Parameter Estimation and Cooperative Effects in Queueing Networks

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Abstract: This paper is devoted to probability-statistical analysis of Jackson opened and closed networks. A problem of an estimation of product limit distributions parameters using load coefficients of network nodes is solved. Cooperative effects in aggregated opened and closed networks are investigated and optimization procedures of their limit deterministic characteristics are constructed. Formulas of a calculation of limit distributions in superpositions of networks (nodes are replaced by opened networks) are obtained.

Key–Words: Phase transition, aggregated networks, superposition of networks, product limit distributions, load coefficients, limit characteristics, parameters estimation.

1 Introduction

Queueing networks are widely used in modelling communication and data transmission systems, data processing systems, etc. [1] - [5]. The most important problem of their investigation is to simplify a calculation and an estimation of their characteristics in spite of large number of their elements. The most widely used mathematical tools for solution of this problem are product-form theorems based on solving a system of balance equations. At the same time, there is a large number of papers devoted to estimation of parameters of particular queueing systems [6] - [8] etc. However, it is quite difficult to transfer these methods to estimation of parameters of queueing networks whose modelling is based on product-form theorems. In this paper statistical estimation of closed and opened queueing networks parameters is based on loading factors which define their product-form limiting distributions. This approach allows to get over a solution of large system of balance equations with inaccurately defined coefficients.

Another possibility to simplify stochastic model with large number of elements is to analyze cooperative effects or effects of subsystems aggregation in stochastic system. It plays large role in different applications. The most well known results are devoted to random graphs with identical edges in which working probabilities of edges decrease to zero with increase of nodes number. In this case there are conditions when connectivity probability of random graph tends to one [11] - [13]. Such effects are accompanied by phenomena analogous to phase transitions in physical statistics [14] -[16]. Continuation of these results contains in [17] - [19]. But analogous phenomena may be established in queueing systems and networks also. The first results in this area have been obtained in [20, chapter II, chapter III, § 4] and are devoted to limit distribution of a number of occupied servers. In this paper cooperative effects have been analyzed as in multiserver queueing systems so in closed and in opened queueing networks in terms of stationary waiting times and queue lengths.

Large number of papers is devoted to an optimization of different queueing networks characteristics by their route matrices (see [21] and references therein). Large complexity of a solution of optimal routing problem in queueing networks is closely connected with a choice of the network characteristics. Nevertheless a concept of the ergodicity in opened queueing networks [22] allows to define an ability to handle customers which is closely connected with an ability to handle in deterministic transportation networks and its equality with maximal flow [23]. So suggested in this paper algorithms of the flow maximization in opened queueing network occur similar to a definition of permissible solutions in the transportation problem of the linear programming [24], [25].

2 Parameter estimation for product-form distributions

The first part of this paper is devoted to a construction of a statistical estimation of parameters of a queueing network with exponential service distributions and

Poisson input flow. Main parameters of the considered queueing network are a route matrix, an input flow intensity and service intensities in the network nodes. To estimate these parameters it is necessary to solve a system of balance equations. But for considered networks limit distributions of numbers of customers in network nodes have product form and are defined by load coefficients of different nodes. In this section load coefficients are estimated using estimation of its limit distribution. This approach allows to avoid a solution of the system of balance equations with incorrectly defined coefficients. This approach is realized for opened and closed Jackson networks: classical, in random environment and with some prohibited transitions between nodes.

2.1 Classical Jackson networks

Consider an open Jackson network consisting of m single-server queueing systems with exponential service times of rates $\mu_1, ..., \mu_m$ fed by a Poisson arrival flow of rate λ. The network routing dynamics is given by a routing matrix $\Theta = ||\theta_{ij}||_{i,j=0}^{m}$ where θ_{ij} is the probability of a transition after getting service from the ith node to the jth, $\theta_{00} = 0$, and 0 is an external source. Assume that the routing matrix is irreducible: $\forall i, j \in \{0, 1, ..., m\}$

$$\exists\, i_1, i_2, ..., i_r \in \{1, ..., m\} \ : \theta_{ii_1},\ \theta_{i_1 i_2}, \dots, \theta_{i_r j} > 0.$$

Then there exists a unique solution $\Lambda = (\lambda, \lambda_1, \lambda_2, ..., \lambda_m)$ of the system $\Lambda = \Lambda\Theta$, $\lambda_1, ..., \lambda_m > 0$.

If

$$\rho_i = \frac{\lambda_i}{\mu_i} < 1,\ i = 1, ..., m, \tag{1}$$

then the discrete Markov process $y(t)$ describing the number of customers at the network nodes with the state set $\boldsymbol{N} = \{\boldsymbol{n} = (n_1, ..., n_m),\ n_i \geq 0\}$ and transition rates

$$L(\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}_k) = \lambda\theta_{0k},\ L(\boldsymbol{n} + \boldsymbol{e}_k, \boldsymbol{n}) = \mu_k\theta_{k0},$$

$$L(\boldsymbol{n}+\boldsymbol{e}_k, \boldsymbol{n}+\boldsymbol{e}_i) = \mu_k\theta_{ki},\ 1 \leq k \neq i \leq m,\ \boldsymbol{n} \in \boldsymbol{N},$$

($\boldsymbol{e}_k$ is the vector whose kth component is 1 and all the others are 0) is ergodic [26] and its limiting distribution [1] is given by

$$\pi(\boldsymbol{n}) = C \prod_{i=1}^{m} \rho_i^{n_i},\ \boldsymbol{n} \in \boldsymbol{N},$$

$$C^{-1} = \sum_{\boldsymbol{n} \in \boldsymbol{N}} \prod_{i=1}^{m} \rho_i^{n_i}. \tag{2}$$

Denote by $\varphi = (\rho_1, \dots, \rho_m, C)$ the vector of parameters of the product formula (2).

Since $\pi(\boldsymbol{0}) = C,\ \pi(\boldsymbol{e}_i) = C\rho_i$ for an open Jackson network we obtain formulas expressing the parameter vector φ by the distribution π, $\varphi = f(\pi)$, in the form

$$\rho_i = \frac{\pi(\boldsymbol{e}_i)}{\pi(\boldsymbol{0})},\ i = 1, ..., m,\ C = \pi(\boldsymbol{0}). \tag{3}$$

In turn the probabilities $\pi(\boldsymbol{n}),\ \boldsymbol{n} \in \boldsymbol{N}$, can be estimated by relative frequencies

$$\pi_T(\boldsymbol{n}) = \frac{1}{T} \int_0^T I(y(t) = \boldsymbol{n})\, dt,$$

where $I(A)$ is the indicator function of an event A, since the convergence in probability takes place ([9], [10, chapter 4], [27], [2, theorem 1.2])

$$\pi_T(\boldsymbol{n}) \to \pi(\boldsymbol{n}),\ T \to \infty. \tag{4}$$

Therefore as an estimate for the parameter vector φ we propose to take $f(\pi_T)$ which converges in probability to $\varphi = f(\pi)$ as $T \to \infty$ by virtue of (3) and (4).

A closed Jackson network differs from an open one in the fact that there is a constant number of customers K in it and no customers arrive from the outside. The behavior of customers in the network is described by a routing matrix $\widehat{\Theta} = ||\theta_{ij}||_{i,j=1}^{m}$, the matrix $\widehat{\Theta}$ is assumed to be irreducible. Then a solution of the system

$$\lambda_1 = a,\ \ (\lambda_1, \lambda_2, ..., \lambda_m) = (\lambda_1, \lambda_2, ..., \lambda_m)\widehat{\Theta}$$

for any $a > 0$ exists, is unique and consists of positive components. Operation of such a network (number of customers at the servers) is described by a discrete Markov process $\widehat{y}(t)$ with state set

$$\boldsymbol{N}_K = \{\boldsymbol{n},\ \boldsymbol{n} \in \boldsymbol{N} :\ \sum_{i=1}^{m} n_i = K\}$$

and transition rates

$$L(\boldsymbol{n}, \boldsymbol{n}-\boldsymbol{e}_k+\boldsymbol{e}_i) = \mu_k\theta_{ki},\ \boldsymbol{n} \in \boldsymbol{N}_K,\ 1 \leq k \neq i \leq m.$$

The process $\widehat{y}(t)$ is ergodic [26] and its stationary distribution [1] is given by

$$\pi(\boldsymbol{n}) = C \prod_{i=1}^{m} \rho_i^{n_i},\ \boldsymbol{n} \in \boldsymbol{N}_K,$$

$$C^{-1} = \sum_{\boldsymbol{n} \in \boldsymbol{N}_K} \prod_{i=1}^{m} \rho_i^{n_i}. \tag{5}$$

If $H = C\rho_1^K,\ \varepsilon_i = \rho_i/\rho_1,\ 2 \le i \le m$, then equation (5) can be rewritten in the form

$$\pi(\boldsymbol{n}) = H\prod_{i=2}^{m}\varepsilon_i^{n_i},\ \boldsymbol{n} \in \boldsymbol{N}_K,\ H^{-1} = \sum_{\boldsymbol{n}\in\boldsymbol{N}_K}\prod_{i=2}^{m}\varepsilon_i^{n_i}. \tag{6}$$

Denote by $\widehat{\varphi} = (\varepsilon_2, \dots, \varepsilon_m, H)$ the vector of parameters of the product formula (6). Since

$$\pi(K\boldsymbol{e}_1) = H,\ \pi((K-1)\boldsymbol{e}_1+\boldsymbol{e}_i) = H\varepsilon_i,\ 2 \le i \le m,$$

for a closed Jackson network we obtain formulas expressing the parameter vector $\widehat{\varphi}$ through the distribution π, $\widehat{\varphi} = \widehat{f}(\pi)$, in the form

$$\varepsilon_i = \frac{\pi((K-1)\boldsymbol{e}_1 + \boldsymbol{e}_i)}{\pi(K\boldsymbol{e}_1)},\ i = 2, ..., m, \tag{7}$$

$$H = \pi(K\boldsymbol{e}_1).$$

In turn the probabilities $\pi(\boldsymbol{n}),\ \boldsymbol{n} \in \boldsymbol{N}_K$, can be estimated by relative frequencies $\pi_T(\boldsymbol{n})$, which satisfy the convergence in probability (4) due to [9], [10, chapter 4], [27], [2, theorem 1.2]. Therefore as an estimate for the parameter vector $\widehat{\varphi}$ we propose to take $\widehat{f}(\pi_T)$ which converges in probability to $\widehat{\varphi} = \widehat{f}(\pi)$ as $T \to \infty$ by virtue of (4) and (7).

2.2 Networks in a random environment

Consider an open Jackson network. Denote by S the set of all nonempty subsets s (with elements in ascending order) of the set $\{1, ..., m\}$. Assume that for a given s the network is determined by the characteristics

$$\lambda = \lambda(s),\ \mu_1 = \mu_1(s), ..., \mu_m = \mu_m(s),\ \Theta = \Theta(s).$$

Operation of the network for a given s is described by a Markov process with state set $\boldsymbol{N}$ and transition rates $L_s(\boldsymbol{n}, \boldsymbol{n}^*), \boldsymbol{n}, \boldsymbol{n}^* \in \boldsymbol{N}$, satisfying the conditions:
1) $\exists\ \boldsymbol{s} \in S$ such that

$$\forall\ \boldsymbol{n}, \boldsymbol{n}^* \in \boldsymbol{N}\ \exists\ \boldsymbol{n}^1, \dots, \boldsymbol{n}^q \in \boldsymbol{N}:\ L_{\boldsymbol{s}}(\boldsymbol{n}, \boldsymbol{n}^1) > 0,$$

$$L_{\boldsymbol{s}}(\boldsymbol{n}^1, \boldsymbol{n}^2) > 0, \dots, L_{\boldsymbol{s}}(\boldsymbol{n}^{q-1}, \boldsymbol{n}^q),\ L_{\boldsymbol{s}}(\boldsymbol{n}^q, \boldsymbol{n}^*) > 0,$$

2) $\exists\ D < \infty : \forall\ \boldsymbol{s} \in S,\ \forall\ \boldsymbol{n} \in \boldsymbol{N}$

$$\sum_{\boldsymbol{n}^*\in\boldsymbol{N}} L_{\boldsymbol{s}}(\boldsymbol{n}, \boldsymbol{n}^*) \le D,$$

3) the function $\pi(\boldsymbol{n}),\ \boldsymbol{n} \in \boldsymbol{N}$, given by (2) satisfies the equality

$$\pi(\boldsymbol{n})\sum_{\boldsymbol{n}^*\in\boldsymbol{N}} L_{\boldsymbol{s}}(\boldsymbol{n}, \boldsymbol{n}^*) = \sum_{\boldsymbol{n}^*\in\boldsymbol{N}} \pi(\boldsymbol{n}^*)L_{\boldsymbol{s}}(\boldsymbol{n}^*, \boldsymbol{n}).$$

Let operation of the network with randomly varying s be described by a discrete Markov process $x(t) = (s(t), y(t))$ with state set $X = S \times \boldsymbol{N}$ and transition rates

$$\Lambda((\boldsymbol{s}, \boldsymbol{n}), (\boldsymbol{s}^*, \boldsymbol{n}^*)) =$$

$$= L_{\boldsymbol{s}}(\boldsymbol{n}, \boldsymbol{n}^*)I(\boldsymbol{s} = \boldsymbol{s}^*) + \nu(\boldsymbol{s}, \boldsymbol{s}^*)I(\boldsymbol{n} = \boldsymbol{n}^*) \tag{8}$$

where $\forall\ \boldsymbol{s}, \boldsymbol{s}^* \in S\ \exists\ \boldsymbol{s}^1, \dots, \boldsymbol{s}^l \in S$:

$$\nu(\boldsymbol{s}, \boldsymbol{s}^1), \nu(\boldsymbol{s}^1, \boldsymbol{s}^2), \dots, \nu(\boldsymbol{s}^{l-1}, \boldsymbol{s}^l), \nu(\boldsymbol{s}^l, \boldsymbol{s}^*) > 0$$

and $\exists\ C < \infty : \forall\ \boldsymbol{s} \in S \quad \sum\limits_{\boldsymbol{s}^*\in S}\nu(\boldsymbol{s}, \boldsymbol{s}^*) \le C$. Assume that there exists a function $A(\boldsymbol{s}) > 0$ satisfying the conditions

$$A(\boldsymbol{s})\sum_{\boldsymbol{s}^*\in S}\nu(\boldsymbol{s}, \boldsymbol{s}^*) = \sum_{\boldsymbol{s}^*\in S}A(\boldsymbol{s}^*)\nu(\boldsymbol{s}^*, \boldsymbol{s}),\ \boldsymbol{s} \in S,$$

$$\sum_{\boldsymbol{s}\in S}A(\boldsymbol{s}) = 1.$$

Then the process $x(t)$ is ergodic [28, theorem 1] and its limiting distribution is of the form

$$\Pi(s, \boldsymbol{n}) = A(s)\pi(\boldsymbol{n}),\ (s, \boldsymbol{n}) \in X = S \times \boldsymbol{N}. \tag{9}$$

It is seen from (8) that the term

$$L_{\boldsymbol{s}}(\boldsymbol{n}, \boldsymbol{n}^*)I(\boldsymbol{s} = \boldsymbol{s}^*)$$

entering the transition rate of the Markov process $x(t)$ and which accounts for changes of the component $y(t)$, depends on the component $s(t)$ varying in time. Therefore, strictly speaking, the component $y(t)$, is not a Markov process. However we can use the law of large numbers for the process $x(t)$ to construct an estimate $\pi_T(\boldsymbol{n}) = \sum\limits_{\boldsymbol{s}\in S}\Pi_T(s, \boldsymbol{n})$, which converges in probability as $T \to \infty$ because of finiteness of S and by virtue of (9) to the sum $\sum\limits_{\boldsymbol{s}\in S}\Pi(s, \boldsymbol{n}) = \pi(\boldsymbol{n})$.

Therefore as an estimate for the parameter vector φ of the limiting distribution $\pi(\boldsymbol{n})$ of the process $y(t)$ one can take $f(\pi_T)$ which converges in probability to $\varphi = f(\pi)$ as $T \to \infty$. This provides rather simple estimates for parameters of open queueing networks in a random environment and allows us to transfer the obtained results to open queueing networks with unreliable elements: nodes, paths between nodes, channels at the nodes, and also to closed networks.

2.3 Networks with blocked transitions

We describe an open blocking network with blocked transitions between some states [29, § 3] by assigning to it a connected undirected graph Γ with edges of the form

$$[\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}_k],\ \ \boldsymbol{n} \in \mathcal{L}, \tag{10}$$

$$[\boldsymbol{n}+\boldsymbol{e}_k, \boldsymbol{n}+\boldsymbol{e}_i], \ \ \boldsymbol{n} \in \mathcal{L}, \ \ n_k > 0, \ k \neq i, \quad (11)$$

where $1 \leq i, \ k \leq m, \ \mathcal{L} \subseteq \boldsymbol{N}$. Denote by $\boldsymbol{N}_{\mathcal{L}}$ the set of all vertices of the graph Γ. The number of customers at nodes of a blocking network is described by a Markov process $y(t)$ with the state set $\boldsymbol{N}_{\mathcal{L}}$ and transition rates

$$L(\boldsymbol{n}, \boldsymbol{n}+\boldsymbol{e}_k) = \lambda\theta_{0k}, \ L(\boldsymbol{n}+\boldsymbol{e}_k, \boldsymbol{n}) = \mu_k\theta_{k0},$$

$$L(\boldsymbol{n}+\boldsymbol{e}_k, \boldsymbol{n}+\boldsymbol{e}_i) = \mu_k\theta_{ki}, \ 1 \leq k \neq i \leq m, \ \boldsymbol{n} \in \mathcal{L}.$$

If the routing matrix Θ is irreducible, condition (1) is fulfilled and $0 < \theta_{0k}, \theta_{k0} < 1, \ 1 \leq k \leq m$, then the limiting distribution of the process $y(t)$ is of the

$$\pi(\boldsymbol{n}) = C\prod_{i=1}^{m}\rho_i^{n_i}, \ \boldsymbol{n} \in \boldsymbol{N}_{\mathcal{L}},$$

form

$$C^{-1} = \sum_{\boldsymbol{n}\in \boldsymbol{N}_{\mathcal{L}}}\prod_{i=1}^{m}\rho_i^{n_i}. \quad (12)$$

If $\boldsymbol{n}^* \in \mathcal{L}$ then (12) can be rewritten as

$$\pi(\boldsymbol{n}) = D\prod_{i=1}^{m}\rho_i^{n_i-n_i^*}, \ \boldsymbol{n} \in \boldsymbol{N}_{\mathcal{L}},$$

$$D^{-1} = \sum_{\boldsymbol{n}\in \boldsymbol{N}_{\mathcal{L}}}\prod_{i=1}^{m}\rho_i^{n_i-n_i^*}. \quad (13)$$

Denote by $\Phi = (\rho_1, \ldots, \rho_m, D)$ the vector of parameters of the product formula (13). Then for an open blocking Jackson network we obtain the following formulas expressing the parameter vector Φ through the distribution $\pi, \ \Phi = F(\pi)$:

$$\rho_i = \frac{\pi(\boldsymbol{n}^*+\boldsymbol{e}_i)}{\pi(\boldsymbol{n}^*)}, \ i = 1, ..., m, \quad (14)$$

$$D = \pi^{-1}(\boldsymbol{n}^*).$$

As an estimate for the parameter vector Φ of the product formula (13) we propose to take $F(\pi_T)$ which converges in probability to $\Phi = F(\pi)$ as $T \to \infty$ by virtue of (4), (14).

To describe a closed blocking network with blocked transitions between some states [29, § 3] we assign to it a connected undirected graph $\widehat{\Gamma}$ with edges of the form (11). Here we have $\mathcal{L} \subset \boldsymbol{N}$ and the vertex set of $\widehat{\Gamma}$ is of the form

$$\widehat{\boldsymbol{N}}_{\mathcal{L}} = \{\boldsymbol{n}+\boldsymbol{e}_i, \ 1 \leq i \leq m, \ \boldsymbol{n} \in \mathcal{L}\} \subseteq \boldsymbol{N}_K.$$

The number of customers at nodes of the closed blocking network is described by a Markov process $\widehat{y}(t)$ with state set $\widehat{\boldsymbol{N}}_{\mathcal{L}}$ and transition rates

$$L(\boldsymbol{n}+\boldsymbol{e}_k, \boldsymbol{n}+\boldsymbol{e}_i) = \mu_k\theta_{ki}, \ \boldsymbol{n} \in \mathcal{L}, \ 1 \leq k \neq i \leq m.$$

If the network routing matrix $\widehat{\Theta} = ||\theta_{ki}||_{k,i=1}^{m}$ is irreducible and satisfies the condition $0 < \theta_{ki} < 1$, $1 \leq k, i \leq m$, then the limiting distribution of the process $\widehat{y}(t)$ is given by formula (12).

Let $\boldsymbol{n}^* \in \boldsymbol{N}_{\mathcal{L}}$ then (12) can be rewritten as

$$\pi(\boldsymbol{n}) = G\prod_{i=2}^{m}\varepsilon_i^{n_i-n_i^*}, \ \boldsymbol{n} \in \widehat{\boldsymbol{N}}_{\mathcal{L}},$$

$$G^{-1} = \sum_{\boldsymbol{n}\in \widehat{\boldsymbol{N}}_{\mathcal{L}}}\prod_{i=2}^{m}\varepsilon_i^{n_i-n_i^*}. \quad (15)$$

Denote by $\widehat{\Phi} = (\varepsilon_2, \ldots, \varepsilon_m, G)$ the vector of parameters of the product formula (15). Then for a closed blocking Jackson network we obtain the following formulas expressing the parameter vector $\widehat{\Phi}$ through the distribution $\pi, \ \widehat{\Phi} = \widehat{F}(\pi)$:

$$\varepsilon_i = \frac{\pi(\boldsymbol{n}^*+\boldsymbol{e}_i-\boldsymbol{e}_1)}{\pi(\boldsymbol{n}^*)}, \ i = 2, ..., m, \quad (16)$$

$$H = \pi(\boldsymbol{n}^*).$$

As an estimate for the parameter vector $\widehat{\Phi}$ of the product formula (15) we propose to take $\widehat{F}(\pi_T)$ which converges in probability to $\widehat{\Phi} = \widehat{F}(\pi)$ as $T \to \infty$ by virtue of (4) and (16).

3 Aggregation of Closed Networks

In the second part a closed cycle of a mass consumption and a renewal of some product is investigated. For this aim a closed queueing network with a number of customers, a number of servers in a consumption node and service intensities proportional to a large parameter n is considered. The parameter n characterizes a network size. For $n \to \infty$ a law of zero and one is established for a probability that all consumers are satisfied. If the limit equals 0 a convergence by a probability to some $b, \ 0 < b < 1$, of a part of satisfied consumers is proved. A problem of b maximization by the network route matrix Θ is solved. The maximization procedure consists of a finding of a route for some one-variable function and includes a definition of permissible solutions of some auxiliary transportation problem.

A model of an aggregated closed queueing network is considered. This model is constructed from n copies of a closed queuing network. Each copy consists of single servers arranged in the nodes $0, 1, \ldots, m$ with service intensities $\nu_0, \nu_1, \ldots, \nu_m$ and contains $\alpha > 1$ customers which circulate in this network in an accordance with the indecomposable route matrix. The customers play a role of products and the node 0 plays a role of a consumer. So in the node 0 the consumer receives a product. The nodes $1,\dots,m$ are places of the products renewal (customers service).

The aggregated system is the closed queueing network with the nodes $0,1,...,m$, which contain

$$r_0 = n, r_1 = 1, \dots, r_m = 1$$

servers correspondingly. $M = \alpha n$ customers circulate in the aggregated network. The service intensities in the servers which are arranged in the nodes $0,1,\dots,m$ are

$$\mu_0 = \nu,\ \mu_1 = n\nu_1, ..., \mu_m = n\nu_m.$$

The customers motions are described by the route matrix $\Theta = ||\theta_{ij}||_{i,j=0}^m$. Then the solution $(1,\Lambda)$ of the system

$$(1,\Lambda) = (1,\Lambda)\Theta,\ \Lambda = (\lambda_1,\dots,\lambda_m), \tag{17}$$

exists and is single [1]. Numbers of customers in the nodes of the aggregated network is described by the ergodic [26, chapter 2] discrete Markov process $\boldsymbol{y}(t) = (y_0(t), y_1(t), \dots, y_m(t))$ with the states set

$$Y = \{\boldsymbol{n} = (n_0, n_1, ..., n_m) :\ n_i \geq 0,\ \sum_{i=0}^m n_i = M\}.$$

The process $\boldsymbol{y}(t)$ limit distribution [1] $\Pi(\boldsymbol{n}),\ \boldsymbol{n} \in Y$, is the following

$$\Pi(\boldsymbol{n}) = C^{-1} \prod_{i=0}^m a_i(n_i),\ C = \sum_{\boldsymbol{n}\in Y} \prod_{i=0}^m a_i(n_i), \tag{18}$$

$$a_i(0) = 1,\ a_i(n_i) = \prod_{k=1}^{n_i} \frac{\lambda_i}{\min(k,\ r_i)\mu_i}\ ,\ 0 < n_i \leq M.$$

Our aim is to analyze the limit for $n \to \infty$ of the stationary distribution

$$P_n = \lim_{t\to\infty} P(y_0(t) \geq n). \tag{19}$$

3.1 Zero-one law for probability of consumers satisfaction

Denote

$$\rho_i = \frac{n\nu\lambda_1}{n\nu_i} = \lambda_i g_i,\ i = 1, ..., m. \tag{20}$$

Without a restriction of a generality assume that

$$\rho_1 \geq \rho_2 \geq \dots \geq \rho_m.$$

Theorem 1 *If*

$$\rho_1 > \rho_2 > \dots > \rho_m > 0 \tag{21}$$

then

$$\lim_{n\to\infty} P_n = \begin{cases} 0, & \rho_1 > 1, \\ 1, & \rho_1 < 1, \end{cases} \tag{22}$$

and the convergence in (22) is geometric.

3.2 Law of large numbers for part of satisfied consumers

Theorem 2 *If the condition (21) is true and* $\rho_1 > 1$ *then for any* $\varepsilon,\ 0 < \varepsilon < 1/2$,

$$\lim_{n\to\infty} P\left(\frac{(1-2\varepsilon)n}{\rho_1} < y_0(t) < \frac{(1+2\varepsilon)n}{\rho_1}\right) = 1 \tag{23}$$

and a convergence in (23) is geometric.

Remark 1 *Using the theorem [2, theorem 1.32] it is possible to prove the theorems 1, 2 statements if the condition (21) is replaced by*

$$\rho_1 = \rho_2 = ... = \rho_{m_1} = \rho^{(1)} > \rho_{m_1+1} = ...$$

$$... = \rho_{m_1+m_2} = \rho^{(2)} > \rho_{m_1+m_2+1} = ... >$$

$$> \rho_{m_1+...+m_{l-1}+1} = ... = \rho_{m_1+...+m_l} = \rho^{(l)} > 0,$$

$$m_1 + ... + m_l = m.$$

Remark 2 *The theorems 1, 2 remain true in the case* $M = [\alpha n]$ *with* $1 < \alpha < \infty$ *also.*

Remark 3 *The law of large numbers proved in the theorem 2 for large* n. *In the aggregated closed queueing network the number*

$$\min\left(1, \frac{1}{\rho^{(1)}}\right) = \frac{1}{\max(1,\rho^{(1)})}$$

characterizes a part of satisfied consumers. So it is worthy to consider different variants of $\rho^{(1)}$ *maximization problem.*

3.3 Minimization of $\rho^{(1)}$ by the route matrix

Consider a problem of a minimization of the function

$$\Phi(\Lambda) = \rho^{(1)} = \max_{1\leq i\leq m} \lambda_i g_i \Longrightarrow \min, \tag{24}$$

$$\Lambda = (\lambda_1, \dots, \lambda_m) \in \mathcal{M},$$

where the set $\mathcal{M}$ consists of vectors Λ so that

$$\lambda_1 > 0, \dots, \lambda_m > 0,\ \sum_{k=1}^m \lambda_k \geq 1, \tag{25}$$

and satisfies the conditions:
a) the set $\mathcal{M}$ is convex and closed and has a smooth boundary,
b) $\exists\ \Lambda^* \in \mathcal{M}$ so that the minimum $\Phi(\Lambda)$ by all $\Lambda \in \mathcal{M}$ which satisfy the equalities $\lambda_1 g_1 = ... = \lambda_m g_m$ is in the point Λ^*.

Theorem 3 *Suppose that the set $\mathcal{M}$ satisfies the conditions a), b). If a tangent plane L to $\mathcal{M}$ in the point Λ^* may be represented by the equation $\Lambda \cdot A = d$ in which the vector $A = (a_1, \dots, a_m)$ satisfies the conditions $a_1 > 0, \dots, a_m > 0$ then a single solution of the problem (24) is the point $\Lambda^* = (\lambda_1^*, \dots, \lambda_m^*)$.*

Remark 4 *The theorem 3 allows to reduce the problem (24) solution to a search of the point Λ^*. This approach is suggested in [25] for deterministic problems of vector optimization and gives a simple calculation procedure.*

Suppose that the set $\mathcal{M}$ satisfies the conditions a), b) and the problem (24) has the single solution Λ^*. For the route matrix Θ with non zero elements (besides of the element with zero subscript) define $\Lambda = \Lambda(\Theta)$ as a single solution of the equations system (17) and put $\Theta(\mathcal{M}) = \{\Theta : \Lambda(\Theta) \in \mathcal{M}\}$. Consider the following optimization problem:

$$\frac{1}{\rho^{(1)}} \Longrightarrow \max, \quad \Theta \in \Theta(\mathcal{M}), \tag{26}$$

equivalent to the problem

$$\Phi(\Lambda(\Theta)) = \rho^{(1)} \Longrightarrow \min, \quad \Theta \in \Theta(\mathcal{M}).$$

Denote $\lambda_0^* = 1$ and put

$$\psi_{i,j} = \lambda_i^* \theta_{ij}, \; 0 \le i, j \le m.$$

Corollary 1 *The route matrix Θ is a solution of the problem (26) if and only if the matrix $||\psi_{i,j}||_{i,j=0}^m$ is a permissable solution of the transportation problem*

$$\sum_{j=0}^{m} \psi_{i,j} = \sum_{j=0}^{m} \psi_{j,i} = \lambda_i^*, \; 0 \le i, j \le m. \tag{27}$$

The condition (25) guarantees a solvability of the transportation problem (27).

Remark 5 *The most convenient form for a solution of the $\rho^{(1)}$ maximization problem is a choice of contingencies not on the route matrix Θ but on the vector $\lambda_1, \dots, \lambda_m$. If $\Phi(\Lambda^*) < 1$ then a transition to a maximization of the index $\max(1, \rho^{(1)})$ may be realized by a finding of the solution Λ^{**} of the system $\lambda_1 g_1 = \dots = \lambda_m g_m = 1$ and by a test of the inclusion $\Lambda^{**} \in \mathcal{M}$.*

Remark 6 *The statements of the theorem 3 and of the corollary 2 are proved in [30] where a similar optimization procedure appears in a maximization of an ability to handle customers in an opened queueing network.*

4 Aggregation of Opened Networks

In the third part a model of an aggregated queuing network and an optimization of its limit characteristics are considered. In the aggregated queueing network numbers of servers in different nodes and an intensity of Poisson input flow are proportional to a large parameter n. Each node of the aggregated opened queueing network is described by stationary occupancy of its servers. These stationary occupancies tend by a probability to some deterministic values for $n \to \infty$. A maximization of a minimum of these limit occupancies consists of a finding of a route for some one-variable function and includes a definition of permissible solutions of some auxiliary transportation problem.

A model of an aggregated opened queueing is constructed from n copies of an opened queuing network. Each copy G consists of the nodes $0, 1, \dots, m$ where the nodes $1, \dots, m$ are one server queueing systems with service intensities $\mu_1, \dots, \mu_m$. The node 0 is a source for customers arriving the network G with the intensity λ and an outflow for customers leaving G. In the network G customers circulate in an accordance with the indecomposable route matrix.

The aggregated opened queueing network G_n has the Poisson input flow with the intensity $n\lambda$ and n-server queueing systems in the nodes $1, \dots, m$. Each server in the node i has the service intensity μ_i. As customers motions in the network G_n are described by the indecomposable route matrix $\Theta = ||\theta_{ij}||_{i,j=0}^m$ so the solution $(1, \Lambda)$, $\Lambda = (\lambda_1, \dots, \lambda_m)$, of the system (17) exists, is single and consists of positive components [1].

Suppose that

$$\rho_i = \frac{\lambda \lambda_i}{\mu_i} < 1, \; i = 1, \dots, m, \tag{28}$$

then a vector of numbers of customers in the nodes $1, \dots, m$ of the aggregated network G_n is described by the ergodic [26, chapter 2] discrete Markov process $\boldsymbol{y}(t) = (y_1(t), \dots, y_m(t))$ with the states set

$$Y = \{\boldsymbol{n} = (n_1, ..., n_m) : \; n_i \ge 0, \; i = 1, ..., m\}.$$

The process $\boldsymbol{y}(t)$ limit distribution [1, § 2]

$$\Pi(\boldsymbol{n}) = \prod_{i=1}^{m} \pi_i(n_i), \; \boldsymbol{n} \in Y, \tag{29}$$

$$\pi_i(n_i) = C_i^{-1} a_i(n_i), \; C_i = \sum_{n_i=0}^{\infty} a_i(n_i),$$

$$a_i(0) = 1, \; a_i(k) = \prod_{1 \le j \le k} \frac{n\rho_i}{\min(j, \; n)}\;, \; k > 0. \tag{30}$$

Here $\pi_i(n_i)$ is a limit distribution of a stationary number of customers in the node i.

4.1 Law of large numbers for distribution $\Pi(\boldsymbol{n})$

Theorem 4 *If the condition (28) is true then for any i and for any $\varepsilon,\ 0<\varepsilon<1/2$ we have that*

$$\lim_{n\to\infty}\sum_{n\rho_i(1-2\varepsilon)<n_i<n\rho_i(1+2\varepsilon),\ 1\le i\le m}\Pi(\boldsymbol{n})=1 \quad (31)$$

and a convergence in (31) is geometric.

The theorem 4 establishes that the vector $(\rho_1,\dots,\rho_m)$ characterizes the limit occupancies of the nodes $1,\dots,m$ in the aggregated network $G_n,\ n\to\infty$.

4.2 Optimization of limit network characteristics

Fix the intensities $\lambda,\mu_1,\dots,\mu_m$ and introduce the functions

$$\Phi(\Lambda)=\max_{1\le i\le m}\rho_i,\ \Psi(\Lambda)=\min_{1\le i\le m}\rho_i,$$
$$\Lambda=(\lambda_1,...,\lambda_m).$$

Define the equalities

$$\rho_1=\ldots=\rho_m. \quad (32)$$

Suppose that the set $\mathcal{M}$ consists of vectors Λ so that

$$\lambda_1>0,\dots,\lambda_m>0,\ \sum_{k=1}^m\lambda_k\ge 1, \quad (33)$$

and satisfies the conditions:
a) the set $\mathcal{M}$ is convex and closed and has a smooth boundary,
b) $\exists\ \Lambda^*\in\mathcal{M}$ so that the minimum $\Phi(\Lambda)$ by all $\Lambda\in\mathcal{M}$ which satisfy the equalities (32) is in the point Λ^*,
c) $\exists\ \Lambda_*\in\mathcal{M}$ so that the maximum $\Psi(\Lambda)$ by all $\Lambda\in\mathcal{M}$ which satisfy the equalities (32) is in the point Λ_*,
d) the inclusion $\mathcal{M}\subseteq\{\Lambda:\ \Phi(\Lambda)<1\}$ is true.

Consider a problem of a minimization of the function

$$\Phi(\Lambda)\Longrightarrow\min,\ \ \Lambda\in\mathcal{M}. \quad (34)$$

Theorem 5 *If a tangent plane L to $\mathcal{M}$ in the point Λ^* may be represented by the equation $\Lambda\cdot A=d$ in which the vector $A=(a_1,\dots,a_m)$ satisfies the conditions $a_1>0,\dots,a_m>0$ then a single solution of the problem (34) is the point $\Lambda^*=(\lambda_1^*,\dots,\lambda_m^*)$.*

Corollary 2. *The route matrix Θ is a solution of the problem (34) if and only if the matrix $\Theta=||\theta_{ij}||_{i,j=0}^m$ is a permissible solution of the transportation problem*

$$\sum_{j=0}^m\theta_{ij}=\sum_{j=0}^m\theta_{ji}=\lambda_i^*,\ 0\le i\le m. \quad (35)$$

The condition (33) guarantees a solvability of the transportation problem (35).

Consider a problem of a maximization of the function

$$\Psi(\Lambda)\Longrightarrow\max,\ \ \Lambda\in\mathcal{M}. \quad (36)$$

Theorem 6 *If a tangent plane L to $\mathcal{M}$ in the point Λ_* may be represented by the equation $\Lambda\cdot C=f$ in which the vector $C=(c_1,\dots,c_m)$ satisfies the conditions $c_1>0,\dots,c_m>0$ then a single solution of the problem (36) is the point $\Lambda_*=(\lambda_*^1,\dots,\lambda_*^m)$.*

Corollary 3 *The route matrix Θ is a solution of the problem (36) if and only if the matrix $\Theta=||\theta_{ij}||_{i,j=0}^m$ is a permissible solution of the transportation problem*

$$\sum_{j=0}^m\theta_{ij}=\sum_{j=0}^m\theta_{ji}=\lambda_*^i,\ 0\le i,j\le m. \quad (37)$$

The condition (33) guarantees a solvability of the transportation problem (37).

Remark 7 *The statements of the theorem 5 and of the corollary 2 are proved in [30] where a similar optimization procedure appears in a maximization of an ability to handle customers in an opened queueing network. This approach is suggested in [25] for deterministic problems of vector optimization and gives a simple calculation procedure. The statements of the theorem 6 and of the corollary 3 may be proved analogously.*

4.3 Stationary characteristics

There are some more properties of aggregated opened queueing networks connected with mean stationary queue length

$$N_i^{(n)}=\sum_{k>n}\pi_i(k)(k-n),\ 1\le i\le m,$$

or mean stationary waiting time

$$W_i^{(n)}=\frac{1}{n\lambda}\sum_{k\ge n}\pi_i(k)(k-n+1),\ 1\le i\le m,$$

in the network nodes in the condition

$$\rho_1=...=\rho_m=\rho. \quad (38)$$

Suppose that $m=1,\ n=1,\ \rho\to 1$ then $N_1^1,W_1^1\to\infty$. But if $m=1,\ n\to\infty$ and $\rho=\rho(n),\ 1-\rho(n)\sim n^{-\alpha}$ then for $0<\alpha<1$ we have $W_1^{(n)}\to\infty$ and for $1<\alpha$ we have $W_1^{(n)}\to 0$ as $n\to\infty$. Analogously if $m=1,\ n\to\infty$ and $\rho=\rho(n),\ 1-\rho(n)\sim n^{-\alpha}$ then for $0<\alpha<1/2$ we have $W_1^{(n)}\to\infty$ and for $1/2<\alpha$ we have $W_1^{(n)}\to 0$

as $n \to \infty$. These cooperative effects have been established for one node queueing network in [31, corollaries 1, 2]. But for opened queueing network with arbitrary number of nodes these results may be generalized in the case of equal nodes load or in the case of approximately equal nodes load as follows.

Theorem 7 *Suppose that the condition (38) is true and $\rho = \rho(n) = const < 1,\ n > 0$, then*

$$N_i^{(n)} \to 0,\ W_i^{(n)} \to 0,\ n \to \infty,\ 1 \le i \le m, \quad (39)$$

with a geometric rate of convergence.

The formula (39) is a corollary of the inequality (48). The conditions of the theorem 7 may be realized in the following way:

$$\frac{\lambda_1}{\mu_1} = \ldots = \frac{\lambda_m}{\mu_m} = a,\ \rho = \lambda a < 1.$$

Theorem 8 *Suppose that the condition (38) is true. If for $n \to \infty$ $1 - \rho(n) \sim n^{-\alpha}$ then for $1 \le i \le m$*

$$N_i^{(n)} \to \begin{cases} 0, & 0 < \alpha < 1/2, \\ \infty, & \alpha > 1/2, \end{cases}$$

$$W_i^{(n)} \to \begin{cases} 0, & 0 < \alpha < 1, \\ \infty, & \alpha > 1. \end{cases}$$

The theorem 8 may be proved using the formulas (29), (30) and [31, corollaries 1, 2]. The conditions of the theorem 8 may be realized in the following way:

$$\frac{\lambda_1}{\mu_1} = \ldots = \frac{\lambda_m}{\mu_m} = a,\ \lambda = \lambda(n) = \frac{1 - n^{-\alpha}}{a}\ .$$

Theorem 9 *If $1 - \rho_i(n) \sim n^{-\alpha_i},\ n \to \infty,$ for $1 \le i \le m$ then*

$$N_i^{(n)} \to \begin{cases} 0, & 0 < \alpha_i < 1/2, \\ \infty, & \alpha_i > 1/2, \end{cases}$$

$$W_i^{(n)} \to \begin{cases} 0, & 0 < \alpha_i < 1, \\ \infty, & \alpha_i > 1. \end{cases}$$

The theorem 9 may be proved using the formulas (29), (30) and [31, corollaries 1, 2]. The conditions of the theorem 9 may be realized in the following way:

$$\lambda(n) \equiv \lambda,\ \mu_i(n) = \frac{\lambda \lambda_i}{1 - n^{-\alpha_i}}.$$

Remark that if we consider multi phase queueing system with loaded nodes then from the theorem 9 it is possible to obtain in last nodes a very small queue and in first nodes - a very large queue. This property may be interpreted as a cooperative network property. An existence of such network properties were noted by some specialists in computer science.

5 Superposition of Queueing Networks

In the fourth part a problem of a calculation of limit distributions in a superposition of queueing networks is solved. A superposition of queueing network is an opened or a closed network in which some nodes are replaced by opened networks. Such constructions allow to calculate their distributions and realized a few steps of networks superposition. In this section an ability to handle customers in a superposition of networks is calculated. Network superpositions may be interpreted as recursively defined structures and are widely used in the nanotechnology.

5.1 Product Theorem

Consider opened Jackson networks $G,\ \ G'$ with the sets of oneserver nodes $\{g_0, g_1, ..., g_m\}$, $\{g_0', g_1', ..., g_r'\}$, with the input Poisson flows (with single intensities) and with the service intensities $\mu_1, ..., \mu_m$ and $\mu_1', ..., \mu_r'$. Denote by $\Theta = ||\theta_{ij}||_{i,j=0}^m,\ \Theta' = ||\theta_{ij}'||_{i,j=0}^r$ the route matrixes of the networks $G,\ \ G'$. Define the superposition $\overline{G} = G \overset{m}{\otimes} G'$ of the networks G, G' by a replacement of the node g_m in G by the network G'. Here an input flow (output flow) of the network G' is created from customers arriving (departing) to the node (from the node) g_m. In the network $\overline{G}$ the input flow is Poisson with the single intensity, the nodes set is

$$\{\overline{g}_0, \overline{g}_1, ..., \overline{g}_{m+r-1}\} = \{g_0, g_1, ..., g_{m-1}, g_1', ..., g_r'\}$$

and the route matrix $\overline{\Theta} = ||\overline{\theta}_{ij}||_{i,j=0}^{m+r-1}$ is defined from the formulas

$$\overline{\theta}_{ij} = \theta_{ij},\ i, j = 0, 1, ..., m-1,$$

$$\overline{\theta}_{m-1+i\ m-1+j} = \theta_{ij}',\ i, j = 1, ..., r,$$

$$\overline{\theta}_{i\ m-1+j} = \theta_{im}\theta_{oj}',\ i = 0, 1, ...m-1,\ j = 1, ..., r,$$

$$\overline{\theta}_{m-1+i\ j} = \theta_{io}'\theta_{mj},\ i = 1, ..., r,\ j = 0, 1, ..., m-1.$$

Lemma 1 *If $\theta_{mm} = 0$ then the matrix $\overline{\Theta} = ||\overline{\theta}_{ij}||_{i,j=0}^{m+r-1}$ satisfies properties of a route matrix and is indivisible.*

From Lemma 1, [1, § 2] there is the single vector $\overline{\Lambda}$ so that

$$\overline{\Lambda} = \overline{\Lambda}\ \overline{\Theta},\ \overline{\Lambda}=(1, \overline{\lambda}_1, ..., \overline{\lambda}_{m+r-1}). \quad (40)$$

Lemma 2 *If $\theta_{mm} = 0$ then*

$$\overline{\lambda}_i = \lambda_i,\ i = 1, ..., m-1,$$

$$\overline{\lambda}_{m-1+i} = \lambda_m \lambda_i',\ i = 1, ..., r.$$

Here $\Lambda = (1, \lambda_1, \lambda_2, ..., \lambda_m),\ \Lambda' = (1, \lambda_1', \lambda_2, ..., \lambda_r')$ are solutions of the systems $\Lambda = \Lambda\Theta,\ \Lambda' = \Lambda'\Theta'$.

Describe a dynamics of the network $\overline{G}$ by the discrete Markov process $\overline{y}(t)$ with the state set

$$\overline{Y} = \{\overline{y} = (n_1, ..., n_{m-1}, n'_1, ..., n'_r) :$$
$$n_1, ..., n_{m-1}, n'_1, ..., n'_r \geq 0\}.$$

Theorem 10 *If* $\theta_{mm} = 0$ *and*

$$\rho_i = \lambda_i/\mu_i < 1,\ i = 1, ..., m-1,$$
$$\rho'_i = \lambda_m\lambda'_i/\mu'_i < 1,\ i = 1, ..., r,$$

then the Markov process $\overline{y}(t)$ *is ergodic and its limit distribution* $\overline{P}(\overline{y}),\ \overline{y} \in \overline{Y}$, *has the form*

$$\overline{P}(\overline{y}) = \prod_{i=1}^{m-1}(1-\rho_i)\rho_i^{n_i}\prod_{i=1}^{r}(1-\rho'_i)(\rho'_i)^{n'_i}.$$

The network $\overline{G}$ is Jackson network and so the theorem 10 statement is a corollary of Lemmas 1, 2 and [1, theorem 2.1].

5.2 Abilities to Handle Customers

Construct the following sequence of the opened networks. Replace each node of the network G by the network G and denote this network by $G^{(1)}$. In the network $G^{(1)}$ replace each node by the network G and obtain the network $G^{(2)}$ and so on. After n steps obtain the network $G^{(n)}$ with m^{n+1} nodes among which there are m^n nodes with the service intensity μ_k. Denote the route matrix of the network $G^{(i)}$ by $\Theta^{(i)}$ and suppose that the single solution of the system

$$\Lambda^{(i)} = \Lambda^{(i)}\Theta^{(i)}$$

is $\Lambda^{(i)} = (1, \lambda_1^{(i)}, \dots, \lambda_{m^{i+1}}^{(i)})$.

If $\theta_{ii} = 0,\ i = 1, \dots, m$, then from the lemma 2 the network $G^{(1)}$ satisfies the formulas

$$\lambda_k^{(1)} = \lambda_k\lambda_1,\ \lambda_{m+k}^{(1)} = \lambda_k\lambda_2, \dots, \lambda_{m^2-m+k}^{(1)} = \lambda_k\lambda_m,$$
$$k = 1, \dots, m,$$

and the network $G^{(2)}$ satisfies the formulas

$$\lambda_k^{(2)} = \lambda_k\lambda_1^{(1)},\ \lambda_{m+k}^{(2)} = \lambda_k\lambda_2^{(1)}, ..., \lambda_{m^3-m+k}^{(2)} =$$
$$= \lambda_k\lambda_{m^2}^{(1)},\ k = 1, \dots, m.$$

and so on. Then the network $G^{(n)}$ satisfies the formulas

$$\lambda_k^{(n)} = \lambda_k\lambda_1^{(n-1)},\ \lambda_{m+k}^{(n)} = \lambda_k\lambda_2^{(n-1)}, ...,$$
$$\lambda_{m^{n+1}-m+k}^{(n)} = \lambda_k\lambda_{m^n}^{(n-1)},\ k = 1, ..., m.$$

So in the network $G^{(n)}$ all nodes with the service intensities μ_k correspond to the vector $\Lambda^{(n)}$ components of the form $\lambda_k\lambda_j^{(n-1)}$ where

$$\lambda_j^{(n-1)} = \lambda_1^{h_1}\lambda_2^{h_2}\cdots\lambda_n^{h_n},\ h_1 + \ldots + h_n = n.$$

Calculate now the ability to handle customers a_n of the network $G^{(n)}$. Here the ability to handle customers equals maximal intensity of input flow for which the network is not overloaded. From [30] obtain

$$a_n = \min_{1\leq k\leq m,\ 1\leq j\leq m^n}\frac{\mu_k}{\lambda_k\lambda_j^{(n-1)}} =$$
$$= \min_{1\leq k\leq n}\frac{\mu_k}{\lambda_k}\left(\min_{1\leq j\leq n}\frac{1}{\lambda_j}\right)^n.$$

If $\min\limits_{1\leq j\leq n}\frac{1}{\lambda_j} < 1$ then $a_n \to 0,\ n \to \infty$, if $\min\limits_{1\leq j\leq n}\frac{1}{\lambda_j} > 1$ then $a_n \to \infty,\ n \to \infty$, if $\min\limits_{1\leq j\leq n}\frac{1}{\lambda_j}=1$ then $a_n \equiv \min\limits_{1\leq k\leq n}\frac{\mu_k}{\lambda_k}$.

6 Proofs of main results

The Proof of Theorem 1. From the formulas (18), (19)

$$P_n = \sum_{k=n}^{\alpha n}\pi_n(k),$$
$$\pi_n(k) = \sum_{\substack{n_1, ..., n_m \geq 0,\\ n_1+...+n_m=\alpha n-k}}\Pi(k, n_1, \dots, n_m) =$$
$$= C^{-1}\psi_n(k)D_n(k),$$
$$\psi_n(k) = \begin{cases} n^n/n!,\ k > n,\\ n^k/k!,\ k \leq n,\end{cases}$$
$$D_n(k) = \sum_{\substack{n_1, ..., n_m \geq 0,\\ n_1+...+n_m=\alpha n-k}}\prod_{i=1}^{m}\rho_i^{n_i},$$
$$C = \sum_{\substack{k, n_1, ..., n_m \geq 0,\\ k+n_1+...+n_m=\alpha n}}\psi_n(k)D_n(k).$$

From [2, theorem 1.31] we have the equality

$$D_n(k) = \sum_{j=1}^{m}c_j\rho_j^{\alpha n+m-k-1} \tag{41}$$

where

$$c_j = \prod_{k\neq j}\left(1-\frac{\rho_k}{\rho_j}\right)^{-1},\ 1 \leq j \leq m,$$

so

$$D_n(k) \le c\rho_1^{\alpha n+m-k-1},\ c = \sum_{j=1}^{m} |c_j|. \quad (42)$$

Suppose that $\rho_1 > 1$ and construct the upper bound of $\pi_n(k),\ k \ge n$, assuming that

$$0 < \varepsilon < 1 - 1/\rho_1.$$

Then it is obvious that ($[a]$ is an integer part of a real number a)

$$\pi_n(k) \le \frac{\pi_n(k)}{\pi_n([n(1-\varepsilon)])} =$$

$$= \frac{n^{n-[n(1-\varepsilon)]}[n(1-\varepsilon)]!\ D_n(k)}{n!\ D_n([n(1-\varepsilon)])}.$$

By the fixed $\varepsilon > 0$ it is possible to choose $N = N(\varepsilon)$: $\forall n > N$

$$\left(\frac{\rho_j}{\rho_1}\right)^{\alpha n+m-[n(1-\varepsilon)]-1} < \varepsilon.$$

Consequently the following inequality is true

$$D_n([n(1-\varepsilon)]) > \rho_1^{\alpha n+m-[n(1-\varepsilon)]-1}(c_1 - c(m-1)\varepsilon).$$

So for $k \ge n > N$ we have

$$\pi_n(k) \le \left(\frac{1}{1-\varepsilon}\right)^{n-[n(1-\varepsilon)]} \frac{c\rho_1^{[n(1-\varepsilon)]-k}}{c_1 - c(m-1)\varepsilon} \le$$

$$\le \left(\frac{1}{\rho_1(1-\varepsilon)}\right)^{n\varepsilon} \frac{c}{(1-\varepsilon)(c_1 - c(m-1)\varepsilon)}$$

then for $n \to \infty$

$$P_n = \sum_{k=n}^{\alpha n} \pi_n(k) \le$$

$$\le \frac{c(\alpha-1)n}{(\rho_1)^{n\varepsilon}(1-\varepsilon)^{n\varepsilon+1}(c_1 - c(m-1)\varepsilon)} \to 0. \quad (43)$$

Suppose now that $\rho_1 < 1$ and construct the upper bound of $\pi_n(k),\ 0 \le k < n$, assuming that

$$0 < \varepsilon < \min(1, \alpha - 1, 1/\rho_1 - 1).$$

By the fixed $\varepsilon > 0$ choose $N = N(\varepsilon) :\ \forall n > N$

$$\left(\frac{\rho_j}{\rho_1}\right)^{\alpha n+m-[n(1+\varepsilon)]-1} < \varepsilon$$

consequently

$$D_n([n(1+\varepsilon)]) > \rho_1^{\alpha n+m-[n(1+\varepsilon)]+1}(c_1 - c(m-1)\varepsilon)$$

then

$$\pi_n(k) \le \frac{\pi_n(k)}{\pi_n([n(1+\varepsilon)])} =$$

$$= \frac{n^k[n(1+\varepsilon)]!\ D_n(k)}{k! n^{[n(1+\varepsilon)]} D_n([n(1+\varepsilon)])} \le$$

$$\le \frac{n^k[n(1+\varepsilon)]! c\rho_1^{[n(1+\varepsilon)]-n}}{k! n^{[n(1+\varepsilon)]}(c_1 - c(m-1)\varepsilon)} \le$$

$$\le \frac{n^k n! [n(1+\varepsilon)]^{[n(1+\varepsilon)]-n} c\rho_1^{[n(1+\varepsilon)]-n}}{k! n^n n^{[n(1+\varepsilon)]-n}(c_1 - c(m-1)\varepsilon)} \le$$

$$\le \frac{[n(1+\varepsilon)]^{[n(1+\varepsilon)]-n} c\rho_1^{[n(1+\varepsilon)]-n}}{n^{[n(1+\varepsilon)]-n}(c_1 - c(m-1)\varepsilon)} \le$$

$$\le \frac{c(\rho_1(1+\varepsilon))^{n\varepsilon}}{c_1 - c(m-1)\varepsilon}.$$

As a result obtain for $n \to \infty$

$$1 - P_n = \sum_{k=0}^{n-1} \pi_n(k) \le$$

$$\le \frac{nc(\rho_1(1+\varepsilon))^{n\varepsilon}}{c_1 - c(m-1)\varepsilon} \to 0.$$

The Proof of Theorem 2. Fix $\varepsilon,\ 0 < \varepsilon < 1/2$, then from the inequality (42) find an upper bound of the probability

$$P\left(\frac{(1+2\varepsilon)n}{\rho_1} \le y_0(t) < n\right) = \sum_{\frac{n(1+2\varepsilon)}{\rho_1} \le k < n} \pi_n(k) \le$$

$$\le \sum_{\frac{(1+2\varepsilon)n}{\rho_1} \le k < n} \frac{\pi_n(k)}{\pi_n\left(\left[\frac{n}{\rho_1}\right]\right)} =$$

$$= \sum_{\frac{(1+2\varepsilon)n}{\rho_1} \le k < n} \frac{\psi_n(k) D_n(k)}{\psi_n\left(\left[\frac{n}{\rho_1}\right]\right) D_n\left(\left[\frac{n}{\rho_1}\right]\right)} \le$$

$$\le \sum_{\frac{(1+2\varepsilon)n}{\rho_1} \le k < n} \frac{\psi_n(k) c\rho_1^{\alpha n+m-k-1}}{\psi_n\left(\left[\frac{n}{\rho_1}\right]\right) c_1 \rho_1^{\alpha n+m-\left[\frac{n}{\rho_1}\right]-1}} =$$

$$= \frac{c}{c_1} \sum_{\frac{(1+2\varepsilon)n}{\rho_1} \le k < n} \frac{n^{k-\left[\frac{n}{\rho_1}\right]} \rho_1^{\left[\frac{n}{\rho_1}\right]-k} \left[\frac{n}{\rho_1}\right]!}{k!} \le$$

$$\le \frac{c}{c_1} \sum_{\frac{(1+2\varepsilon)n}{\rho_1} \le k < n} \frac{n^{k-\left[\frac{n}{\rho_1}\right]} \rho_1^{\left[\frac{n}{\rho_1}\right]-k}}{k(k-1)\dots\left(\left[\frac{n}{\rho_1}\right]+1\right)} \le$$

$$\leq \sum_{\frac{(1+2\varepsilon)n}{\rho_1}\leq k<n} \frac{\frac{c}{c_1}\left(\frac{n}{\rho_1}\right)^{k-\left[\frac{n}{\rho_1}\right]}}{\left(\frac{n}{\rho_1}\right)^{\left[\frac{(1+\varepsilon)n}{\rho_1}\right]-\left[\frac{n}{\rho_1}\right]}\left(\frac{(1+\varepsilon)n}{\rho_1}\right)^{k-\left[\frac{(1+\varepsilon)n}{\rho_1}\right]}}.$$

So we have for $n\to\infty$ that

$$P\left(\frac{(1+2\varepsilon)n}{\rho_1}\leq y_0(t)<n\right)\leq$$

$$\leq \frac{\frac{nc}{c_1}\left(1-\frac{1+2\varepsilon}{\rho_1}\right)}{(1+\varepsilon)^{\left[\frac{(1+2\varepsilon)n}{\rho_1}\right]-\left[\frac{(1+\varepsilon)n}{\rho_1}\right]}}\to 0. \qquad (44)$$

Unite the formulas (43), (44) and obtain the limit

$$P\left(\frac{(1+2\varepsilon)n}{\rho_1}\leq y_0(t)<\alpha n\right)\to 0,\ n\to\infty, \quad (45)$$

with a geometric rate of a convergence.

Using the inequality (42) analogously to the previous case construct an upper bound of the probability

$$P\left(0\leq y_0(t)\leq \frac{(1-2\varepsilon)n}{\rho_1}\right)=\sum_{0\leq k\leq \frac{(1-2\varepsilon)n}{\rho_1}}\pi_n(k)\leq$$

$$\leq \sum_{0\leq k\leq \frac{(1-2\varepsilon)n}{\rho_1}}\frac{\pi_n(k)}{\pi_n\left(\left[\frac{n}{\rho_1}\right]\right)}\leq$$

$$\leq \frac{c}{c_1}\sum_{0\leq k\leq \frac{(1-2\varepsilon)n}{\rho_1}}\frac{\left(\frac{n}{\rho_1}\right)^{k-\left[\frac{n}{\rho_1}\right]}\left[\frac{n}{\rho_1}\right]!}{k!}=$$

$$=\frac{c}{c_1}\sum_{0\leq k\leq \frac{(1-2\varepsilon)n}{\rho_1}}\left(\frac{n}{\rho_1}\right)^{k-\left[\frac{n}{\rho_1}\right]}(k+1)\cdot\ldots\cdot\left[\frac{n}{\rho_1}\right].$$

As for $0\leq k\leq\left[\frac{n}{\rho_1}\right]$ the product

$$\left(\frac{n}{\rho_1}\right)^{k-\left[\frac{n}{\rho_1}\right]}(k+1)\cdot\ldots\cdot\left[\frac{n}{\rho_1}\right]$$

increases by k so the following inequalities are true

$$P\left(0\leq y_0(t)\leq\frac{(1-2\varepsilon)n}{\rho_1}\right)\leq$$

$$\leq\frac{(1-2\varepsilon)nc}{\rho_1c_1}\left(\frac{n}{\rho_1}\right)^{\left[\frac{(1-2\varepsilon)n}{\rho_1}\right]-\left[\frac{n}{\rho_1}\right]}\times$$

$$\times\left(\left[\frac{(1-2\varepsilon)n}{\rho_1}\right]+1\right)\times\ldots\times\left[\frac{n}{\rho_1}\right]\leq$$

$$\leq\frac{(1-2\varepsilon)nc}{\rho_1c_1}\left(\frac{n}{\rho_1}\right)^{\left[\frac{(1-2\varepsilon)n}{\rho_1}\right]-\left[\frac{n}{\rho_1}\right]}\times$$

$$\times\frac{\left(\frac{(1-\varepsilon)n}{\rho_1}\right)^{\left[\frac{(1-\varepsilon)n}{\rho_1}\right]-\left[\frac{(1-2\varepsilon)n}{\rho_1}\right]}}{\left(\frac{n}{\rho_1}\right)^{-\left[\frac{n}{\rho_1}\right]+\left[\frac{(1-\varepsilon)n}{\rho_1}\right]}}\leq$$

$$\leq\frac{(1-2\varepsilon)nc}{\rho_1c_1}(1-\varepsilon)^{\left[\frac{(1-\varepsilon)n}{\rho_1}\right]-\left[\frac{(1-2\varepsilon)n}{\rho_1}\right]}\leq$$

$$\leq\frac{(1-2\varepsilon)nc}{\rho_1c_1}(1-\varepsilon)^{\frac{(1-\varepsilon)n}{\rho_1}-\frac{(1-2\varepsilon)n}{\rho_1}-1}.$$

As a result obtain for $n\to\infty$ that

$$P\left(0\leq y_0(t)\leq\frac{(1-2\varepsilon)n}{\rho_1}\right)\leq$$

$$\leq\frac{(1-2\varepsilon)nc}{\rho_1c_1}(1-\varepsilon)^{\frac{\varepsilon n}{\rho_1}-1}\to 0. \qquad (46)$$

From the formulas (45), (46) obtain the formula (23).

The Proof of Theorem 4. Fix $\varepsilon,\ 0<\varepsilon<1/2$ then from the definition (30)

$$\pi_i(k)\leq\frac{\pi_i(k)}{\pi_i([n\rho_i])}$$

and consequently

$$\sum_{0\leq k\leq n\rho_i(1-2\varepsilon)}\pi_i(k)\leq\sum_{0\leq k\leq n\rho_i(1-2\varepsilon)}\frac{\pi_i(k)}{\pi_i([n\rho_i])}\leq$$

$$\leq\sum_{0\leq k\leq n\rho_i(1-2\varepsilon)}\prod_{k<j\leq n\rho_i}\frac{j}{n\rho_i}\leq$$

$$\leq\sum_{0\leq k\leq n\rho_i(1-2\varepsilon)}\prod_{n\rho_i(1-2\varepsilon)<j\leq n\rho_i(1-\varepsilon)}\frac{j}{n\rho_i}.$$

As a result obtain that for fixed $\varepsilon,\ 0<\varepsilon<1/2$, and for $n\to\infty$

$$\sum_{0\leq k\leq n\rho_i(1-2\varepsilon)}\pi_i(k)\leq n\rho_i(1-\varepsilon)^{n\rho_i\varepsilon}\to 0. \quad (47)$$

Analogously it is easy to prove that

$$\sum_{n\rho_i(1+2\varepsilon)\leq n_i\leq n,\ 1\leq i\leq m}\Pi(\boldsymbol{n})\leq$$

$$\leq n(1-\rho_i)(1+\varepsilon)^{-n\rho_i\varepsilon}\to 0 \qquad (48)$$

and

$$\pi_i(k)\leq(1+\varepsilon)^{-n\rho_i\varepsilon}\rho_i^{k-n},\ k>n \qquad (49)$$

so from the condition (28)

$$\sum_{n<k<\infty} \pi_i(k) \leq \frac{(1+\varepsilon)^{-n\rho_i\varepsilon}}{1-\rho_i} \to 0. \qquad (50)$$

The limits (47) - (50) have a geometric (by n) rate convergence and lead to the formula

$$\lim_{n\to\infty} \sum_{n\rho_i(1-2\varepsilon)<k<n\rho_i(1+2\varepsilon)} \pi_i(k) = 1. \qquad (51)$$

Then the formulas (29), (51) give the limit (31) with a geometric rate convergence.

The Proof of Lemma 1. The matrix $\overline{\Theta}$ is route as all its elements are nonnegative and

$$\sum_{j=0}^{m+r-1} \overline{\theta}_{ij} = \sum_{j=0}^{m-1} \theta_{ij} + \sum_{j=1}^{r} \theta_{im}\theta'_{0j} =$$

$$= 1 - \theta_{im} + \theta_{im} = 1,\ i = 0, 1, ..., m-1,$$

$$\sum_{j=0}^{m+r-1} \overline{\theta}_{m-1+i\, j} = \sum_{j=0}^{m-1} \theta'_{io}\theta_{mj} + \sum_{j=1}^{r} \theta'_{ij} =$$

$$= \theta'_{io}(1-\theta_{mm}) + 1 - \theta'_{i0} = 1,\ i = 1, ..., r.$$

Show that for $\forall\ i, j \in \{0, 1, ..., m+r-1\}$

$$\exists\ k_1, ..., k_s \in \{1, ..., m+r-1\} :$$

$$\overline{\theta}_{ik_1}, \overline{\theta}_{k_1k_2}, ..., \overline{\theta}_{k_sj} > 0. \qquad (52)$$

If

$$i, j \in \{0, 1, ..., m-1\}\ (i, j \in \{m, ..., m+r-1\})$$

then the formulas (52) may be obtained from the matrix Θ (from the matrix Θ') indivisibility. If

$$i \in \{0, 1, ..., m-1\},\ j \in \{m, ..., m+r-1\}$$

then from the matrix Θ indivisibility $\exists i_1, ..., i_s \in \{1, ..., m-1\}$:

$$\overline{\theta}_{ii_1} = \theta_{ii_1} > 0, ..., \overline{\theta}_{i_{s-1}i_s} = \theta_{i_{s-1}i_s} > 0, \theta_{i_sm} > 0,$$

and from the matrix Θ' indivisibility $\exists\ j_1, ..., j_n \in \{1, ..., r\}$:

$$\theta'_{0\, m-1+j_1} > 0,\ \overline{\theta}_{m-1+j_1\, m-1+j_2} = \theta'_{j_1j_2} > 0, ...,$$

$$\overline{\theta}_{m-1+j_n\, j} = \theta'_{j_n\, j-m+1} > 0.$$

As $\overline{\theta}_{i_s\, m-1+j_1} = \theta_{i_sm}\theta'_{0\, m-1+j_1}$ then

$$\overline{\theta}_{ii_1} > 0, ..., \overline{\theta}_{i_{s-1}i_s} > 0,\ \overline{\theta}_{i_s\, m-1+j_1} > 0,$$

$$\overline{\theta}_{m-1+j_1\, m-1+j_2} > 0, ..., \overline{\theta}_{m-1+j_n\, j} > 0.$$

The case $i \in \{m, ..., m+r-1\},\ j \in \{0, 1, ..., m-1\}$ is considered similar.

The Proof of Lemma 2. Indeed if $i \in \{1, ..., m-1\}$ then from the formula (40) obtain

$$\lambda_i = \overline{\lambda}_i = \overline{\theta}_{0i} + \sum_{j=1}^{m+r-1} \overline{\lambda}_j\overline{\theta}_{ji} = \theta_{0i} + \sum_{j=1}^{m-1} \lambda_j\theta_{ji} +$$

$$+ \sum_{j=1}^{r} \lambda_m\lambda'_j\theta'_{j0}\theta_{mi} = \lambda_i.$$

If $i \in \{1, .., r\}$ then

$$\lambda_m\lambda'_i = \overline{\lambda}_{m-1+i} = \overline{\theta}_{0\, m-1+i} + \sum_{j=1}^{m+r-1} \overline{\lambda}_j\overline{\theta}_{j\, m-1+i} =$$

$$= \theta'_{0i}\theta_{0m} + \sum_{j=1}^{m-1} \lambda_j\theta'_{0i}\theta_{jm} + \sum_{j=1}^{r} \lambda_m\lambda'_j\theta'_{ji} =$$

$$= \theta'_{0i}\theta_{0m} + \theta'_{0i}(\lambda_m - \theta_{0m} - \lambda_m\theta_{mm}) +$$

$$+ \lambda_m\lambda'_i - \lambda_m\theta'_{0i} = \lambda_m\lambda'_i.$$

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