A new iterative method for equilibrium problems, fixed point problems of infinitely nonexpansive mappings and a general system of variational inequalities

Jing Zhao
College of Science
Civil Aviation University of China
Tianjin, 300300
P.R. China
zhaojing200103@163.com

Caiping Yang
College of Science
Civil Aviation University of China
Tianjin, 300300
P.R. China
cpyang@cauc.edu.cn

Guangxuan Liu
College of Science
Civil Aviation University of China
Tianjin, 300300
P.R. China
guangxuanliu@163.com

Abstract: In this paper, we introduce a new iterative scheme for finding the common element of the set of common fixed points of infinitely many nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of a general system of variational inequalities for inverse-strongly monotone mappings in Hilbert spaces. We prove that the sequence converges strongly to a common element of the above three sets under some parameters controlling conditions. This main result improve and extend the corresponding results announced by many others. Using this theorem, we obtain three corollaries.

Key–Words: Nonexpansive mapping, Equilibrium problem, Fixed point, Inverse-strongly monotone mapping, General system of variational inequality, Iterative algorithm.

1 Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. $\rightharpoonup$ and $\rightarrow$ denote weak and strong convergence, respectively. Let C be a nonempty closed convex subset of H and $F : C \times C \to R$ be a bifunction of $C \times C$ into R, where R is the set of real numbers. The equilibrium problem for $F : C \times C \to R$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \tag{1}$$

for all $y \in C$. The set of solutions of (1) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. For solving the equilibrium problem for a bifunction $F : C\times C \to R$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

It is well known that for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the metric projection of H onto C. P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{3}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{4}$$

for all $x \in H, y \in C$.

Let $A : C \to H$ be a mapping. The classical variational inequality, denoted by $VI(A, C)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^*\rangle \geq 0 \tag{5}$$

for all $v \in C$. In this paper, $u \in VI(C, A)$ denotes u is a point of the set of solutions of the variational inequality $VI(C, A)$. It is easy to see that the following is true:

$$u \in VI(A, C) \Leftrightarrow u = P_C(u - \lambda Au), \;\; \lambda > 0. \tag{6}$$

A mapping A of C into H is called α-inverse-strongly monotone if there exists a positive real number α such that $\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$ for all $u, v \in C$. It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous. A mapping T of C into itself is called nonexpansive if $\|Tu - Tv\| \leq \|u - v\|$ for all $u, v \in C$. We denoted by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$.

For finding an element of $F(T) \cap VI(A, C)$, Takahashi and Toyoda [1] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T P_C(x_n - \lambda_n A x_n), \quad (7)$$

where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. Motivated by the idea of Korpelevich [2], Nadezhkina and Takahashi [3], Zeng and Yao [4] and Yao and Yao [5] proposed some so-called extragradient methods for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem.

Let $A, B : C \to H$ be two mappings. Now we concern the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \quad (8)$$

which is called a general system of variational inequalities where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (8) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu A x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \quad (9)$$

which is defined by Verma [6] (see also [7]) and is called the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then problem (9) reduces to the classical variational inequality problem (5). For solving problem (8), recently, Ceng et al. [8] introduced and studied a relaxed extragradient method. Based on the relaxed extragradient method and the viscosity approximation method, W. Kumam and P. Kumam [9] constructed a new viscosity relaxed extragradient approximation method. Very recently, based on the extragradient method, Yao et al. [10] proposed an iterative method for finding a common element of the set of a general system of variational inequalities and the set of fixed points of a strictly pseudocontractive mapping in a real Hilbert space.

On the other hand, let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself and let $\{t_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0, 1]$. For each $n \geq 1$, define a mapping W_n of C into itself as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = t_n T_n U_{n,n+1} + (1 - t_n) I, \\ U_{n,n-1} = t_{n-1} T_{n-1} U_{n,n} + (1 - t_{n-1}) I, \\ \vdots \\ U_{n,k} = t_k T_k U_{n,k+1} + (1 - t_k) I, \\ \vdots \\ U_{n,2} = t_2 T_2 U_{n,3} + (1 - t_2) I, \\ W_n = U_{n,1} = t_1 T_1 U_{n,2} + (1 - t_1) I. \end{cases} \quad (10)$$

Such a mapping W_n is called the W_n-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $t_n, t_{n-1}, \ldots, t_1$; see [11]. For finding a common element of the set of solutions of the equilibrium problem (1) and the set of common fixed points of infinitely many nonexpansive mappings in H, Yao et al. [12] introduced the following iterative scheme:

$$\begin{cases} F(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n \rangle \geq 0, \ \ \forall x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n u_n, \end{cases}$$

where $x_0 \in H$, $\{t_n\}$ is a sequence in $(0, b]$ for some $b \in (0, 1)$, f is a contraction of H into itself and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. They obtained a strong convergence theorem.

Motivated and inspired by the above works, in this paper, we introduce an iterative method based on the extragradient method and viscosity method for finding the common element of the set of common fixed points of infinitely many nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of a general system of variational inequalities for inverse-strongly monotone mappings in real Hilbert spaces. We establish some strong convergence theorems for our iterative scheme.

In order to prove our main results, we also need the following lemmas.

Lemma 1 *([13]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty} (\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|) \leq 0$. Then $\lim_{n\to\infty} \| y_n - x_n \| = 0$.*

Lemma 2 *([14]) Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 3 *([15]) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \ \ n \geq 0,$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} \mid \delta_n \mid < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 4 *([16]) Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into R satisfying $(A1)-(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*
$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0$ *for all* $y \in C$.

Lemma 5 *([17]) Assume that $F : C \times C \to R$ satisfies $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \ \forall y \in C\}$

for all $x \in H$. Then, the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,* $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x-y\rangle$;
3. $F(T_r) = EP(F)$;
4. *$EP(F)$ is closed and convex.*

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself, where C is a nonempty closed convex subset of a real Hilbert space H. Given a sequence $\{t_n\}_{n=1}^{\infty}$ in $[0,1]$, we define a sequence $\{W_n\}_{n=1}^{\infty}$ of self-mapping on C by (10). Then we have the following results.

Lemma 6 *([11]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $\{t_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$ the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 7 *([12]) It can be known from Lemma 6 that if D is a nonempty bounded subset of C, then for $\varepsilon > 0$ there exists $n_0 \geq k$ such that for all $n > n_0$*

$\sup_{x\in D} \|U_{n,k}x - U_k x\| \leq \varepsilon,$

where $U_k x = \lim_{n\to\infty} U_{n,k}x$.

Remark 8 *([12]) Using Lemma 6, we define a mapping $W : C \to C$ as follows:*

$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$

for all $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, \cdots$ and $t_1, t_2, \cdots$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive. Indeed, observe that for any $x, y \in C$,
$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \leq \|x-y\|$.

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 0\}$. Hence, it is clear from Remark 7 that for any arbitrary $\varepsilon > 0$ there exists $N_0 \geq 1$ such that for all $n > N_0$

$$\begin{aligned} \|W_n x_n - W x_n\| &= \|U_{n,1}x_n - U_1 x_n\| \\ &\leq \sup_{x\in D} \|U_{n,1}x - U_1 x\| \leq \varepsilon. \end{aligned}$$

This implies that $\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0$.

Lemma 9 *([11]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$, let $\{t_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

Lemma 10 *([8]) For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (8) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)]$$

for all $x \in C$, where $y^* = P_C(x^* - \mu B x^*)$.

Note that the mapping G is nonexpansive provided $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$. Throughout this paper, the set of fixed points of the mapping G is denoted by Γ.

Lemma 11 *In a real Hilbert space H, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \ \ \forall x, y \in H.$$

2 Main Results

Theorem 12 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to R$ satisfying $(A1)-(A4)$, the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse strongly monotone, respectively. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\Omega := \cap_{n=1}^{\infty} F(T_n) \bigcap EP(F) \bigcap \Gamma \neq \emptyset$. Let $f : C \to C$ be a ρ-contraction with $\rho \in [0, \frac{1}{2})$. Suppose $x_1 \in C$ and $\{x_n\}$ is generated by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ z_n = P_C(u_n - \mu B u_n), \\ y_n = \alpha_n f(x_n) + (1-\alpha_n) P_C(z_n - \lambda A z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda A z_n) + \delta_n W_n y_n, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, W_n is defined by (10) for $\{t_n\}$ in $(0,b] \subset (0,1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$

are sequences in $[0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *is a real sequence such that*

(i) $\beta_n+\gamma_n+\delta_n = 1$ *and* $\liminf_{n\to\infty}((1-2\rho)\delta_n-\gamma_n) > 0$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(iv) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$

(v) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} \mid r_{n+1} - r_n \mid= 0$,

then $\{x_n\}$ *converges strongly to* $x^* = P_\Omega f(x^*)$ *and* (x^*, y^*) *is a solution of the general system of variational inequalities* (8)*, where* $y^* = P_C(x^* - \mu Bx^*)$.

Proof. Let $Q = P_\Omega$. Then Qf is a contraction of C into itself. Since C is a closed set of H, there exists a unique element of $x^* \in C$ such that $x^* = Qf(x^*)$. For any $x, y \in C$ and $\lambda \in (0, 2\alpha)$, we note that

$$\begin{aligned} & \|(I-\lambda A)x-(I-\lambda A)y\|^2 \\ = & \|x-y-\lambda(Ax-Ay)\|^2 \\ = & \|x-y\|^2-2\lambda\langle x-y, Ax-Ay\rangle \\ & +\lambda^2\|Ax-Ay\|^2 \\ \le & \|x-y\|^2+\lambda(\lambda-2\alpha)\|Ax-Ay\|^2 \\ \le & \|x-y\|^2, \end{aligned} \tag{11}$$

which implies that $I - \lambda A$ is nonexpansive. In the same way we can obtain that $I - \mu B$ is also nonexpansive and

$$\begin{aligned} & \|(I-\mu B)x-(I-\mu B)y\|^2 \\ \le & \|x-y\|^2+\mu(\mu-2\beta)\|Bx-By\|^2 \end{aligned} \tag{12}$$

for all $x, y \in C$ and $\mu \in (0, 2\beta)$. Let $\{T_{r_n}\}$ be a sequence of mapping defined as in Lemma 5 and let $x^* \in \Omega$. Then $x^* = W_n x^* = T_{r_n}x^*$ and $x^* = P_C[P_C(x^* - \mu Bx^*) - \lambda AP_C(x^* - \mu Bx^*)]$. Putting $y^* = P_C(x^* - \mu Bx^*)$, we have $x^* = P_C(y^* - \lambda Ay^*)$. Let $v_n = P_C(z_n - \lambda Az_n)$, we have that

$$\|u_n - x^*\| = \|T_{r_n}x_n - T_{r_n}x^*\| \le \|x_n - x^*\|,$$

$$\begin{aligned} & \|z_n-y^*\| \\ = & \|P_C(u_n-\mu Bu_n)-P_C(x^*-\mu Bx^*)\| \\ \le & \|u_n-x^*\| \le \|x_n-x^*\|, \end{aligned}$$

$$\begin{aligned} & \|v_n-x^*\| \\ = & \|P_C(z_n-\lambda Az_n)-P_C(y^*-\lambda Ay^*)\| \\ \le & \|z_n-y^*\| \le \|x_n-x^*\| \end{aligned}$$

and

$$\begin{aligned} & \|y_n-x^*\| \\ \le & \alpha_n\|f(x_n)-f(x^*)\|+\alpha_n\|f(x^*)-x^*\| \\ & +(1-\alpha_n)\|v_n-x^*\| \\ \le & \alpha_n\rho\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\| \\ & +(1-\alpha_n)\|x_n-x^*\|. \end{aligned}$$

Since W_n is nonexpansive, we obtain

$$\begin{aligned} & \|x_{n+1}-x^*\| \\ \le & \beta_n\|x_n-x^*\|+\gamma_n\|v_n-x^*\| \\ & +\delta_n\|W_ny_n-x^*\| \\ \le & \beta_n\|x_n-x^*\|+\gamma_n\|x_n-x^*\|+\delta_n\|y_n-x^*\| \\ \le & (\beta_n+\gamma_n)\|x_n-x^*\|+\delta_n\alpha_n\|f(x^*)-x^*\| \\ & +\delta_n(1-\alpha_n+\alpha_n\rho)\|x_n-x^*\| \\ = & (1-\alpha_n\delta_n(1-\rho))\|x_n-x^*\| \\ & +\delta_n\alpha_n\|f(x^*)-x^*\| \\ \le & \max\{\|x_n-x^*\|, \tfrac{1}{1-\rho}\|f(x^*)-x^*\|\}. \end{aligned}$$

By induction, we have that

$$\|x_n - x^*\| \le \max\{\|x_1 - x^*\|, \frac{1}{1-\rho}\|f(x^*) - x^*\|\}$$

for all $n \ge 1$. Thus the sequence $\{x_n\}$ is bounded. Consequently, the sets $\{u_n\}, \{z_n\}, \{v_n\}, \{y_n\}, \{W_ny_n\}, \{Bu_n\}$ and $\{Az_n\}$ are also bounded.

Next, we claim that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Indeed, we define a sequence $\{s_n\}$ by $x_{n+1} = \beta_n x_n + (1 - \beta_n)s_n$, $\forall n \ge 1$. Thus, we have

$$\begin{aligned} & s_{n+1}-s_n \\ = & \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_n} \\ = & \frac{\gamma_{n+1}v_{n+1}+\delta_{n+1}W_{n+1}y_{n+1}}{1-\beta_{n+1}} \\ & -\frac{\gamma_nv_n+\delta_nW_ny_n}{1-\beta_n} \\ = & \frac{\gamma_{n+1}(v_{n+1}-v_n)}{1-\beta_{n+1}} \\ & +\frac{\delta_{n+1}(W_{n+1}y_{n+1}-W_{n+1}y_n)}{1-\beta_{n+1}} \\ & +(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n})v_n \\ & +(\frac{\delta_{n+1}}{1-\beta_{n+1}}-\frac{\delta_n}{1-\beta_n})W_{n+1}y_n \\ & +\frac{\delta_n}{1-\beta_n}(W_{n+1}y_n-W_ny_n). \end{aligned}$$

We note that

$$\begin{aligned} & \|v_{n+1}-v_n\| \\ = & \|P_C(I-\lambda A)z_{n+1}-P_C(I-\lambda A)z_n\| \\ \le & \|z_{n+1}-z_n\| \\ = & \|P_C(I-\mu B)u_{n+1}-P_C(I-\mu B)u_n\| \\ \le & \|u_{n+1}-u_n\|. \end{aligned} \tag{13}$$

From $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we note that

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C \tag{14}$$

and

$$F(u_{n+1},y)+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle\geq 0 \tag{15}$$

for all $y\in C$. Putting $y=u_{n+1}$ in (14) and $y=u_n$ in (15) respectively, we have

$$F(u_n,u_{n+1})+\frac{1}{r_n}\langle u_{n+1}-u_n,u_n-x_n\rangle\geq 0$$

and

$$F(u_{n+1},u_n)+\frac{1}{r_{n+1}}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle\geq 0.$$

So, from $(A2)$ we have

$$\langle u_{n+1}-u_n,\frac{u_n-x_n}{r_n}-\frac{u_{n+1}-x_{n+1}}{r_{n+1}}\rangle\geq 0.$$

Hence

$$\langle u_{n+1}-u_n,u_n-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\rangle\geq 0$$

and

$$\begin{array}{rl}\langle & u_{n+1}-u_n,u_n-u_{n+1}\\ + & u_{n+1}-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\rangle\geq 0.\end{array}$$

Since $\liminf_{n\to\infty}r_n>0$, without loss of generality, we may assume that there exists a real number c such that $r_n>c>0$ for all $n\geq 1$. Then we have

$$\begin{array}{rl}& \|u_{n+1}-u_n\|^2\\ \leq & \langle u_{n+1}-u_n,x_{n+1}-x_n\\ & +(1-\frac{r_n}{r_{n+1}})(u_{n+1}-x_{n+1})\rangle\\ \leq & \|u_{n+1}-u_n\|\{\|x_{n+1}-x_n\|\\ & +\mid 1-\frac{r_n}{r_{n+1}}\mid\|u_{n+1}-x_{n+1}\|\}\end{array}$$

and hence

$$\begin{array}{rl}& \|u_{n+1}-u_n\|\\ \leq & \|x_{n+1}-x_n\|\\ & +\frac{1}{r_{n+1}}\mid r_{n+1}-r_n\mid\|u_{n+1}-x_{n+1}\|\\ \leq & \|x_{n+1}-x_n\|+\frac{L_1}{c}\mid r_{n+1}-r_n\mid,\end{array} \tag{16}$$

where $L_1=\sup\{\|u_n-x_n\|:n\geq 1\}$. Substituting (16) into (13), we have

$$\|v_{n+1}-v_n\|\leq\|x_{n+1}-x_n\|+\frac{L_1}{c}\mid r_{n+1}-r_n\mid. \tag{17}$$

Moreover, we have

$$\begin{array}{rl}& \|W_{n+1}y_{n+1}-W_{n+1}y_n\|\\ \leq & \|y_{n+1}-y_n\|\\ \leq & \|v_{n+1}-v_n\|\\ & +\alpha_{n+1}\|f(x_{n+1})-v_{n+1}\|\\ & +\alpha_n\|f(x_n)-v_n\|\\ \leq & \|x_{n+1}-x_n\|+\frac{L_1}{c}\mid r_{n+1}-r_n\mid\\ & +(\alpha_{n+1}+\alpha_n)L_2,\end{array} \tag{18}$$

where $L_2=\sup\{\|f(x_n)-v_n\|:n\geq 1\}$. From (10), since T_i and $U_{n,i}$ are nonexpansive, we deduce that for each $n\geq 1$,

$$\begin{array}{rl}& \|W_{n+1}y_n-W_ny_n\|\\ = & \|t_1T_1U_{n+1,2}y_n-t_1T_1U_{n,2}y_n\|\\ \leq & t_1\|U_{n+1,2}y_n-U_{n,2}y_n\|\\ = & t_1\|t_2T_2U_{n+1,3}y_n-t_2T_2U_{n,3}y_n\|\\ \leq & t_1t_2\|U_{n+1,3}y_n-U_{n,3}y_n\|\\ & \cdots\\ \leq & (\prod_{i=1}^n t_i)\|U_{n+1,n+1}y_n-U_{n,n+1}y_n\|\\ \leq & L_3\prod_{i=1}^n t_i\end{array} \tag{19}$$

for some constant $L_3>0$. Combining (17), (18) and (19), we have

$$\begin{array}{rl}& \|s_{n+1}-s_n\|\\ \leq & \|x_{n+1}-x_n\|+\frac{L_1}{c}\mid r_{n+1}-r_n\mid\\ & +\frac{\delta_{n+1}L_2}{1-\beta_{n+1}}(\alpha_{n+1}+\alpha_n)\\ & +\mid\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\mid(\|v_n\|+\|W_{n+1}y_n\|)\\ & +\frac{L_3\delta_n}{1-\beta_n}\prod_{i=1}^n t_i.\end{array}$$

Thus it follows from conditions $(i)-(v)$ that (noting that $0<t_i\leq b<1,\forall i\geq 1$)

$$\limsup_{n\to\infty}(\|s_{n+1}-s_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

By Lemma 1 we get $\lim_{n\to\infty}\|s_n-x_n\|=0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|s_n-x_n\|=0.$$

Further, we can obtain that $\lim_{n\to\infty}\|Az_n-Ay^*\|=0$ and $\lim_{n\to\infty}\|Bu_n-Bx^*\|=0$. Indeed, from (11) and (12) we get that

$$\begin{array}{rl}& \|y_n-x^*\|^2\\ \leq & \alpha_n\|f(x_n)-x^*\|^2\\ & +(1-\alpha_n)\|P_C(I-\lambda A)z_n-P_C(I-\lambda A)y^*\|^2\end{array}$$

$$
\begin{aligned}
&\leq \alpha_n L_4 + (1-\alpha_n)(\|z_n - y^*\|^2 \\
&\quad +\lambda(\lambda - 2\alpha)\|Az_n - Ay^*\|^2) \\
&\leq \alpha_n L_4 + (1-\alpha_n)(\mu(\mu-2\beta)\|Bu_n - Bx^*\|^2 \\
&\quad +\|u_n - x^*\|^2 + \lambda(\lambda-2\alpha)\|Az_n - Ay^*\|^2) \\
&\leq \alpha_n L_4 + \mu(\mu-2\beta)\|Bu_n - Bx^*\|^2 \\
&\quad +\|x_n - x^*\|^2 + \lambda(\lambda-2\alpha)\|Az_n - Ay^*\|^2,
\end{aligned}
$$

where $L_4 = \sup\{\|f(x_n) - x^*\|^2 : n \geq 1\}$. So, we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
\leq\ & \beta_n\|x_n - x^*\|^2 + \gamma_n\|v_n - x^*\|^2 \\
& +\delta_n\|W_n y_n - x^*\|^2 \\
=\ & \beta_n\|x_n - x^*\|^2 + \delta_n\|y_n - x^*\|^2 \\
& +\gamma_n\|(y_n - x^*) + \alpha_n(v_n - f(x_n))\|^2 \\
\leq\ & \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 \\
& +\alpha_n L_5 \\
\leq\ & \|x_n - x^*\|^2 + (1-\beta_n)\alpha_n L_4 + \alpha_n L_5 \\
& +(1-\beta_n)[\mu(\mu-2\beta)\|Bu_n - Bx^*\|^2 \\
& +\lambda(\lambda-2\alpha)\|Az_n - Ay^*\|^2],
\end{aligned} \tag{20}
$$

where L_5 is some appropriate constant. It follows that

$$
\begin{aligned}
& (1-\beta_n)[\mu(2\beta-\mu)\|Bu_n - Bx^*\|^2 \\
& +\lambda(2\alpha-\lambda)\|Az_n - Ay^*\|^2] \\
\leq\ & \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
& +\alpha_n((1-\beta_n)L_4 + L_5) \\
\leq\ & \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
& +\alpha_n((1-\beta_n)L_4 + L_5).
\end{aligned}
$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$ and $\limsup_{n\to\infty} \beta_n < 1$, we obtain $\lim_{n\to\infty} \|Az_n - Ay^*\| = 0$ and $\lim_{n\to\infty} \|Bu_n - Bx^*\| = 0$.

Now we show that $\|Wy_n - y_n\| \to 0$ as $n \to \infty$. Noting that P_C is firmly nonexpansive, from $\|u_n - x^*\| \leq \|x_n - x^*\|$ we have

$$
\begin{aligned}
& \|z_n - y^*\|^2 \\
=\ & \|P_C(I-\mu B)u_n - P_C(I-\mu B)x^*\|^2 \\
\leq\ & \langle (I-\mu B)u_n - (I-\mu B)x^*, z_n - y^*\rangle \\
=\ & \frac{1}{2}[\|(I-\mu B)u_n - (I-\mu B)x^*\|^2 + \|z_n - y^*\|^2 \\
& -\|(I-\mu B)u_n - (I-\mu B)x^* - (z_n - y^*)\|^2] \\
\leq\ & \frac{1}{2}[\|u_n - x^*\|^2 + \|z_n - y^*\|^2 \\
& -\|u_n - z_n - \mu(Bu_n - Bx^*) - (x^* - y^*)\|^2] \\
\leq\ & \frac{1}{2}[\|x_n - x^*\|^2 + \|z_n - y^*\|^2 \\
& -\|u_n - z_n - (x^* - y^*)\|^2 - \mu^2\|Bu_n - Bx^*\|^2 \\
& +2\mu\langle u_n - z_n - (x^* - y^*), Bu_n - Bx^*\rangle]
\end{aligned}
$$

and from $\|z_n - y^*\| \leq \|x_n - x^*\|$ we also have

$$
\begin{aligned}
& \|v_n - x^*\|^2 \\
=\ & \|P_C(I-\lambda A)z_n - P_C(I-\lambda A)y^*\|^2 \\
\leq\ & \langle (I-\lambda A)z_n - (I-\lambda A)y^*, v_n - x^*\rangle \\
=\ & \frac{1}{2}[\|(I-\lambda A)z_n - (I-\lambda A)y^*\|^2 + \|v_n - x^*\|^2 \\
& -\|(I-\lambda A)z_n - (I-\lambda A)y^* - (v_n - x^*)\|^2] \\
\leq\ & \frac{1}{2}[\|z_n - y^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n \\
& -\lambda(Az_n - Ay^*) - (y^* - x^*)\|^2] \\
\leq\ & \frac{1}{2}[\|x_n - x^*\|^2 + \|v_n - x^*\|^2 \\
& -\|z_n - v_n + (x^* - y^*)\|^2 \\
& +2\lambda\langle z_n - v_n + (x^* - y^*), Az_n - Ay^*\rangle \\
& -\lambda^2\|Az_n - Ay^*\|^2].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
& \|z_n - y^*\|^2 \\
\leq\ & \|x_n - x^*\|^2 - \|u_n - z_n - (x^* - y^*)\|^2 \\
& +2\mu\langle u_n - z_n - (x^* - y^*), Bu_n - Bx^*\rangle
\end{aligned} \tag{21}
$$

and

$$
\begin{aligned}
& \|v_n - x^*\|^2 \\
\leq\ & \|x_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 \\
& +2\lambda\langle z_n - v_n + (x^* - y^*), Az_n - Ay^*\rangle.
\end{aligned} \tag{22}
$$

By (21) we get

$$
\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
\leq\ & \beta_n\|x_n - x^*\|^2 + \gamma_n\|v_n - x^*\|^2 \\
& +\delta_n\|W_n y_n - x^*\|^2 \\
\leq\ & \beta_n\|x_n - x^*\|^2 + \gamma_n\|z_n - y^*\|^2 + \delta_n\|y_n - x^*\|^2 \\
\leq\ & \beta_n\|x_n - x^*\|^2 + \gamma_n\|z_n - y^*\|^2 \\
& +\delta_n(\alpha_n\|f(x_n) - x^*\|^2 + (1-\alpha_n)\|z_n - y^*\|^2) \\
\leq\ & \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|z_n - y^*\|^2 + \delta_n\alpha_n L_4 \\
\leq\ & \|x_n - x^*\|^2 - (1-\beta_n)\|u_n - z_n - (x^* - y^*)\|^2 \\
& +2(1-\beta_n)\mu\|u_n - z_n - (x^* - y^*)\| \cdot \|Bu_n - Bx^*\| \\
& +\delta_n\alpha_n L_4,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& (1-\beta_n)\|u_n - z_n - (x^* - y^*)\|^2 \\
\leq\ & \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
& +2(1-\beta_n)\mu L_6\|Bu_n - Bx^*\| + \alpha_n L_7 \\
\leq\ & \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
& +2(1-\beta_n)\mu L_6\|Bu_n - Bx^*\| + \alpha_n L_7
\end{aligned} \tag{23}
$$

for approximate constants L_6 and L_7. It follows from (20) and (22) that

$$
\|x_{n+1} - x^*\|^2
$$

$$\begin{aligned}
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + \alpha_n L_5 \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\alpha_n\|f(x_n) - x^*\|^2 \\
&\quad +(1-\alpha_n)\|v_n - x^*\|^2] + \alpha_n L_5 \\
&\leq \|x_n - x^*\|^2 + \alpha_n L_5 + \alpha_n(1-\beta_n)L_4 \\
&\quad -(1-\beta_n)\|z_n - v_n + (x^* - y^*)\|^2 \\
&\quad +2(1-\beta_n)\lambda\langle z_n - v_n + (x^* - y^*) \\
&\quad , Az_n - Ay^*\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-\beta_n)\|z_n - v_n + (x^* - y^*)\|^2 \\
\leq\ & \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
& +L_8\|Az_n - Ay^*\| + \alpha_n L_9 \\
\leq\ & \|x_n - x_{n+1}\|(\|x_n - x^*\| \\
& +\|x_{n+1} - x^*\|) \\
& +L_8\|Az_n - Ay^*\| + \alpha_n L_9
\end{aligned} \tag{24}$$

for approximate constants L_8 and L_9. Note that $\|x_n - x_{n+1}\| \to 0$, $\alpha_n \to 0$, $\|Bu_n - Bx^*\| \to 0$and $\|Az_n - Ay^*\| \to 0$. From (23) and (24) we deduce

$$\lim_{n\to\infty} \|u_n - z_n - (x^* - y^*)\| = 0 \tag{25}$$

and

$$\lim_{n\to\infty} \|z_n - v_n + (x^* - y^*)\| = 0. \tag{26}$$

Since T_{r_n} is firmly nonexpansive for each $n \geq 1$, we have

$$\begin{aligned}
&\|u_n - x^*\|^2 \\
=\ & \|T_{r_n}x_n - T_{r_n}x^*\|^2 \\
\leq\ & \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^*\rangle \\
=\ & \langle u_n - x^*, x_n - x^*\rangle \\
=\ & \frac{1}{2}(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2)
\end{aligned}$$

and hence $\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2$. It follows that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
\leq\ & \beta_n\|x_n - x^*\|^2 + \gamma_n\|v_n - x^*\|^2 \\
& +\delta_n\|W_n y_n - x^*\|^2 \\
\leq\ & \beta_n\|x_n - x^*\|^2 + \gamma_n\|z_n - y^*\|^2 + \delta_n\|y_n - x^*\|^2 \\
\leq\ & \beta_n\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 \\
& +\delta_n(\alpha_n\|f(x_n) - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2) \\
\leq\ & \beta_n\|x_n - x^*\|^2 + (\gamma_n + \delta_n)\|u_n - x^*\|^2 \\
& +\delta_n\alpha_n L_4 \\
\leq\ & \|x_n - x^*\|^2 - (1-\beta_n)\|x_n - u_n\|^2 + \delta_n\alpha_n L_4,
\end{aligned}$$

and hence

$$\begin{aligned}
&(1-\beta_n)\|x_n - u_n\|^2 \\
\leq\ & \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \delta_n\alpha_n L_4 \\
\leq\ & \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
& +\delta_n\alpha_n L_4
\end{aligned}$$

for $L_4 = \sup\{\|f(x_n) - x^*\|^2 : n \geq 1\}$. So, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{27}$$

From $y_n = \alpha_n f(x_n) + (1-\alpha_n)v_n$, we get $\|y_n - v_n\| = \alpha_n\|f(x_n) - v_n\| \to 0$ as $n \to \infty$. It follows from (25), (26) and (27) that

$$\begin{aligned}
&\lim_{n\to\infty} \|u_n - v_n\| = \lim_{n\to\infty} \|x_n - v_n\| \\
=\ & \lim_{n\to\infty} \|x_n - y_n\| = 0.
\end{aligned} \tag{28}$$

Since

$$\begin{aligned}
&\delta_n\|W_n y_n - x_n\| \\
=\ & \|x_{n+1} - \beta_n x_n - \gamma_n v_n - \delta_n x_n\| \\
\leq\ & \|x_{n+1} - x_n\| + \gamma_n\|x_n - v_n\|,
\end{aligned}$$

from $\|x_{n+1} - x_n\| \to 0$ and (28) we have $\|W_n y_n - x_n\| \to 0$ and hence $\|W_n y_n - y_n\| \to 0$ as $n \to \infty$. Moreover, we get $\|W y_n - y_n\| \to 0$ as $n \to \infty$ from Remark 8.

Next, we show that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle \leq 0,$$

where $x^* = P_\Omega f(x^*)$. As $\{y_n\}$ is bounded, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup z \in C$ and

$$\begin{aligned}
&\limsup_{n\to\infty} \langle f(x^*) - x^*, y_n - x^*\rangle \\
=\ & \lim_{i\to\infty} \langle f(x^*) - x^*, y_{n_i} - x^*\rangle.
\end{aligned}$$

From $\|W y_n - y_n\| \to 0$ and Lemma 2, we obtain $z \in F(W)$. It follows from Lemma 9 that $z \in \cap_{n=1}^{\infty} F(T_n)$. Let us show $z \in EP(F)$. Since $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

From $(A2)$, we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}).$$

From $\|u_n - x_n\| \to 0$ and $\|x_n - y_n\| \to 0$ we get $u_{n_i} \rightharpoonup z$. Since $\|u_{n_i} - x_{n_i}\| \to 0$ and $\liminf_{n\to\infty} r_n > 0$, it follows from condition $(A4)$ that

$$0 \geq F(y, z), \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty+(1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $F(y_t, z) \leq 0$. So from $(A1)$ and $(A4)$ we have

$$\begin{aligned} 0 = F(y_t, y_t) &\leq tF(y_t, y) + (1-t)F(y_t, z) \\ &\leq tF(y_t, y) \end{aligned}$$

and hence $0 \leq F(y_t, y)$. From $(A3)$, we get $0 \leq F(z, y)$ for all $y \in C$ and $z \in EP(F)$. We shall show $z \in \Omega$. We note that

$$\begin{aligned} & \|y_n - G(y_n)\| \\ \leq\ & \alpha_n \|f(x_n) - G(y_n)\| \\ & +(1-\alpha_n)\|P_C[P_C(u_n - \mu B u_n) \\ & -\lambda A P_C(u_n - \mu B u_n)] - G(y_n)\| \\ =\ & \alpha_n \|f(x_n) - G(y_n)\| \\ & +(1-\alpha_n)\|G(u_n) - G(y_n)\| \\ \leq\ & \alpha_n \|f(x_n) - G(y_n)\| + (1-\alpha_n)\|u_n - y_n\| \\ \to\ & 0. \end{aligned}$$

From Lemma 2 we have $z \in F(G)$ and hence $z \in \Gamma$. Hence $z \in \Omega$. It follows from $\|x_n - y_n\| \to 0$ and (3) that

$$\begin{aligned} & \limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^* \rangle \\ =\ & \limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - y_n + y_n - x^* \rangle \\ \leq\ & \limsup_{n\to\infty} \langle f(x^*) - x^*, y_n - x^* \rangle \qquad (29) \\ =\ & \lim_{i\to\infty} \langle f(x^*) - x^*, y_{n_i} - x^* \rangle \\ =\ & \langle f(x^*) - x^*, z - x^* \rangle \leq 0. \end{aligned}$$

At last, we show that $\lim_{n\to\infty} x_n = x^*$. From Lemma 11 we get that

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ \leq\ & \|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) \\ & +\delta_n(W_n y_n - x^*) + \gamma_n\alpha_n(v_n - f(x_n))\|^2 \\ \leq\ & \|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) \\ & +\delta_n(W_n y_n - x^*)\|^2 \\ & +2\gamma_n\alpha_n\langle v_n - f(x_n), x_{n+1} - x^*\rangle \\ \leq\ & \beta_n\|x_n - x^*\|^2 + (\gamma_n + \delta_n)\|y_n - x^*\|^2 \\ & +2\gamma_n\alpha_n\langle v_n - x^*, x_{n+1} - x^*\rangle \\ & +2\gamma_n\alpha_n\langle x^* - f(x_n), x_{n+1} - x^*\rangle \\ \leq\ & \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[(1-\alpha_n)\|v_n - x^*\|^2 \\ & +2\alpha_n\langle f(x_n) - x^*, y_n - x^*\rangle] \\ & +2\gamma_n\alpha_n\|v_n - x^*\| \cdot \|x_{n+1} - x^*\| \\ & +2\gamma_n\alpha_n\langle x^* - f(x_n), x_{n+1} - x^*\rangle. \end{aligned}$$

It follows from $\|v_n - x^*\| \leq \|x_n - x^*\|$ that

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ \leq\ & \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - x^*\|^2 \\ & +2\alpha_n(\gamma_n + \delta_n)\langle f(x_n) - x^*, y_n - x^*\rangle \\ & +2\gamma_n\alpha_n\|x_n - x^*\| \cdot \|x_{n+1} - x^*\| \\ & +2\gamma_n\alpha_n\langle x^* - f(x_n), x_{n+1} - x^*\rangle \\ \leq\ & [1 - \alpha_n(1-\beta_n)]\|x_n - x^*\|^2 \\ & +2\alpha_n\gamma_n\langle f(x_n) - x^*, y_n - x_{n+1}\rangle \\ & +2\alpha_n\delta_n\langle f(x_n) - x^*, y_n - x^*\rangle \\ & +2\gamma_n\alpha_n\|x_n - x^*\| \cdot \|x_{n+1} - x^*\| \\ \leq\ & [1 - \alpha_n(1-\beta_n)]\|x_n - x^*\|^2 \\ & +2\alpha_n\gamma_n L_4\|y_n - x_{n+1}\| \\ & +2\alpha_n\delta_n\langle f(x_n) - x^*, y_n - x_n\rangle \\ & +2\alpha_n\delta_n\langle f(x_n) - x^*, x_n - x^*\rangle \\ & +2\gamma_n\alpha_n\|x_n - x^*\| \cdot \|x_{n+1} - x^*\| \\ \leq\ & [1 - \alpha_n(1-\beta_n)]\|x_n - x^*\|^2 \\ & +2\alpha_n L_4(\gamma_n\|y_n - x_{n+1}\| + \delta_n\|y_n - x_n\|) \\ & +2\alpha_n\delta_n\rho\|x_n - x^*\| \\ & +2\alpha_n\delta_n\langle f(x^*) - x^*, x_n - x^*\rangle \\ & +\gamma_n\alpha_n[\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2], \end{aligned}$$

which implies

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ \leq\ & [1 - \frac{(1-2\rho)\delta_n - \gamma_n}{1 - \alpha_n\gamma_n}\alpha_n]\|x_n - x^*\|^2 \\ & +\frac{(1-2\rho)\delta_n - \gamma_n}{1 - \alpha_n\gamma_n}\alpha_n \times \{\frac{2L_4}{(1-2\rho)\delta_n - \gamma_n} \\ & (\gamma_n\|y_n - x_{n+1}\| + \delta_n\|y_n - x_n\|) \\ & +\frac{2\delta_n}{(1-2\rho)\delta_n - \gamma_n}\langle f(x^*) - x^*, x_n - x^*\rangle\} \end{aligned}$$

where $L_4 = \sup\{\|f(x_n) - x^*\|^2 : n \geq 1\}$.

Note that $\liminf_{n\to\infty} \frac{(1-2\rho)\delta_n - \gamma_n}{1-\alpha_n\gamma_n} > 0$. We have $\sum_{n=1}^{\infty} \frac{(1-2\rho)\delta_n - \gamma_n}{1-\alpha_n\gamma_n}\alpha_n = \infty$. It follows from $\|y_n - x_n\| \to 0$, $\|x_n - x_{n+1}\| \to 0$, (29) and Lemma 3 that $x_n \to x^*$. This completes the proof. □

As direct consequences of Theorem 12, we obtain three corollaries.

Corollary 13 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to R$ satisfying $(A1)$ – $(A4)$, the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse strongly monotone, respectively. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\Omega := \cap_{n=1}^{\infty} F(T_n) \bigcap EP(F) \bigcap \Gamma \neq \emptyset$. For fixed $u \in C$*

and given $x_1 \in C$ arbitrarily, $\{x_n\}$ is generated by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ z_n = P_C(u_n - \mu B u_n), \\ y_n = \alpha_n u + (1-\alpha_n) P_C(z_n - \lambda A z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda A z_n) + \delta_n W_n y_n, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, W_n is defined by (10) *for $\{t_n\}$ in $(0, b] \subset (0, 1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset (0, \infty)$ is a real sequence such that*

(i) $\beta_n + \gamma_n + \delta_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(iv) $\lim_{n\to\infty} (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$

(v) $\liminf_{n\to\infty} (\delta_n - \gamma_n) > 0$,

(vi) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} \mid r_{n+1} - r_n \mid = 0$.

Then $\{x_n\}$ converges strongly to $x^* = P_\Omega u$ and (x^*, y^*) is a solution of the general system of variational inequalities (8)*, where $y^* = P_C(x^* - \mu B x^*)$.*

Corollary 14 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse strongly monotone, respectively and $\{T_n\}_{n=1}^{\infty}$ a sequence of nonexpansive self-mappings on C such that $\Omega := \cap_{n=1}^{\infty} F(T_n) \bigcap \Gamma \neq \emptyset$. Let $f : C \to C$ be a ρ-contraction with $\rho \in [0, \frac{1}{2})$. Suppose $x_1 \in C$ and $\{x_n\}$ is generated by*

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ y_n = \alpha_n f(x_n) + (1-\alpha_n) P_C(z_n - \lambda A z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda A z_n) + \delta_n W_n y_n, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, W_n is defined by (10) *for $\{t_n\}$ in $(0, b] \subset (0, 1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are sequences in $[0, 1]$ such that*

(i) $\beta_n + \gamma_n + \delta_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(iv) $\lim_{n\to\infty} (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$

(v) $\liminf_{n\to\infty} ((1-2\rho)\delta_n - \gamma_n) > 0$,

then $\{x_n\}$ converges strongly to $x^* = P_\Omega f(x^*)$ and (x^*, y^*) is a solution of the general system of variational inequalities (8)*, where $y^* = P_C(x^* - \mu B x^*)$.*

Corollary 15 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse strongly monotone, respectively and $\{T_n\}_{n=1}^{\infty}$ a sequence of nonexpansive self-mappings on C such that $\Omega := \cap_{n=1}^{\infty} F(T_n) \bigcap \Gamma \neq \emptyset$. For fixed $u \in C$ and given $x_1 \in C$ arbitrarily, $\{x_n\}$ is generated by*

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ y_n = \alpha_n u + (1-\alpha_n) P_C(z_n - \lambda A z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda A z_n) + \delta_n W_n y_n, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, W_n is defined by (10) *for $\{t_n\}$ in $(0, b] \subset (0, 1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are sequences in $[0, 1]$ such that*

(i) $\beta_n + \gamma_n + \delta_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(iv) $\lim_{n\to\infty} (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$

(v) $\liminf_{n\to\infty} (\delta_n - \gamma_n) > 0$.

Then $\{x_n\}$ converges strongly to $x^* = P_\Omega f(x^*)$ and (x^*, y^*) is a solution of the general system of variational inequalities (8)*, where $y^* = P_C(x^* - \mu B x^*)$.*

Remark 16 *We note that the results in Theorem 12 improved and extended the corresponding results in Yao et al.* [12] *from equilibrium problem and infinitely many nonexpansive mappings to equilibrium problem, general system of variational inequalities and infinitely many nonexpansive mappings.*

Remark 17 *Next, we can extend the main results of this paper from Hilbert spaces to the general Bnanch spaces.*

Acknowledgements: The research was supported by Fundamental Research Funds for the Central Universities (Program No. ZXH2011D005), it was also supported by the science research foundation program in Civil Aviation University of China (2011kys02).

References:

[1] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* Vol.118, 2003, pp. 417-428.

[2] G. M. Korpelevich, An extragradient method for finding saddle points and for other problems, *Ekon. Mat. Metody*,Vol.12, 1976, pp. 747-756.

[3] N. Nadezhkina, W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* Vol.128, 2006, pp. 191-201.

[4] L. C. Zeng, J. C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, *Taiwanese J. Math.* Vol.10, 2006, pp. 1293-1303.

[5] Y. Yao, J. C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings, *Appl. Math. Comput.* Vol.186, 2007, 2:pp. 1551-1558.

[6] R. U. Verma, On a new system of nonlinear variational inequalities and associated iterative algorithms, *Math. Sci. Res. Hot-Line*, Vol.3, 1999, pp. 65-68.

[7] R. U. Verma, Iterative algorithms and a new system of nonlinear quasivariational inequalities, *Adv. Nonlinear Var. Inequal.* Vol.4, 2001, pp. 117-124.

[8] L. C. Ceng, C. Wang, J. C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, *Math. Methods Oper. Res.* Vol.67, 2008, pp. 375-390.

[9] W. Kumam, P. Kumam, Hybrid iterative scheme by a relaxed extragradient method for solutions of equilibrium problems and a general system of variational inequalities with application to optimization, *Nonlinear Anal.: Hybrid System*, Vol.3, 2009, pp. 640-656.

[10] Y. Yao, Y. C. Liou, S. M. Kang, Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method, *Comput. Math. Appl.*, Vol.59, 2010, pp. 3472-3480.

[11] J. G. O'Hara, P. Pillay, H. K. Xu, Iterative approaches to convex feasibility problems in Banach spaces, *Nonlinear Anal.*, Vol. 64, 2006, 9:pp. 2022-2042.

[12] Y. Yao, Y. C. Liou, J. C. Yao, Convergence theorem for equilibrium problem and fixed point problems of infinite family of nonexpansive mappings, *Fixed Point Theory and Applications* Vol.2007, 2007.

[13] T. Suzuki, Strong convergence of Krasnoselskii and mann's type sequences for one-parameter nonexpansive semigroups without bochner intrgrals, *J. Math. Anal. Appl.* Vol.305, 2005, pp. 227-239.

[14] H. H. Bauschke, The approximation of fixed points of composition of nonexpansive mappings in Hilbert space, *J. Math. Anal. Appl.* Vol.202, 1996, pp. 150-159.

[15] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* Vol.298, 2004, pp. 279-291.

[16] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student.* Vol.63, 1994, pp. 123-145.

[17] P. L. Combettes, S. A. Hirstoaga, Equilibrium progamming using proximal-like algorithms, *Math. Program* Vol.79, 1997, pp. 29-41.