Computing rupture degrees of some graphs

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Abstract: Computer or communication networks are so designed that they do not easily get disrupted under external attack and, moreover, these are easily reconstructed when they do get disrupted. These desirable properties of networks can be measured by various parameters such as connectivity, toughness, tenacity and rupture degree. Among these parameters, rupture degree is comparatively better parameter to measure the vulnerability of networks. In this paper, the authors give the exact values for the rupture degree of the Cartesian product of a path and a cycle. After that, we discuss the rupture degree of total graphs of paths and cycles. Finally, we study the values for rupture degree of powers of paths and cycles.

Key–Words: Rupture degree, Vulnerability, Cartesian product, Total graph, Powers of graphs, R-set.

1 Introduction

Throughout this paper, a graph $G = (V, E)$ always means a simple connected graph with vertex set V and edge set E. For $S \subseteq V(G)$, let $\omega(G - S)$ and $m(G - S)$, respectively, denote the number of components and the order of a largest component in $G-S$. A set $S \subseteq V(G)$ is a cut set of G, if either $G - S$ is disconnected or $G - S$ has only one vertex. We shall use $\lceil x \rceil$ for the smallest integer not smaller than x, and $\lfloor x \rfloor$ for the largest integer not larger than x. We use Bondy and Murty [2] for terminology and notations not defined here.

A communication network is composed of processors and communication links. Network designers attach importance the reliability and stability of a network. If the network begins losing communication links or processors, then there is a loss in its effectiveness. This event is called as the vulnerability of communication networks.

The vulnerability of communication networks measures the resistance of a network to a disruption in operation after the failure of certain processors and communication links. Cable cuts, processor interruptions, software errors, hardware failures, or transmission failure at various points can interrupt service for a long period of time. But network designs require greater degrees of stability and reliability or less vulnerability in communication networks. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconfiguration of the network.

The communication network often has as considerable an impact on a network's performance as the processors themselves. Performance measures for communication networks are essential to guide the designers in choosing an appropriate topology. In order to measure the performance, we are interested in the following performance metrics (there may be others):

(1) the number of elements that are not functioning,

(2) the number of remaining connected subnetworks,

(3) the size of a largest remaining group within which mutual communication can still occur.

The communication network can be represented as an undirected and unweighted graph, where a processor (station) is represented as a node and a communication link between processors (stations) as an edge between corresponding nodes. If we use a graph to model a network, there are many graph theoretical parameters used to describe the vulnerability of communication networks.

One of the vulnerability parameters determined above is connectivity which deals with the quantity (1). The other parameters, toughness and scattering number take into account of the quantities (1) and (2). The integrity deals with the quantities (1) and (3). The rupture degree is a measure which deals with all the quantities, (1), (2) and (3).

Definition 1 *Let G be an incomplete graph. Then the*

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rupture degree $r(G)$ of G is defined as

$$r(G) = max\{\omega(G-S) - |S| - m(G-S) : S \subset V(G),\ \omega(G-S) \geq 2\}.$$

In particular, the rupture degree of a complete graph K_n is defined to be $1-n$.

Definition 2 *Let G be an incomplete connected graph, a set $S \subset V(G)$ is called an R-*set *if it satisfies $r(G) = \omega(G-S) - |S| - m(G-S)$.*

The concept of rupture degree was first introduced by Li, Zhang and Li in [10], where the authors determined the rupture degrees of several classes of graphs, and gave formulas for the rupture degrees of join graphs and some bounds of rupture degrees. Some Nordhaus-Goddard-type results for the rupture degree are also deduced.

Of all the above parameters, rupture degree are comparatively appropriate for measuring the vulnerability of networks, for rupture degree gives us more knowledge about the network to disruption. On the other hand, the tenacity which is introduced by M. Cozzens et al. [5] is also a measure which deals with the quantities, (1), (2) and (3). The rupture degree can be regarded as additive dual of tenacity. Li et al. [10] proved that the rupture degree is a better parameter than tenacity. As a consequence, the rupture degree is a better parameter to measure the vulnerability of a communication network. In [9], we proposed the following decision problem.

Decision Problem Not r-Rupture

Let G be an incomplete connected graph and r be an integer. Does there exist an $X \subset V(G)$ with $\omega(G-X) \geq 2$ such that $\omega(G-X) > |X| + m(G-X) + r$?

For this decision problem, we proved that computing the rupture degree of a graph is NP-hard in general, so it is an interesting problem to determine this parameter for some special classes of interesting or practically useful graphs. It is easy to see that the less the rupture degree of a network the more stable it is considered to be. In [8], A. Kirlangic studied the rupture degree of gear graphs. In this paper, we give some results on the rupture degree of some specific classes of graphs.

The rest is organized as follows. Formulas for computing the rupture degree of the Cartesian product of a path and a cycle is determined in section 2. In section 3, we determine the rupture degree for total graphs of some special graphs. In the section 4, we give exact values for the rupture degree of powers of paths and cycles. Finally, in section 5, we give a conclusion remark.

2 Rupture degree for the Cartesian product of a path and a cycle

In this section, we focus our attention to rupture degree of the Cartesian product of a path P_n and a cycle C_m.

The Cartesian product of two graphs G_1 and G_2, denoted by $G_1 \times G_2$, is defined as follows:

$$V(G_1 \times G_2) = V(G_1) \times V(G_2),$$

two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_1v_1 \in E(G_1)$ and $u_2 = v_2$. Observe that if G_1 and G_2 are connected, then $G_1 \times G_2$ is connected. In particular, we study the rupture degree of the Cartesian product of a path P_n and a cycle C_m, denoted by $P_n \times C_m$.

It is easy to see that if we denote

$$P_n = u_1u_2\cdots u_n,$$

$$C_m = v_1v_2\cdots v_mv_1.$$

Then the vertex set of $P_n \times C_m$ is

$$\{(u_i, v_j)|u_i \in P_n, v_j \in C_m, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

An undirected graph $G = (V, E)$ is said to be hamiltonian if it contains a hamiltonian cycle. Firstly we give two properties of the graph $P_n \times C_m$.

Proposition 3 *The graph $P_n \times C_m$ is a hamiltonian graph.*

Proof. It is easy to see that if a graph G contains a hamiltonian cycle, then so does graph $P_n \times G$. It follows that $P_n \times C_m$ contains a hamiltonian cycle, so $P_n \times C_m$ is a hamiltonian graph. □

Proposition 4 *The graph $P_n \times C_m$ is a bipartite graph when m is even.*

Proof. It is well known that if G is a bipartite graph with bipartition $[A, B]$ and H is bipartite graph with bipartition $[C, D]$, then the Cartesian product of these two bipartite graphs G and H, $G \times H$ is a bipartite graph with bipartition

$$[(A \times C) \cup (B \times D), (A \times D) \cup (B \times C)].$$

Hence, it follows that when m is even, the cycle C_m is a bipartite graph, and for any positive integer n, the path P_n is always a bipartite graph, so, on this condition, the Cartesian product of a path and a cycle $P_n \times C_m$ is a bipartite graph. □

If S is an R-set of $P_n \times C_m$, we shall show that the components of $P_n \times C_m - S$ satisfy the following properties.

Proposition 5 *If S is an R-set of $P_n \times C_m$, then the components of $P_n \times C_m - S$ must be K_1 or K_2.*

Proof. Let S be an R-set of $P_n \times C_m$. We assume that there exist k components whose order is larger than 2. In order to prove this assumption is not true, we distinguish the following three cases.

Case 1. If any components of $P_n \times C_m - S$ whose order is larger than 2 do not contain cycles, then these components must be trees. We let $(u,v)_1, (u,v)_2, \cdots, (u,v)_k$ be vertices of maximum degree in such k components, respectively. We let

$$S' = S \cup \{(u,v)_1, (u,v)_2, \cdots, (u,v)_k\},$$

then

$$\begin{aligned} |S'| &= |S| + k, \\ m(P_n \times C_m - S') &\le m(P_n \times C_m - S) - 1, \\ \omega(P_n \times C_m - S') &\ge \omega(P_n \times C_m - S) + k, \end{aligned}$$

so, we have

$$\begin{aligned} &\omega(P_n \times C_m - S') - |S'| - m(P_n \times C_m - S') \\ &\ge \omega(P_n \times C_m - S) - |S| - m(P_n \times C_m - S) + 1 \\ &> \omega(P_n \times C_m - S) - |S| - m(P_n \times C_m - S), \end{aligned}$$

a contradiction to the definition of rupture degree of graphs.

Case 2. If any components of $P_n \times C_m - S$ whose order is larger than 2 contains a cycle.

Subcase 2.1 If every such k components contain a cut edge, then, it is easy to see that every cut edge must have a end vertex whose degree is larger than 1, we denote these vertices be $(u,v)_1, (u,v)_2, \cdots, (u,v)_k$, let

$$S' = S \cup \{(u,v)_1, (u,v)_2, \cdots, (u,v)_k\},$$

then

$$|S'| = |S| + k,$$

$$m(P_n \times C_m - S') \le m(P_n \times C_m - S) - 2,$$

$$\omega(P_n \times C_m - S') \ge \omega(P_n \times C_m - S) + k,$$

so, we have

$$\begin{aligned} &\omega(P_n \times C_m - S') - |S'| - m(P_n \times C_m - S') \\ &\ge \omega(P_n \times C_m - S) - |S| - m(P_n \times C_m - S) + 2 \\ &> \omega(P_n \times C_m - S) - |S| - m(P_n \times C_m - S), \end{aligned}$$

a contradiction to the definition of rupture degree of graphs.

Subcase 2.2 If there exist no cut edge in such k components, we let vertex set

$$S' = \left\{ (u_i, v_j) \;\middle|\; \begin{array}{l} 1 \le i \le n; 1 \le j \le m; \\ \text{if i is odd, then j is even;} \\ \text{if i is even, then j is odd} \end{array} \right\},$$

It is easily seen that :

(1) all components of $P_n \times C_m - S'$ are K_1, when m is even;

(2) every components of $P_n \times C_m - S'$ is K_1 or K_2, when m is odd. And we have

$$m(P_n \times C_m - S) > m(P_n \times C_m - S') \ge 1,$$

$$\omega(P_n \times C_m - S) < \omega(P_n \times C_m - S'),$$

so, we have the following two cases:

Subcase 2.2.1 If $|S| \ge |S'|$, contradicts to the assumption.

Subcase 2.2.2 If $|S| < |S'|$, we can characterize graph $P_n \times C_m - S'$ by considering the condition of graph $P_n \times C_m - S$. Because every components of $P_n \times C_m - S'$ is either K_1 or K_2, If we want to get k components without cut edge and with order no less than 3, we must release vertices in k components, and we must add vertices from $P_n \times C_m - S'$ to S', by this method, we know that if the number of adding vertices is less than the number of vertices releasing from S', then we have $|S'| < |S|$, and we can see that , when we release one vertex from $|S'|$, then $\omega(P_n \times C_m - S')$ will reduce at least 1. If we denote

$$|S'| = |S| + \Delta S,$$

then

$$\omega(P_n \times C_m - S') \ge \omega(P_n \times C_m - S) + \Delta S,$$

so, we have

$$\begin{aligned} &\omega(P_n \times C_m - S') - |S'| - m(P_n \times C_m - S') \\ &> \omega(P_n \times C_m - S) + \Delta S - |S| \\ &\quad - \Delta S - m(P_n \times C_m - S) \\ &= \omega(P_n \times C_m - S) - |S| - m(P_n \times C_m - S). \end{aligned}$$

This is a contradiction to the definition of rupture degree of graphs. The proof is thus completed. □

Definition 6 *In graph $P_n \times C_m$, for $1 \le k \le n$, let vertex set $V_k = \{(u_k, v_1), (u_k, v_2), \cdots, (u_k, v_m)\}$, we denote C_m^k, an m-order cycle, be subgraph of $P_n \times C_m$ with vertex set V_k. If $1 \le i < i+1 < j \le n$, we call two cycles C_m^i and C_m^j are nonadjacent in $P_n \times C_m$.*

Definition 7 *In graph $P_n \times C_m$, the induced subgraph $\{(u_h, v_i), (u_h, v_{i+1})\}$ and $\{(u_j, v_k), (u_j, v_{k+1})\}$ are both K_2, $1 \le h < h+1 < j \le n$, $1 \le i \le m$,$1 \le j \le m$. If $i = j$, we call that $\{(u_h, v_i), (u_h, v_{i+1})\}$ and $\{(u_j, v_k), (u_j, v_{k+1})\}$ are located at the same position of nonadjacent cycles C_m^i and C_m^j.*

Proposition 8 *If S is a minimal R-set of $P_n \times C_m$, then in the components of $P_n \times C_m - S$, there exist at most $\lceil \frac{n}{2} \rceil$ number of K_2, and they are located at the same position of nonadjacent cycles.*

Proof. (1) By the definition of rupture degree $r(G) = max\{\omega(G - S) - |S| - m(G - S) : \omega(G - S) > 1\}$. we can see that if we want to get $r(G)$, we must let the R-set S satisfying that $|S|$ and $m(P_n \times C_m - S)$ are small and $\omega(P_n \times C_m - S)$ is large.

(a) When m is even, by proposition 3, we know that $P_n \times C_m$ is a bipartite graph, so, the order of components in $P_n \times C_m - S$ is not larger than 1.

(b) When m is odd, if all K_2 of $m(P_n \times C_m - S)$ are located on path P_n, and $\omega(P_n \times C_m - S)$ is as larger as possible, then, it is easy to see that S must contain 4 adjacent vertices, so, on this condition, S does not satisfy that it is as small as possible. So, in $P_n \times C_m$, all K_2 are not located on path P_n, i.e. all K_2 of $P_n \times C_m - S$ must be located on cycle C_m^r.

(2) If there exists a component K_2 which is located at the different position from that of other components K_2, we will distinguish three cases.

(a) If there exist two K_2 in $P_n \times C_m - S$ which are located on the same cycle C_m^k, we denote these two K_2 as $\{(u_k, v_i), (u_k, v_{i+1})\}$ and $\{(u_k, v_j), (u_k, v_{j+1})\}$, $1 < k < n$, $1 \leq i \leq m$,$j \geq (i + 3)mod(m)$. Among these two K_2, we assume that $\{(u_k, v_i), (u_k, v_{i+1})\}$ is located on the same position of that of other K_2. Now, let

$$S' = S \cup \{(u_k, v_i)\} - \{(u_{k-1}, v_i),$$
$$(u_k, v_{i+1}), (u_{k+1}, v_i)\},$$

or

$$S' = S \cup \{(u_k, v_{i+1})\} - \{(u_{k-1}, v_{i+1}),$$
$$(u_k, v_{i+2}), (u_{k+1}, v_{i+1})\},$$

then, we have

$$|S'| = |S| - 2,$$
$$m(P_n \times C_m - S') = m(P_n \times C_m - S) = 2,$$
$$\omega(P_n \times C_m - S') = \omega(P_n \times C_m - S) + 2,$$

so we have

$$\begin{aligned}&\omega(P_n \times C_m - S') - |S'| - m(P_n \times C_m - S')\\ &= \omega(P_n \times C_m - S) + 2 - |S| + 2 - m(P_n \times C_m - S)\\ &> \omega(P_n \times C_m - S) - |S| - m(P_n \times C_m - S).\end{aligned}$$

This contradicts to the definition of rupture degree of graphs.

(b) If there exist two K_2 in $P_n \times C_m - S$ which are located on two adjacent cycles, then, their position must be different from each other, we denote these two K_2 as $\{(u_k, v_i), (u_k, v_{i+1})\}$ and $\{(u_{k+1}, v_j), (u_{k+1}, v_{j+1})\}$, $0 < k < n - 1$, $1 \leq i \leq m$,$j \geq (i + 2)modm$ *or* $j \leq (i - 2 + m)modm$.

Among these two K_2, we assume that $\{(u_k, v_i), (u_k, v_{i+1})\}$ is located on the same position of that of other K_2. Now, let

$$S' = S \cup \{(u_{k+1}, v_j)\} - \{(u_k, v_j), (u_{k+1}, v_{j-1}),$$
$$(u_{k+2}, v_j)\},$$

or

$$S' = S \cup \{(u_{k+1}, v_{i+1})\} - \{(u_k, v_{j+1}),$$
$$(u_{k+1}, v_{j+1}), (u_{k+2}, v_{j+1})\},$$

then, we have

$$|S'| = |S| - 2,$$
$$m(P_n \times C_m - S') = m(P_n \times C_m - S) = 2,$$
$$\omega(P_n \times C_m - S') = \omega(P_n \times C_m - S) + 2.$$

So we have

$$\begin{aligned}&\omega(P_n \times C_m - S') - |S'| - m(P_n \times C_m - S')\\ &= \omega(P_n \times C_m - S) + 2 - |S| + 2 - m(P_n \times C_m - S)\\ &> \omega(P_n \times C_m - S) - |S| - m(P_n \times C_m - S),\end{aligned}$$

a contradiction to the definition of rupture degree of graphs.

(c) If all K_2 in $P_n \times C_m - S$ are located on nonadjacent cycles, and there exists a K_2 whose position is different from that of others, we denotes this K_2 as $\{(u_k, v_i), (u_k, v_{i+1})\}$, $0 < k < n$, $1 \leq i \leq m$. And we let the other K_2 be $\{(u_h, v_j), (u_h, v_{j+1})\}$, ($1 \leq k \leq n$, and $h \neq k$, $1 \leq j \leq m$, $j \neq i$). Now, let

$$S' = S \cup \{(u_k, v_j)\} - \{(u_{k-1}, v_i), (u_k, v_{i-1}),$$
$$(u_k, v_j), (u_{k+1}, v_i)\},$$

or

$$S' = S \cup \{(u_k, v_{i+1})\} - \{(u_{k-1}, v_{i+1}), (u_k, v_{i+2}),$$
$$(u_k, v_j), (u_{k+1}, v_{i+1})\},$$

or

$$S' = S \cup \{(u_k, v_{i+1})\} - \{(u_{k-1}, v_{i+1}), (u_k, v_{i+2}),$$
$$(u_k, v_{j+1}), (u_{k+1}, v_{i+1})\},$$

then, we have

$$|S'| = |S| - 3,$$
$$m(P_n \times C_m - S') = m(P_n \times C_m - S) = 2,$$

$$\omega(P_n \times C_m - S') = \omega(P_n \times C_m - S) + 3,$$

so, we have

$$\omega(P_n \times C_m - S') - |S'| - m(P_n \times C_m - S')$$

$$= \omega(P_n \times C_m - S) + 6 - |S| - m(P_n \times C_m - S)$$

$$> \omega(P_n \times C_m - S) - |S| - m(P_n \times C_m - S).$$

This also contradicts to the definition of rupture degree of graphs.

So, by cases (a), (b), (c), we know that all K_2 in $P_n \times C_m - S$ must be located on nonadjacent cycles, hence, there exist at most $\lceil \frac{n}{2} \rceil$ number of K_2 in $P_n \times C_m - S$. The proof is then completed. □

Lemma 9 [9] *If* $1 \le m \le n$, *then* $r(K_{m,n}) = n - m - 1$.

Lemma 10 [10] *If* C_n *is an* n $(n \ge 3)$ *cycle, then*

$$r(C_n) = \begin{cases} -1, & \text{if } n \text{ is even} \\ -2, & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 11 [10] *If* H *is a spanning subgraph of graph* G, *then* $r(G) \le r(H)$.

Theorem 12 *The rupture degree of the Cartesian product of a path and a cycle* $P_n \times C_m$ *is as follows*

$$r(P_n \times C_m) = \begin{cases} -1, & \text{if } m \text{ is even} \\ -\frac{n+3}{2}, & \text{if } m \text{ and } n \text{ are both odd} \\ -\frac{n+4}{2}, & \text{if } m \text{ is odd and } n \text{ is even.} \end{cases}$$

Proof. We distinguish the following three cases:

Case 1. If m is even and n is any positive integer. So, on one hand, by proposition 3 we know that $P_n \times C_m$ contains a hamiltonian cycle C_{mn}, then by Lemma 11 we know that

$$r(P_n \times C_m) \le r(C_{mn}).$$

For m is even, by Lemma 10, $r(C_{mn}) = -1$ and so

$$r(P_n \times C_m) \le -1.$$

On the other hand, by proposition 4 we know that $P_n \times C_m$ is a bipartite graph, and

$$P_n \times C_m \subseteq K_{\frac{mn}{2}, \frac{mn}{2}}.$$

so, by Lemma 9 and Lemma 11 we have

$$r(P_n \times C_m) \ge r(K_{\frac{mn}{2}, \frac{mn}{2}}) = -1.$$

So, in this case, $r(P_n \times C_m) = -1$.

Case 2. If m and n are both odd.

On one hand, let

$$S' = \left\{ (u_i, v_j) \;\middle|\; \begin{array}{l} 1 \le i \le n; 1 \le j \le m; \\ \text{if i is odd, then j is even;} \\ \text{if i is even, then j is odd} \end{array} \right\},$$

then

$$|S| = \frac{m-1}{2} \times \frac{n+1}{2} + \frac{m+1}{2} \times \frac{n-1}{2} = \frac{mn-1}{2},$$

$$\omega(P_n \times C_m - S) = \frac{n(m-1)}{2},$$

$$m(P_n \times C_m - S) = 2.$$

So we have

$$r(P_n \times C_m) \ge$$

$$\omega(P_n \times C_m - S) - |S| - m(P_n \times C_m - S) =$$

$$\frac{n(m-1)}{2} - \frac{mn-1}{2} - 2 = -\frac{n+3}{2}.$$

On the other hand, by proposition 8 we know that there exist at most $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ number of K_2 in $P_n \times C_m - S$, and the other components are all K_1. We have

$$|S| \ge \lceil \frac{mn - \frac{n+1}{2} 2}{2} \rceil = \lceil \frac{mn - n - 1}{2} \rceil.$$

But, in order to get at most $\frac{n+1}{2}$ number of K_2, we must delete at least $\frac{n-1}{2}$ number of K_2 from $P_n \times C_m$, and so

$$|S| \ge \lceil \frac{mn - \frac{n+1}{2} 2}{2} \rceil + \frac{n-1}{2} = \frac{mn-1}{2},$$

and

$$\omega(P_n \times C_m - S) \le \frac{n+1}{2} + \lfloor \frac{mn - \frac{n+1}{2} 2}{2} \rfloor - \frac{n-1}{2},$$

$$m(P_n \times C_m - S) = 2,$$

so we have

$$r(P_n \times C_m) \le \frac{mn-n}{2} - \frac{mn-1}{2} - 2 = -\frac{n+3}{2}.$$

In this case, $r(P_n \times C_m) = -\frac{n+3}{2}$.

Case 3. If m is odd and n is even.

On one hand, let

$$S' = \left\{ (u_i, v_j) \;\middle|\; \begin{array}{l} 1 \le i \le n; 1 \le j \le m; \\ \text{if i is odd, then j is odd;} \\ \text{if i is even, then j is even} \end{array} \right\},$$

then

$$|S| = \frac{mn}{2},$$

$$\omega(P_n \times C_m - S) = \frac{n(m-1)}{2},$$

$$m(P_n \times C_m - S) = 2.$$

We have

$$r(P_n \times C_m) \geq$$

$$\omega(P_n \times C_m - S) - |S| - m(P_n \times C_m - S)$$

$$= \frac{n(m-1)}{2} - \frac{mn}{2} - 2 = -\frac{n+4}{2}.$$

On the other hand, by proposition 8 we know that there exist at most $\lceil \frac{n}{2} \rceil = \frac{n}{2}$ number of K_2 in $P_n \times C_m - S$, and the other components are all K_1. So we have

$$|S| \geq \lceil \frac{mn - \frac{n}{2}2}{2} \rceil = \lceil \frac{mn-n}{2} \rceil,$$

However, in order to get at most $\frac{n}{2}$ number of K_2, we must delete at least $\frac{n}{2}$ number of K_2 from $P_n \times C_m$. Therefore,

$$|S| \geq \lceil \frac{mn-n}{2} \rceil + \frac{n}{2} = \frac{mn}{2},$$

and

$$\omega(P_n \times C_m - S) \leq \frac{n}{2} + \lfloor \frac{mn - \frac{n}{2}2}{2} \rfloor - \frac{n}{2} = \frac{n(m-1)}{2},$$

$$m(P_n \times C_m - S) = 2,$$

so we have

$$\begin{array}{rcl} r(P_n \times C_m) & \leq & \frac{mn-n}{2} - \frac{mn}{2} - 2 \\ & = & -\frac{n+4}{2}. \end{array}$$

So, in this case, $r(P_n \times C_m) = -\frac{n+4}{2}$. The proof is then complete. □

3 Rupture degree of total graphs

The total graph $T(G)$ of a graph G is a graph such that the vertex set of $T(G)$ corresponds to the vertices and edges of G and two vertices are adjacent in $T(G)$ if and only if their corresponding elements are either adjacent or incident in G. It is easy to see that $T(G)$ always contains both G and Line graph $L(G)$ as a induced subgraphs. Total graph is the largest graph that is formed by the adjacent relations of elements of a graph. It is highly recommended for the design of interconnection networks. In [7], P. Dündar and A. Aytaç determined the integrity of Total graphs via some parameters. In this section, we give formulas for rupture degree of total graph of path P_n, cycle C_n, and star $S_{1,n}$.

Definition 13 *[2] A subset S of V is called an independent set of G if no two vertices of S are adjacent in G. An independent set S is a maximum if G has no independent set S' with $|S'| > |S|$. The independence number of G, $\beta(G)$, is the number of vertices in a maximum independent set of G.*

Definition 14 *[2] A subset S of V is called a covering of G if every edge of G has at least one end in S. A covering S is a minimum covering if G has no covering S' with $|S'| > |S|$. The covering number, $\alpha(G)$, is the number of vertices in a minimum covering of G.*

Firstly, we will study rupture degree of $T(P_n)$, the total graph of P_n.

Lemma 15 *[7] Let $T(P_n)$ be the total graph of P_n, then the independence number of $T(P_n)$ is $\beta(T(P_n)) = \lceil \frac{2n-1}{3} \rceil$.*

Lemma 16 *[7] Let $T(P_n)$ be the total graph of P_n, then the covering number of $T(P_n)$ is $\alpha(T(P_n)) = \lceil \frac{2(2n-1)}{3} \rceil$.*

Theorem 17 *Let $T(P_n)$ be the total graph of P_n with order n, then the rupture degree of $T(P_n)$ is*

$$r(T(P_n)) = \max\{2\lceil \tfrac{2n-1}{3} \rceil - 2n, \lceil \tfrac{n}{2} \rceil - n - 1\}.$$

Proof. The number of vertices, the independence number and the covering number of $T(P_n)$ are $|V(T(P_n))| = 2n-1$, $\beta(T(P_n)) = \lceil \frac{2n-1}{3} \rceil$, $\alpha(T(P_n)) = \lceil \frac{2(2n-1)}{3} \rceil$ respectively. Let S be an R-set of $T(P_n)$, then $m(T(P_n) - S) \leq 2$, for, otherwise, if there is a component G_i of $T(P_n) - S$ such that $|G_i| = m \geq 3$. Then we select a cut-set S_i of G_i and let the order of every component of $G_i - S_i$ is not larger than 2. Clearly, $|S_i| \leq 2\lfloor \frac{m+1}{4} \rfloor$ and $\omega(G_i - S_i) \geq \lfloor \frac{m+1}{4} \rfloor$. Thus we get a new R-set $S' = S_i \cup S$. Since $m \geq 2 + 2\lfloor \frac{m+1}{4} \rfloor$. Then we get that

$$\omega(T(P_n) - S') - |S'| - m(T(P_n) - S')$$
$$\geq \begin{cases} \omega(T(P_n) - S) - \lfloor \frac{m+1}{4} \rfloor - |S| - 2 \\ = \omega(T(P_n) - S) - |S| - m(T(P_n) - S), \\ if\ m = 3 \\ \omega(T(P_n) - S) - \lfloor \frac{m+1}{4} \rfloor - |S| - 3 \\ > \omega(T(P_n) - S) - |S| - m(T(P_n) - S), \\ if\ m > 3 \end{cases}$$

This contradicts to S is a R-set of $T(P_n)$. Therefore $m(T(P_n) - S) \leq 2$. The following we distinguish two cases to complete the proof.

Case 1 If $m(T(P_n) - S) = 1$. Clearly, $|S| = \alpha(T(P_n))$ and $\omega(T(P_n) - S) = \beta(T(P_n))$ Thus by Lemmas 3.1 and 3.2 we get

$$r(T(P_n) = \omega(T(P_n) - S) - |S| - m(T(P_n) - S)$$

$$= \beta(T(P_n)) - \alpha(T(P_n)) - 1 = 2\lceil\frac{2n-1}{3}\rceil - 2n$$

Case 2 If $m(T(P_n) - S) = 2$. Clearly, $|S| = n - 1$ and $\omega(T(P_n) - S) = \lceil\frac{n}{2}\rceil$, hence

$$r(T(P_n) = \omega(T(P_n) - S) - |S| - m(T(P_n) - S)$$

$$= \lceil\frac{n}{2}\rceil - n - 1.$$

Therefore

$$r(G) = max\{2\lceil\frac{2n-1}{3}\rceil - 2n, \lceil\frac{n}{2}\rceil - n - 1\}.$$

In the next, we will give the formula for computing rupture degree of $T(C_n)$, the total graph of cycle C_n.

Lemma 18 *[7] Let $T(C_n)$ is the total graph of C_n, then the independence number of $T(C_n)$ is $\beta(T(C_n)) = \lceil\frac{2n}{3}\rceil$.*

Lemma 19 *[7] Let $T(C_n)$ is the total graph of C_n, then the covering number of $T(C_n)$ is $\alpha(T(C_n)) = \lceil\frac{4n}{3}\rceil$.*

Theorem 20 *Let $T(C_n)$ be the total graph of C_n with order n, then the rupture degree of $T(C_n)$ is*

$$r(T(C_n)) = \begin{cases} \max\{2\lceil\frac{2n}{3}\rceil - 2n - 1, \frac{n}{2} - n - 2\}, \\ \textit{if } n \textit{ is even} \\ \max\{2\lceil\frac{2n}{3}\rceil - 2n - 1, \lfloor\frac{n}{2}\rfloor - n - 3\}, \\ \textit{if } n \textit{ is odd} \end{cases}$$

Proof. The number of vertices, the independence number and the covering number of $T(C_n)$ are $|V(T(C_n))| = 2n$, $\beta(T(C_n)) = \lceil\frac{2n}{3}\rceil$, $\alpha(T(C_n)) = \lceil\frac{4n}{3}\rceil$ respectively. Let S be an R-set of $T(C_n)$, then similar to Theorem 17, $m(T(C_n) - S) \le 2$. The following we distinguish two cases to complete the proof.

Case 1 If n is even.

Subcase 1.1 If $m(T(C_n) - S) = 1$. Clearly, $|S| = \alpha(T(C_n))$ and $\omega(T(C_n) - S) = \beta(T(C_n))$. Thus by Lemmas 18 and 19, we get

$$\begin{aligned} r(T(C_n) &= \omega(T(C_n)-S)-|S|-m(T(C_n)-S) \\ &= \beta(T(C_n)) - \alpha(T(C_n)) - 1 \\ &= \lceil\tfrac{2n}{3}\rceil - 2n - 1. \end{aligned}$$

Subcase 1.2 If $m(T(C_n) - S) = 2$. Clearly, $|S| = n$ and $\omega(T(C_n) - S) = \frac{n}{2}$, hence

$$r(T(C_n) = \omega(T(C_n) - S) - |S| - m(T(C_n) - S)$$

$$= \frac{n}{2} - n - 2.$$

Case 2 If n is odd. The proof is similar to that of Case 1.

Hence

$$r(T(C_n)) = \begin{cases} \max\{2\lceil\frac{2n}{3}\rceil - 2n - 1, \frac{n}{2} - n - 2\}, \\ \text{if } n \text{ is even} \\ \max\{2\lceil\frac{2n}{3}\rceil - 2n - 1, \lfloor\frac{n}{2}\rfloor - n - 3\}, \\ \text{if } n \text{ is odd} \end{cases}$$

In the following, we will give the rupture degree of the total graph of star $S_{1,n}$. Firstly, we give three necessary lemmas.

Lemma 21 *Let G be an incomplete connected graph of order n, $\beta(G)$ is the independence number of G and $T(G)$ is the tenacity of G, then we have*

$$2\beta(G) - n - 1 \le r(G) \le \beta(G)(1 - T(G)).$$

Proof. On one hand, let S_1 be a largest independent set of G. Then $|S_1| = \beta(G)$, and it is easily seen that $S' = V \setminus S_1$ is a vertex cut set of G, $m(G - S') = 1$ and $\omega(G - S') = \beta(G)$. Hence

$$\begin{aligned} r(G) &= \omega(G - S) - |S| - m(G - S) \\ &\ge \omega(G - S') - |S'| - m(G - S') \\ &= 2\beta(G) - n - 1. \end{aligned}$$

On the other hand, we suppose that S is an R-set of G. Then, by the definition, we have

$$r(G) = \omega(G - S) - |S| - m(G - S),$$

where $\omega(G - S) \ge 2$. So we have

$$\frac{r(G)}{\omega(G - S)} = 1 - \frac{|S| + m(G - S)}{\omega(G - S)}.$$

By the definition of tenacity, we know that

$$\frac{|S| + m(G - S)}{\omega(G - S)} \ge T(G).$$

On the other hand, it is obvious that $\omega(G - S) \le \beta(G)$. So we have $r(G) \le \beta(G)(1 - T(G))$. □

Lemma 22 *[7] Let $T(S_{1,n})$ be the total graph of $S_{1,n}$, then the independence number of $T(S_{1,n})$ is $\beta(T(S_{1,n})) = n$.*

Lemma 23 *[1] Let $T(S_{1,n})$ be the total graph of star $S_{1,n}$. Then the tenacity of $T(S_{1,n})$ is*

$$T(T(S_{1,n})) = \frac{n+2}{n}.$$

Theorem 24 *Let $T(S_{1,n})$ be the total graph of star $S_{1,n}$. Then the rupture degree of $T(S_{1,n})$ is*

$$r(T(S_{1,n})) = -2.$$

Proof. The number of vertices, the independence number of $T(S_{1,n})$ are $V(T(S_{1,n})) = 2n+1$, $\beta(T(S_{1,n})) = n$, respectively. Then by Lemma 21 we have

$$-2 \leq r(T(S_{1,n})).$$

On the other hand, by Lemmas 21, 22 and 23, we have

$$\begin{aligned} r(T(S_{1,n})) &\leq \beta(T(S_{1,n}))(1 - T(T(S_{1,n}))) \\ &= n(1 - \tfrac{n+2}{n}) = -2. \end{aligned}$$

So, we have $r(T(S_{1,n})) = -2$. □

4 Rupture degree of powers of graphs

For an integer $k \geq 1$, the k-th power of a graph G, denoted by G^k, is a supergraph with $V(G^k) = V(G)$ and $E(G^k) = \{(u,v) : u, v \in V(G), u \neq v$ and $d_G(u,v) \leq k\}$. The second power of a graph is also called its square.

Remark 25 *We notice that G^1 is just G itself. So, we let $k \geq 2$ in the following.*

As a useful network, power of cycles and paths have arouse interests for many network designers. C. A. Barefoot, et al. gave the exact values of integrity of powers of cycles in [3], and determined the connectivity, binding number and toughness of powers of cycles [4]. Vertex-neighbor-integrity of powers of cycles were studied in [6] by M. B. Cozzens and Shu-Shih Y. Wu. In [12] Dara Moazzami gave the exact values for the tenacity of powers of cycles. X. K. Zhang and C. M. Yang [11] studied the binding number of the Powers of Paths and cycles.

In this section, we consider the problem of computing the rupture degree of powers of cycles and paths.

Firstly, we determine the rupture degree of powers of cycles.

It is easy to see that $C_n^k \cong K_n$ if $n \leq 2k+1$. So, in the following lemmas, we assume that $2 \leq k < \frac{n-1}{2}$.

Lemma 26 *If S is a minimal R-set for the graph C_n^k, $2 \leq k < \frac{n-1}{2}$, then S consists of the union of sets of k consecutive vertices such that there exists at least one vertex not in S between any two sets of consecutive vertices in S.*

Proof. We assume that the vertices of C_n^k are labeled by $0, 1, 2, \cdots, n-1$. Let S be a minimal R-set of C_n^k and j be the smallest integer such that $T = \{j, j+1, \cdots, j+t-1\}$ is a maximum set of consecutive vertices such that $T \subseteq S$. Relabel the vertices of C_n^k as $v_1 = j, v_2 = j+1, \cdots, v_t = j+t-1, \cdots, v_n = j-1$. Since $S \neq V(C_n^k)$ and $T \neq V(C_n^k)$, v_n does not belong to S. Since S must leave at least two components of $G - S$, we have $t \neq n-1$, and so $v_{t+1} \neq v_n$. Therefore, $\{v_{t+1}, v_n) \cap S = \emptyset$. Now suppose $t < k$. Choose v_i such that $1 \leq i \leq t$, and delete v_i from S yielding a new set $S' = S - \{v_i\}$ with $|S'| = |S| - 1$. By the definition of C_n^k $(1 \leq k \leq \frac{n}{2})$ we know that the edges $v_i v_n$ and $v_i v_{t+1}$ are in $C_n^k - S'$. Consider a vertex v_p adjacent to v_i in $C_n^k - S'$. If $p \geq t+1$, then $p < t+k$. So, v_p is also adjacent to v_{t+1} in $C_n^k - S'$. If $p < n$, then $p \geq n-k+1$ and v_p is also adjacent to v_n in $C_n^k - S'$. Since $t < k$, then v_n and v_{t+1} are adjacent in $C_n^k - S'$. Therefore, we can conclude that deleting the vertex v_i from S does not change the number of components, and so $\omega(C_n^k - S') = \omega(C_n^k - S)$ and $m(C_n^k - S') \leq m(C_n^k - S) + 1$. Thus, we have

$$\begin{aligned} &\omega(C_n^k - S') - |S'| - m(C_n^k - S') \\ &\geq \omega(C_n^k - S) - |S| + 1 - m(C_n^k - S) - 1 \\ &= \omega(C_n^k - S) - |S| - m(C_n^k - S) \\ &= r(C_n^k). \end{aligned}$$

This is contrary to our choice of S. Thus we must have $t \geq k$. Now suppose $t > k$. Delete v_t from the set S yielding a new set $S_1 = S - \{v_t\}$. Since $t > k$, the edge $v_t v_n$ is not in $C_n^k - S_1$. Consider a vertex v_p adjacent to v_t in $C_n^k - S_1$. Then, $p \geq t+1$ and $p \leq t+k$, and so v_p is also adjacent to v_{t+1} in $C_n^k - S_1$. Therefore, deleting v_t from S yields $\omega(C_n^k - S_1) = \omega(C_n^k - S)$ and $m(C_n^k - S_1)$=$m(C_n^k - S) + 1$. So,

$$\begin{aligned} &\omega(C_n^k - S_1) - |S_1| - m(C_n^k - S_1) \\ &\geq \omega(C_n^k - S) - |S| + 1 - m(C_n^k - S) - 1 \\ &= \omega(C_n^k - S) - |S| - m(C_n^k - S) \\ &= r(C_n^k), \end{aligned}$$

which is again contrary to our choice of S. Thus, $t = k$, and so S consists of the union of sets of exactly k consecutive vertices. □

Lemma 27 *There is an R-set S for the graph C_n^k, $2 \leq k < \frac{n-1}{2}$, such that all components of $C_n^k - S$ have order $m(C_n^k - S)$ or $m(C_n^k - S) - 1$.*

Proof. Among all R-sets of minimum order, consider those sets with maximum number of minimum order components, and we let s denote the order of a minimum component. Among these sets, let S be one with the fewest components of order s in C_n^k. Suppose $s \leq m(C_n^k - S) - 2$. Note that all of the components must be sets of consecutive vertices. Assume that C_p is a smallest component. Then $|V(C_p)| = s$, and without loss of generality, let $C_p = \{v_1, v_2, \cdots, v_s\}$. Suppose C_e is a largest component, and so $|V(C_e)| = m(C_n^k - S) = m$ and let $C_e = \{v_j, v_{j+1}, \cdots, v_{j+m-1}\}$. Let $C_1, C_2, \cdots, C_a$ be the components with vertices between v_s of C_k and v_j of C_e, such that $|C_i| = p_i$ for $1 \leq i \leq a$, and let $C_i = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{p_i}}\}$. Now we construct the vertex set S' as $S' = S - \{v_{s+1}, v_{1_{p_1+1}}, v_{2_{p_2+1}}, \cdots, v_{a_{p_a+1}}\} \cup \{v_{1_1}, v_{2_2}, \cdots, v_{a_1}, v_j\}$. Therefore, $|S'| = |S|$, $m(C_n^k - S') \leq m(C_n^k - S)$ and $\omega(C_n^k - S') = \omega(C_n^k - S)$. So we have

$$\omega(C_n^k - S') - |S'| - m(C_n^k - S')$$
$$\geq \omega(C_n^k - S) - |S| - m(C_n^k - S).$$

Therefore,

$$\omega(C_n^k - S') - |S'| - m(C_n^k - S')$$
$$= \omega(C_n^k - S) - |S| - m(C_n^k - S).$$

But, $C_n^k - S'$ has one less components of order s than $C_n^k - S$, a contradiction. Thus, all components of $C_n^k - S$ have order $m(C_n^k - S)$ or $m(C_n^k - S) - 1$. So, $m(C_n^k - S) = \lceil \frac{n-k\omega}{\omega} \rceil$. □

By the above two lemmas we give the exact values for rupture degrees of the powers of cycles.

Theorem 28 *Let C_n^k be k-th ($k \geq 2$)power of a cycle and $n = r(k+1) + s$ for $0 \leq s < k+1$. Then*

$$r(C_n^k) = \begin{cases} 1-n, & if\, n \leq 2k+1; \\ 2-k-\lceil \frac{p}{2} \rceil, & if\, 2k+1 < n \leq 8(k-1); \\ \max\{\underline{\omega} - \underline{\omega}k - \lceil \frac{n-k\underline{\omega}}{\underline{\omega}} \rceil \overline{\omega} - \overline{\omega}k - \lceil \frac{n-k\overline{\omega}}{\overline{\omega}} \rceil\} \\ if\ n > 8(k-1) \end{cases}$$

where $\underline{\omega} = \lfloor \sqrt{\frac{n}{k-1}} \rfloor$, $\overline{\omega} = \lceil \sqrt{\frac{n}{k-1}} \rceil$.

Proof. If $n \leq 2k+1$, then $C_n^k = K_n$, so, $r(C_n^k) = 1-n$. If $n > 2k+1$, let S be a minimum R-set of C_n^k. By Lemmas 2.1 and 2.2 we know that $|S| = k\omega$ and $m(C_n^k - S) = \lceil \frac{n-r\omega}{\omega} \rceil$. Thus, from the definition of rupture degree we have

$$r(C_n^k) = max\{\omega - k\omega - \lceil \frac{n-k\omega}{\omega} \rceil | 2 \leq \omega \leq k\}.$$

Now we consider the function

$$f(\omega) = \omega - k\omega - \lceil \frac{n-k\omega}{\omega} \rceil.$$

It is easy to see that $f'(\omega) = 1 - k - \lceil \frac{-n}{\omega^2} \rceil = \lceil \frac{(1-k)\omega^2+n}{\omega^2} \rceil$. Since $\omega^2 > 0$, we have $f'(\omega) \geq 0$ if and only if $g(\omega) = (1-k)\omega^2 + n \geq 0$. Since the two roots of the equation $g(\omega) = (1-k)\omega^2 + n = 0$ are $\omega_1 = -\sqrt{\frac{n}{k-1}}$ and $\omega_2 = \sqrt{\frac{n}{k-1}}$. But $\omega_1 < 0$, and so it is deleted. Then if $0 < \omega \leq \lfloor \omega_2 \rfloor$, we have $f'(\omega) \geq 0$, and so $f(\omega)$ is an increasing function; if $\omega \geq \lceil \omega_2 \rceil$, then $f'(\omega) \leq 0$, and so $f(\omega)$ is a decreasing function. Thus, we have the following cases:

Case 1 If $n \leq 8(k-1)$, then $\lfloor \omega_2 \rfloor \leq 2$. Since we know that $2 \leq \omega \leq r$, we have that $f(\omega)$ is a decreasing function and the maximum value occurs at the boundary. So, $\omega = 2$ and $r(C_n^k) = f(2) = 2 - r - \lceil \frac{n}{2} \rceil$.

Case 2 If $n > 8(k-1)$, then $\lfloor \omega_2 \rfloor \geq 2$. So, we have

Subcase 2.1 If $2 \leq \omega \leq \lfloor \omega_2 \rfloor$, then $f(\omega)$ is an increasing function.

Subcase 2.2 If $\lceil \omega_2 \rceil \leq \omega \leq r$, then $f(\omega)$ is a decreasing function.

Thus the maximum value occurs when $\omega = \lfloor \omega_2 \rfloor$ or $\omega = \lceil \omega_2 \rceil$. Then, on this condition, $r(C_n^k) = f(\lfloor \omega_2 \rfloor) = max\{\underline{\omega} - \underline{\omega}k - \lceil \frac{n-k\underline{\omega}}{\underline{\omega}} \rceil, \overline{\omega} - \overline{\omega}k - \lceil \frac{n-k\overline{\omega}}{\overline{\omega}} \rceil\}$, where $\underline{\omega} = \lfloor \sqrt{\frac{n}{k-1}} \rfloor$, $\overline{\omega} = \lceil \sqrt{\frac{n}{k-1}} \rceil$.

The proof is now complete. □

In the next theorem, we shall consider the problem of computing the rupture degree of powers of paths.

It is easy to see that $P_n^k \cong K_n$ if $n \leq k+1$. So, in the following lemmas, we suppose that $2 \leq k \leq n-2$.

Lemma 29 *If S is a minimal R-set for the graph P_n^k, $2 \leq k \leq n-2$, then S consists of the union of sets of k consecutive vertices such that there exists at least one vertex not in S between any two sets of consecutive vertices in S.*

Proof. We assume that the vertices of P_n^k are labeled by $v_1, v_2, \cdots, v_n$. Let S be a minimal R-set of P_n^k and j be the smallest integer such that $T = \{v_j, v_{j+1}, \cdots, v_{j+t-1}\}$ is a maximum set of consecutive vertices such that $T \subseteq S$. Since $S \neq V(P_n^k)$ and $T \neq V(P_n^k)$, then $v_{j-1} \notin S$. Since S must leave at least two components of $G - S$, we have $v_{j+t} \notin S$, and so $v_{j+t} \neq v_{j-1}$. Therefore, $\{v_{j+t}, v_{j-1}) \cap S = \emptyset$. Now suppose $t < k$. Choose v_{j+i} such that $1 \leq i \leq t$, and delete v_{j+i} from S yielding a new set $S^{'} = S - \{v_{j+i}\}$ with $|S^{'}| = |S| - 1$. By the definition of P_n^k ($2 \leq k < \frac{n-1}{2}$) we know that the edges $v_{j+i}v_{j-1}$ and $v_{j+i}v_{j+t}$ are in $P_n^k - S^{'}$. Consider a vertex v_p adjacent to v_{j+i} in $P_n^k - S^{'}$. If $p \geq t+j+1$, then $p < t+j+k$. So, v_p is also adjacent to v_{t+j} in $P_n^k - S^{'}$. If $p < j-1$, then $p \geq j-k$ and v_p is also adjacent to v_{j-1} in $P_n^k - S^{'}$. Since $t < k$, then v_{j-1} and v_{j+t} are adjacent in $P_n^k - S^{'}$. Therefore, we can conclude that deleting the vertex v_{j+i} from S does not change the number of components, and so $\omega(P_n^k - S^{'}) = \omega(P_n^k - S)$ and $m(P_n^k - S^{'}) \leq m(P_n^k - S) + 1$. Thus, we have

$$\begin{aligned}&\omega(P_n^k - S^{'}) - |S^{'}| - m(P_n^k - S^{'})\\ &\geq \omega(P_n^k - S) - |S| + 1 - m(P_n^k - S) - 1\\ &= \omega(P_n^k - S) - |S| - m(P_n^k - S)\\ &= r(P_n^k).\end{aligned}$$

This is contrary to our choice of S. Thus we must have $t \geq k$. Now suppose $t > k$. Delete v_{j+t-1} from the set S yielding a new set $S_1 = S - \{v_{j+t-1}\}$. Since $t > k$, the edge $v_{j+t-1}v_{j-1}$ is not in $P_n^k - S_1$. Consider a vertex v_p adjacent to v_{j+t-1} in $P_n^k - S_1$. Then, $p \geq j+t$ and $p \leq j+t+k-2$, and so v_p is also adjacent to v_{j+t} in $P_n^k - S_1$. Therefore, deleting v_{j+t-1} from S yields $\omega(P_n^k - S_1) = \omega(P_n^k - S)$ and $m(P_n^k - S_1)$=$m(P_n^k - S) + 1$. So,

$$\begin{aligned}&\omega(P_n^k - S_1) - |S_1| - m(P_n^k - S_1)\\ &\geq \omega(P_n^k - S) - |S| + 1 - m(P_n^k - S) - 1\\ &= \omega(P_n^k - S) - |S| - m(P_n^k - S)\\ &= r(P_n^k),\end{aligned}$$

which is again contrary to our choice of S. Thus, $t = k$, and so S consists of the union of sets of exactly k consecutive vertices. $\square$

Lemma 30 *There is an R-set S for the graph P_n^k, $2 \leq k \leq n-2$, such that all components of $P_n^k - S$ have order $m(P_n^k - S)$ or $m(P_n^k - S) - 1$.*

Proof. Among all R-sets of minimum order, consider those sets with maximum number of minimum order components, and we let s denote the order of a minimum component. Among these sets, let S be one with the fewest components of order s in P_n^k. Suppose $s \leq m(P_n^k - S) - 2$. Note that all of the components must be sets of consecutive vertices. Assume that C_p is a smallest component. Then $|V(C_p)| = s$, and let $C_p = \{v_1, v_2, \cdots, v_s\}$. Suppose C_e is a largest component, and without loss of generality, we assume that C_e is on the right side of C_p, so $|V(C_e)| = m(P_n^k - S) = m$ and let $C_e = \{v_j, v_{j+1}, \cdots, v_{j+m-1}\}$. Let $C_1, C_2, \cdots, C_a$ be the components with vertices between v_s of C_k and v_j of C_e, such that $|C_i| = p_i$ for $1 \leq i \leq a$, and let $C_i = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{p_i}}\}$. Now we construct the vertex set $S^{'}$ as $S^{'} = S - \{v_{s+1}, v_{1_{p_1+1}}, v_{2_{p_2+1}}, \cdots, v_{a_{p_a+1}}\} \cup \{v_{1_1}, v_{2_2}, \cdots, v_{a_1}, v_j\}$. Therefore, $|S^{'}| = |S|$, $m(P_n^k - S^{'}) \leq m(P_n^k - S)$ and $\omega(P_n^k - S^{'}) = \omega(P_n^k - S)$. So we have

$$\begin{aligned}&\omega(P_n^k - S^{'}) - |S^{'}| - m(P_n^k - S^{'})\\ &\geq \omega(P_n^k - S) - |S| - m(P_n^k - S).\end{aligned}$$

Therefore,

$$\begin{aligned}&\omega(P_n^k - S^{'}) - |S^{'}| - m(P_n^k - S^{'})\\ &= \omega(P_n^k - S) - |S| - m(P_n^k - S).\end{aligned}$$

But, $P_n^k - S^{'}$ has one less components of order s than $P_n^k - S$, a contradiction. Thus, all components of $P_n^k - S$ have order $m(P_n^k - S)$ or $m(P_n^k - S) - 1$. So, $m(P_n^k - S) = \lceil \frac{n-k(\omega-1)}{\omega} \rceil$. $\square$

By the above two lemmas we give the exact values for rupture degrees of the powers of paths.

Theorem 31 *Let P_n^k be the k-th ($k \geq 2$)power of a path P_n, $n = r(k+1) + s$ for $0 \leq s < k+1$. Then*

$$r(P_n^k) = \begin{cases} 1-n, \quad \text{if } n \leq k+1; \\ 2-k-\lceil \frac{n-k}{2} \rceil, \\ \text{if } k+1 < n \leq 7k-8; \\ \max\{\underline{\omega} - (\underline{\omega}-1)k - \lceil \frac{n-k(\underline{\omega}-1)}{\underline{\omega}} \rceil \\ \overline{\omega} - (\overline{\omega}-1)k - \lceil \frac{n-k(\overline{\omega}-1)}{\overline{\omega}} \rceil\}, \\ \text{if } n > 7k-8 \end{cases}$$

where $\underline{\omega} = \lfloor \sqrt{\frac{n+k}{k-1}} \rfloor$, $\overline{\omega} = \lceil \sqrt{\frac{n+k}{k-1}} \rceil$.

Proof. Let S be a minimum R-set of P_n^k. By Lemmas 15 and 16 we know that $|S| = k(\omega - 1)$ and $m(P_n^k - S) = \lceil \frac{n-k(\omega-1)}{\omega} \rceil$. Thus, from the definition of rupture degree we have

$$\begin{aligned}&r(P_n^k) =\\ &\max\{\omega - k(\omega-1) - \lceil \tfrac{n-k(\omega-1)}{\omega} \rceil \mid 2 \leq \omega \leq r+1\}.\end{aligned}$$

Now we consider the function

$$f(\omega) = \omega - k(\omega-1) - \lceil \frac{n-k(\omega-1)}{\omega} \rceil.$$

It is easy to see that $f^{'}(\omega) = 1 - k - \lceil \frac{-n-k}{\omega^2} \rceil = \lceil \frac{(1-k)\omega^2+n+k}{\omega^2} \rceil$. Since $\omega^2 > 0$, we have $f^{'}(\omega) \geq 0$ if and only if $g(\omega) = (1-k)\omega^2 + n + k \geq 0$. Since the two roots of the equation $g(\omega) = (1-k)\omega^2+n+k = 0$ are

$$\omega_1 = -\sqrt{\tfrac{n+k}{k-1}} \quad \text{and} \quad \omega_2 = \sqrt{\tfrac{n+k}{k-1}}.$$

But $\omega_1 < 0$, and so it is deleted. Then if $0 < \omega \leq \lfloor\omega_2\rfloor$, we have $f'(\omega) \geq 0$, and so $f(\omega)$ is an increasing function; if $\omega \geq \lceil\omega_2\rceil$, then $f'(\omega) \leq 0$, and so $f(\omega)$ is a decreasing function. Thus, we have the following cases:

Case 1 If $n \leq 7k-8$, then $\lfloor\omega_2\rfloor \leq 2$. Since we know that $2 \leq \omega \leq r+1$, we have that $f(\omega)$ is a decreasing function and the maximum value occurs at the boundary. So, $\omega = 2$ and $r(C_n^k) = f(2) = 2-k-\lceil\frac{n-k}{2}\rceil$.

Case 2 If $n > 7k-8$, then $\lfloor\omega_2\rfloor \geq 2$. So, we have

Subcase 2.1 If $2 \leq \omega \leq \lfloor\omega_2\rfloor$, then $f(\omega)$ is an increasing function.

Subcase 2.2 If $\lceil\omega_2\rceil \leq \omega \leq r+1$, then $f(\omega)$ is a decreasing function.

Thus the maximum value occurs when $\omega = \lfloor\omega_2\rfloor$ or $\omega = \lceil\omega_2\rceil$. Then, $r(P_n^k) = f(\lfloor\omega_2\rfloor) = max\{\underline{\omega} - (\underline{\omega}-1)k - \lceil\frac{n-k(\underline{\omega}-1)}{\underline{\omega}}\rceil, \overline{\omega} - (\overline{\omega}-1)k - \lceil\frac{n-k(\overline{\omega}-1)}{\overline{\omega}}\rceil\}$, where where $\underline{\omega} = \lfloor\sqrt{\frac{n+k}{k-1}}\rfloor$, $\overline{\omega} = \lceil\sqrt{\frac{n+k}{k-1}}\rceil$. The proof is now complete. □

5 Conclusion

If a system such as a communication network is modeled by a graph G, there are many graph theoretical parameters used to describe the stability and reliability of communication networks including connectivity, integrity, toughness, binding number, tenacity and rupture degree. Two ways of measuring the stability of a network is through the ease with which one can disrupt the network, and the cost of a disruption. Connectivity has the least cost as far as disrupting the network, but it does not take into account what remains after disruption. One can say that the disruption is less harmful if the disconnected network contains more components and much less harmful if the affected components are small. One can associate the cost with the number of the vertices destroyed to get small components and the reward with the number of the components remaining after destruction. The rupture degree measure is compromise between the cost and the reward by minimizing the cost: reward ratio. Thus, a network with a small rupture degree performs better under external attack. The results of this paper suggest that rupture degree is a more suitable measure of stability in that it has the ability to distinguish between graphs that intuitively should have different measures of stability. In this paper, we have obtained the exact values for the rupture degree of some special graphs. To make further progress in this direction, one could try to characterize the graphs with given rupture degree.

Acknowledgements: This research is supported by the NSFC(No.10871226). The authors are thankful to anonymous referees for their constructive suggestions and critical comments, which led to this improved version.

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