)$, $w = (w_1, w_2, \cdots, w_n, \cdots)$, $f = (f_1, f_2, \cdots, f_n, \cdots)$, $I_1 = (I_{11}, I_{12}, \cdots, I_{1n}, \cdots)$, $\bar{I}_1 = (\bar{I}_{11}, \bar{I}_{12}, \cdots, \bar{I}_{1n}, \cdots)$ and

$$f_n(t,x,y,z,w) = \frac{t}{2}(t+x_n)+\frac{3t}{5}y_n+t^2z_n+\frac{t^3}{18}\sin^3 w_n$$

$$I_{1n}(x,y) = \frac{1}{10}\cos^2 x_n+\frac{1}{5}y_n, \ \bar{I}_{1n}(x,y) = \frac{1}{4}x_n+\frac{1}{4}y_n$$

where $m = 1, t_1 = \frac{1}{2}$, the assumption (H1) holds and

$$\|f_n(t,x,y,Tx,Sx)\| \le \frac{1}{2}\|x\| + \frac{3}{5}\|y\| + \|Tx\| + \frac{5}{9} \le \frac{3}{2}\|x\| + \frac{3}{5}\|y\| + \frac{5}{9}.$$

i.e. the assumption (H3) holds too. On the other hand, for any bounded set B_i $(i = 1, 2, 3, 4)$, since

$$\alpha(f(t, B_1, B_2, B_3, B_4) \le \frac{t}{2}\alpha(B_1) + \frac{3t}{5}\alpha(B_2) + t^2\alpha(B_3) + \frac{t^3}{6}\alpha(B_4)$$

$$\alpha(I_1(B_1,B_2)) = \tfrac{1}{5}\alpha(B_1) + \tfrac{1}{5}\alpha(B_2),$$
$$\alpha(\bar{I}_1(B_1,B_2)) = \tfrac{1}{4}\alpha(B_1) + \tfrac{1}{4}\alpha(B_2)$$

the assumption (H2) hold and $L_1(t) = \frac{t}{2}, L_2(t) = \frac{3t}{5}, L_3(t) = t^2, L_4(t) = \frac{t^3}{6}, a_1 = b_1 = \frac{1}{5}, \bar{a}_1 = \bar{b}_1 = \frac{1}{4}$. So $\delta_0 = \frac{11}{80}, \delta_1 = \frac{33}{80}, \mu_0 = \frac{1}{384}, \mu_1 = \frac{1}{24}, \sigma_{00} = \frac{17}{384}, \sigma_{01} = \frac{1}{12}, \sigma_{11} = \frac{1}{3}, \lambda_1 = \frac{13}{20}$. Thus

$$M_0 = \frac{1}{3840}\begin{pmatrix} 613 & 5 \\ 3184 & 2224 \end{pmatrix} = \begin{pmatrix} \frac{613}{3840} & \frac{1}{768} \\ \frac{199}{240} & \frac{139}{240} \end{pmatrix}.$$

Calculating the row norm of the matrix M, we have

$$\|M_0\|_1 = \max\{\frac{3797}{3840}, \frac{743}{1280}\} < 1.$$

So the formula (23) holds. Thus the conclusions of our example hold from theorem 10.

Remark 4. Farther calculating two rest common norms in the example, we have

$$\rho(M_0^T M_0) > 1.04 > 1 \text{ and } \|M_0\|_\infty = \frac{169}{120} > 1.$$

In addition, set

$$f_n(t,x,y,z,w) = \frac{t}{2}(t+x_n+y_n) + \frac{t^2}{8}z_n + \frac{t^3}{24}\sin^3 w_n$$

$$I_{1n}(x,y) = \frac{1}{8}\cos^2 x_n + \frac{1}{4}y_n,\ \ \bar{I}_{1n}(x,y) = \frac{1}{4}x_n + \frac{1}{4}y_n,$$

we have

$$\rho(M_0^T M_0) < 0.98 < 1,\ \ \|M_0\|_1 = \frac{3139}{3072} > 1$$

and

$$\|M_0\|_\infty = \frac{125}{96} > 1.$$

As well as we set

$$f_n(t,x,y,z,w) = \frac{t}{8}(t+x_n) + \frac{t}{4}y_n + \frac{4t^2}{3}z_n + \frac{t^3}{3}\sin^3 w_n$$

$$I_{1n}(x,y) = \frac{1}{16}\cos^2 x_n + \frac{1}{8}y_n,\ \ \bar{I}_{1n}(x,y) = \frac{1}{8}x_n + \frac{1}{8}y_n$$

and $t_1 = \frac{7}{8}$, then

$$\rho(M_0^T M_0) > 1.1 > 1,\ \ \|M_0\|_1 = \frac{1694431}{1179648} > 1$$

and

$$\|M_0\|_\infty = \frac{4597}{4608} < 1.$$

So each norm of the matrix M_0 in theorem 10 is corresponding to one of the conclusions. Thus theorem 10 include many different results.

5 An Annotation

This idea, that differential equations with impulses are transferred into differential system and are studied, can be used to the following boundary value problem (BVP) for second order impulsive integro-differential equations of mixed type in a real Banach space E

$$\begin{cases} u'' = f(t,u,u',Tu,Su) \quad \forall t \in J = [0,1], t \neq t_k \\ \triangle u|_{t=t_k} = I_k(u(t_k), u'(t_k)) \\ \triangle u'|_{t=t_k} = \bar{I}_k(u(t_k), u'(t_k)) \\ u(0) = \tilde{x}_0, u(1) = \tilde{x}_1 \quad (k = 1,2,\cdots,m), \end{cases} \tag{30}$$

where the symbols is identical with that of IVP(1). In this section, we use the following assumption:

(**H4**). There exist non-negative Lebesgue integrable functions $L_i \in L[J, R^+] (i = 1,2,3,4)$ such that

$$\begin{aligned} &\|f(t,x,y,u,v) - f(t,\bar{x},\bar{y},\bar{u},\bar{v})\|_E \\ &\le L_1(t)\|x-\bar{x}\|_E + L_2(t)\|y-\bar{y}\|_E + \\ &L_3(t)\|u-\bar{u}\|_E + L_4(t)\|v-\bar{v}\|_E, \\ &\quad t \in J, x,\bar{x},y,\bar{y},u,\bar{u},v,\bar{v} \in E, \\ &\|I_k(x,y) - I_k(\bar{x},\bar{y})\|_E \le a_k\|x-\bar{x}\|_E + b_k\|y-\bar{y}\|_E, \\ &\quad x,\bar{x},y,\bar{y} \in E \ \ (k = 1,2,\ldots,m) \end{aligned}$$

and

$$\begin{aligned} &\|\bar{I}_k(x,y) - \bar{I}_k(\bar{x},\bar{y})\|_E \le \bar{a}_k\|x-\bar{x}\|_E + \bar{b}_k\|y-\bar{y}\|_E, \\ &\qquad x,\bar{x},y,\bar{y} \in E \ \ (k = 1,2,\ldots,m). \end{aligned}$$

Theorem 12 *If the assumption (H4) holds and the spectral radius of matrix $M_1^T M_1$ satisfy*

$$\rho(M_1^T M_1) < 1, \tag{31}$$

where

$$M_1 = \begin{pmatrix} \delta_0+\mu_1 & \delta_1+\mu_2 & \cdots & \delta_m \\ \delta_0+\mu_1+\lambda_1 & \delta_1+\mu_2 & \cdots & \delta_m \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots \\ \delta_0+\mu_1+\lambda_1 & \delta_1+\mu_2+\lambda_2 & \cdots & \delta_m \end{pmatrix} \tag{32}$$

and

$$\begin{aligned} &\mu_k = (a_k+b_k) + (1-t_k)(\bar{a}_k+\bar{b}_k), \\ &\lambda_k = \max\{\mu_k, (\bar{a}_k+\bar{b}_k)\}, \\ &\qquad k = 1,2,\ldots,m, \\ &\delta_i = (t_{i+1}-t_i)\int_{t_i}^1 (L_3(s)K_i + L_4(s)H_i)ds + \\ &\qquad (t_{i+1}-t_i)\int_0^{t_i} L_4(s)H_i ds + \\ &\qquad \int_{t_i}^{t_{i+1}}[L_1(s)+L_2(s)]ds, \\ &\qquad i = 0,1,2,\ldots,m, \end{aligned} \tag{33}$$

then BVP(30) has an unique solution $u \in PC^1[J,E] \cap C^2[J',E]$. *Moreover, for any* $z_0 \in PC^1[J,E]$*, the iterative sequence defined by*

$$\begin{gathered} z_n(t) = \varphi(t) + \int_0^1 G(t,s)F(s,z_{n-1}(s))ds + \\ t\sum_{k=1}^{m}[Q_k(z_{n-1}(t_k)) + (1-t_k)\bar{Q}_k(z_{n-1}(t_k))] + \\ \sum_{0<t_k<t}[Q_k(z_{n-1}(t_k)) + (t-t_k)\bar{Q}_k(z_{n-1}(t_k))], \\ n = 1,2,3,\ldots \end{gathered} \tag{34}$$

converges to $u(t)$ *uniformly on* $t \in J$*, and the sequence*

$$\begin{gathered} z'_n(t) = \varphi'(t) + \int_0^1 G'_t(t,s)F(s,z_{n-1}(s))ds + \\ \sum_{k=1}^{m}[Q_k(z_{n-1}(t_k)) + (1-t_k)\bar{Q}_k(z_{n-1}(t_k))] + \\ \sum_{0<t_k<t}\bar{Q}_k(z_{n-1}(t_k)),\;\; n = 1,2,3,\ldots \end{gathered} \tag{35}$$

converges to $u'(t)$ *uniformly on* $t \in J$*. Here*

$$\begin{gathered} F(s,u(s)) = f(s,u(s),u'(s),Tu(s),Su(s)), \\ Q_k(u(t_k)) = I_k(u(t_k),u'(t_k)), \\ \bar{Q}_k(u(t_k)) = \bar{I}_k(u(t_k),u'(t_k)). \end{gathered}$$

Proof: At first, (30) is equivalent to the following first-order nonlinear impulsive integro-differential equation

$$\begin{gathered} u(t) = \varphi(t) + \int_0^1 G(t,s)F(s,u(s))ds - \\ t\sum_{k=1}^{m}[Q_k(u(t_k)) + (1-t_k)\bar{Q}_k(u(t_k))] + \\ \sum_{0<t_k<t}[Q_k(u(t_k)) + (t-t_k)\bar{Q}_k(u(t_k))] \end{gathered} \tag{36}$$

where

$$G(t,s) = \begin{cases} s(t-1) & 0 \le s < t \\ (s-1)t & t \le s < 1 \end{cases},$$

and

$$\varphi(t) = \tilde{x}_0 + t(\tilde{x}_1 - \tilde{x}_0).$$

For any $x, y \in PC^1[J,E]$, let $x_k(t) = x(t), y_k(t) = y(t)$ as $t \in \bar{J}_k$ ($k = 0,1,\ldots,m$), where $x_k(t_k) = x(t_k^+), y_k(t_k) = y(t_k^+)$ at the left point of each subinterval $\bar{J}_k$ ($k = 1,2,\ldots,m$). Since $\max\limits_{t,s\in J}\{|G(t,s)|,|G'(t,s)|\} \le 1$, from (H1) and (36), we have

$$\begin{aligned} &\|(A_0x)(t) - (A_0y)(t)\|_E \\ \le\;& \sum_{i=0}^{m}\int_{t_i}^{t_{i+1}} |G_0(t,s)|[L_1(s)\|x_i(s) - y_i(s)\|_E \\ &+L_2(s)\|x'_i(s) - y'_i(s)\|_E \\ &+L_3(s)\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}} K_j(s,r)\|x_j(r) - y_j(r)\|_E dr \\ &+L_4(s)\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}} H_j(s,r)\|x_j(r) - y_j(r)\|_E dr]ds \\ +\;& t\sum_{i=1}^{m}[a_i\|x_{i-1}(t_i) - y_{i-1}(t_i)\|_E \\ &+b_i\|x'_{i-1}(t_i) - y'_{i-1}(t_i)\|_E \\ &+(1-t_i)(\bar{a}_i\|x_{i-1}(t_i) - y_{i-1}(t_i)\|_E \\ &+\bar{b}_i\|x'_{i-1}(t_i) - y'_{i-1}(t_i)\|_E)] \\ \le\;& \sum_{i=0}^{m}\int_{t_i}^{t_{i+1}} \{[L_1(s) + L_2(s)]\|x_i - y_i\|_{C^1_{J_i}} \\ &+\sum_{j=0}^{i}(t_{j+1} - t_j)L_3(s)K_j\|x_j - y_j\|_{C^1_{J_j}} \\ &+\sum_{j=0}^{m}(t_{j+1} - t_j)L_4(s)H_j\|x_j - y_j\|_{C^1_{J_j}}\}ds \\ +\;& \sum_{i=1}^{m}[a_i + b_i + (1-t_i)(\bar{a}_i + \bar{b}_i)]\|x_{i-1} - y_{i-1}\|_{C^1_{J_{i-1}}} \end{aligned}$$

and

$$\begin{aligned} &\|(A_0x)'(t) - (A_0y)'(t)\|_E \\ \le& \sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\{[L_1(s) + L_2(s)]\|x_i - y_i\|_{C^1_{J_i}} + \\ &\sum_{j=0}^{i}(t_{j+1} - t_j)L_3(s)K_j\|x_j - y_j\|_{C^1_{J_j}} + \\ &\sum_{j=0}^{m}(t_{j+1} - t_j)L_4(s)H_j\|x_j - y_j\|_{C^1_{J_j}}\}ds + \\ &\sum_{i=1}^{m}[a_i + b_i + (1-t_i)(\bar{a}_i + \bar{b}_i)]\|x_{i-1} - y_{i-1}\|_{C^1_{J_{i-1}}}. \end{aligned}$$

So we have

$$\begin{gathered} \|(A_0x) - (A_0y)\|_{C^1_{J_0}} \le \sum_{i=0}^{m}\delta_i\|x_i - y_i\|_{C^1_{J_i}} + \\ \sum_{i=1}^{m}\mu_i\|x_{i-1} - y_{i-1}\|_{C^1_{J_{i-1}}}. \end{gathered} \tag{37}$$

Let $Q(x,y) = \sum\limits_{i=0}^{m}[\int_{t_i}^{t_{i+1}}(L_1(s) + L_2(s))\|x_i - y_i\|_{C^1_{J_i}} + \sum\limits_{j=0}^{i}(t_{j+1} - t_j)L_3(s)K_j\|x_j - y_j\|_{C^1_{J_j}} +$

$\sum_{j=0}^{m}(t_{j+1}-t_j)L_4(s)H_j\|x_j-y_j\|_{C^1_{J_j}}]ds+\sum_{i=1}^{m}[a_i+b_i+(1-t_i)(\bar{a}_i+\bar{b}_i)]\|x_{i-1}-y_{i-1}\|_{C^1_{J_{i-1}}}$. We obtain

$$\|(A_1x)(t)-(A_1y)(t)\|_E\le Q(x,y)+ [(a_1+b_1)+(1-t_1)(\bar{a}_1+\bar{b}_1)]\|x_0-y_0\|_{C^1_{J_0}},$$

$$\|(A_1x)'(t)-(A_1y)'(t)\|\le Q(x,y)+ (\bar{a}_1+\bar{b}_1)\|x_0-y_0\|_{C^1_{J_0}}.$$

and then

$$\|(A_1x)-(A_1y)\|_{C^1_{J_1}}\le Q(x,y)+\lambda_1\|x_0-y_0\|_{C^1_{J_0}}. \tag{38}$$

In general, we obtain that

$$\|(A_kx_k)-(A_ky_k)\|_{C^1_{J_k}}\le Q(x,y)+ \sum_{i=1}^{k}\lambda_i\|x_{i-1}-y_{i-1}\|_{C^1_{J_{i-1}}}, \quad k=2,3,\dots,m. \tag{39}$$

So from (37),(38) and (39), we get

$$\begin{pmatrix}\|A_0x_0-A_0y_0\|_{C^1_{J_0}}\\ \|A_1x_1-A_1y_1\|_{C^1_{J_1}}\\ \dots\\ \|A_mx_m-A_my_m\|_{C^1_{J_m}}\end{pmatrix} \le M_1\begin{pmatrix}\|x_0-y_0\|_{C^1_{J_0}}\\ \|x_1-y_1\|_{C^1_{J_1}}\\ \dots\\ \|x_m-y_m\|_{C^1_{J_m}}\end{pmatrix}, \tag{40}$$

where M_1 is defined by (32)(Here and in what follows the vector inequality $x\le y$ denotes that all of the corresponding components of vectors satisfy $x_i\le y_i$ $(i=0,1,\dots,m)$). Then we have

$$\left\|\begin{matrix}\|A_0x_0-A_0y_0\|_{C^1_{J_0}}\\ \|A_1x_1-A_1y_1\|_{C^1_{J_1}}\\ \dots\\ \|A_mx_m-A_my_m\|_{C^1_{J_m}}\end{matrix}\right\|_2 \le\sqrt{\rho(M_1^TM_1)}\left\|\begin{matrix}\|x_0-y_0\|_{C^1_{J_0}}\\ \|x_1-y_1\|_{C^1_{J_1}}\\ \dots\\ \|x_m-y_m\|_{C^1_{J_m}}\end{matrix}\right\|_2. \tag{41}$$

From (31),(41) and the Banach fixed point theorem, the operator $A=(A_0,A_1,\dots,A_m)$ has an unique fixed point. Thus BVP(30) has an unique solution $u(t)\in PC^1[J,E]\cap C^2[J',E]$.

Moreover, if $u(t)$ is the unique solution of BVP(30) and $z_n(t)$ is defined by (34), let $u_k(t)=u(t),t\in J_k$ and $z_{n,k}(t)=z_n(t),t\in J_k$ $(k=0,1,\dots,m)$. Similar to the reduction process of (40), we can get

$$\begin{pmatrix}\|A_0z_{0,0}-A_0u_0\|_{C^1_{J_0}}\\ \|A_1z_{0,1}-A_1u_1\|_{C^1_{J_1}}\\ \dots\\ \|A_mz_{0,m}-A_mu_m\|_{C^1_{J_m}}\end{pmatrix} \le M_1\begin{pmatrix}\|z_{0,0}-u_0\|_{C^1_{J_0}}\\ \|z_{0,1}-u_1\|_{C^1_{J_1}}\\ \dots\\ \|z_{0,m}-u_m\|_{C^1_{J_m}}\end{pmatrix}.$$

Considering that the components of A are nonnegative, from mathematical induction, it is easy to obtain that

$$\begin{pmatrix}\|A_0z_{n,0}-A_0u_0\|_{C^1_{J_0}}\\ \|A_1z_{n,1}-A_1u_1\|_{C^1_{J_1}}\\ \dots\\ \|A_mz_{n,m}-A_mu_m\|_{C^1_{J_m}}\end{pmatrix} \le M_1^{n+1}\begin{pmatrix}\|z_{0,0}-u_0\|_{C^1_{J_0}}\\ \|z_{0,1}-u_1\|_{C^1_{J_1}}\\ \dots\\ \|z_{0,m}-u_m\|_{C^1_{J_m}}\end{pmatrix}.$$

So $z_n(t),z_n'(t)$ uniformly converge to $u(t),u'(t)$ respectively for any $t\in J$. In other words, the conclusion of Theorem 12 holds.

If we replace the norm in (40) by p-norm of matrix, we can obtain following conclusion easily.

Theorem 13 *If the assumption (H4) holds and the matrix M_1 defined by (32) satisfies*

$$\|M_1\|_p<1, \tag{42}$$

where $1\le p\le+\infty$, then we have the conclusions of theorem 12.

Remark 5. Let $M=\max_{(t,s)\in J\times J}|K(t,s)|$ and $N=\max_{(t,s)\in J\times J}|H(t,s)|$. Since

$$\sum_{i=0}^{m}\delta_i\le\int_0^1[L_1(s)+L_2(s)]ds+N\int_0^1L_4(s)ds+ M\sum_{i=0}^{m}(t_{i+1}-t_i)\int_{t_{i+1}}^1L_3(s)ds \le\int_0^1[L_1(s)+L_2(s)+ML_3(s)+NL_4(s)]ds,$$

then we have

$$\begin{aligned} \|M\|_\infty &= \sum_{i=0}^{m} \delta_i + \sum_{i=1}^{m} (\mu_i + \lambda_i) \\ &\leq \int_0^1 L_0(s) ds + \sum_{i=1}^{m} (\mu_i + \lambda_i), \end{aligned}$$

where $L_0(s) = L_1(s) + L_2(s) + ML_3(s) + NL_4(s)$. The condition (42) is more general than one obtained directly by (36). So the conclusion of Theorem 13 is an extension of those in [4] for initial value problems.

Remark 6. Most of those conclusions of theorem 12 and 13 are new, since the conditions (H4) involve both the derivative x' and the linear integral operator Su. Usually, for convenience, we can use $\|\cdot\|_1, \|\cdot\|_2$ or $\|\cdot\|_\infty$ as the operator norm in (42).

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