ABOUT THE WEAK EFFICIENCIES IN VECTOR OPTIMIZATION

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Abstract: We present the principal properties of the weak efficient points given in the literature. We study a vector optimization problems for multifunctions, defined with infimal and supremal efficient points in locally convex spaces ordered by convex, pointed closed cones with nonempty interior. We introduce and study the solutions for these problems using the algebraic and topological results for the efficient points. Also, we'll present the links between our problems and 2 special problems, the scalar and the approximate problems as well as some saddle points theorems and duality results using a suitable Lagrangian adapted for the INFSUP problem, a generalization of the MINMAX problem.

Key–Words: order vector spaces, convex cones, efficient points.

1 Introduction

It is a well known fact that the Pareto and the weak Pareto efficiencies was intensively used for the study of the optimal problems in the literature of mathematical economics.

The special conditions imposed for the existence of such points led to the study of other kinds of efficient points, the so called supremal and infimal points. These points are natural generalizations for ordinary concepts of infimum and supremum known in real analysis. Firstly, these notion was given in vector lattices (see [21]) or in a multi-dimensional Euclidian space using the lower and upper bounds (see [3]) or efficiencies for different sets (see [4], [9], [7]).

Nieuwenhuis, firstly introduced ([8]) the supremal and infimal points in Banach spaces ordered by a closed convex cone with nonempty interior and Tanino ([18]) present such points in a linear topological space ordered by a nonempty interior cone.

In the general form, these notions was given by Postolica ([10]) and the above discussed points can be found in the weak efficiencies introduced here.

A general vector optimization problem with multifunctions can be presented in the following form:

$$(VP)\ \ Eff \bigcup_{x \in G} F(x)$$

where $F : X \rightrightarrows Z$ is a multifunction between two vector spaces X, Z, $G \subset X$ and Eff denotes one's of the efficient points set, the minimum and maximum Pareto points set, usually.

These problems was intensively studied in the last 20 years taking into account the interest generated in mathematical economics. Thus lagrangean and conjugate duality results was obtained as well as weak and strong duality theorems and scalarization results.

Nevertheless, the authors continue the study of the Problem (VP) by replacing the Pareto efficient points with other efficient points. Thus, in [20], [11], we can find the ε-Lagrangean multiplicators, ε-weak saddle points, ε-duality problem studied in connection with the approximate efficient points.

We consider in the following the optimization problem obtained with the efficient points given in [10] and it will be called the INF (or SUP) problem. The essential property of these efficient points set is that it is nonempty for a large class of sets and thus the problem has often nonempty values but it was not easy to find a suitable notion of a solution for this problem since an infimum point is not really a point of the set. The final solution was the consideration of a sequence as a solution for such vector problem as we will see in the section 3. The paper is structured in 6 parts. The second section present the principal properties of the infimal and the supremal points which will be used in what follows.

The third part present the notion of the solution for our problem and the links between this solution and some notion introduced in [2]. A study of some optimization problems closely related with our problem, i.e. the approximate problem (P^a) and the scalar problem (P^*) is given in the fourth section. The perturbed problem is also studied in connection with the INFSUP problem, a generalization of the MINMAX problem; a notion of saddle point is given and the weak and the strong duality theorems are proved. The lagrangean and the conjugate duality are studied in the fifth section.

2 Background to the efficient points

Let Z be a locally convex space ordered by a closed, convex, pointed cone Z_+. We will write $a \leq b$ if $b - a \in Z_+$ and $a < b$ if $b - a \in Z_+ \setminus \{0\}$. As usually, we can consider a smallest element denoted $-\infty$ and a biggest element denoted $+\infty$ and $\bar{Z} = Z \cup \{+\infty\} \cup \{-\infty\}$. The complementary set of A will be denoted $A^c = Z \setminus A$. We'll denote $\mathbb{R}^p_+$ the positive cone of the euclidian space $\mathbb{R}^p$. The closure (resp. the interior and the boundary) of the set $A \subset Z$ will be denoted $cl\ A$ (resp. $\text{Int}\ A$, $Fr\ A$) and if $\text{Int}\ Z_+ \neq \emptyset$ then $K = \text{Int}\ Z_+ \cup \{0\}$. In this case, the efficiencies considered with respect to the cone K will be called weak efficiencies and will be denoted $wEFF\ A$. As usually, a fundamental system of neighborhoods for a point x will be denoted by $\mathcal{V}(x)$. If the interior of the cone Z_+ is nonempty, we can consider a fundamental system of neighborhoods for $-\infty$ denoted $\mathcal{V}(-\infty)$ (resp. for $+\infty$ denoted $\mathcal{V}(+\infty)$) given by the sets $V = [-\infty, a) = \{x \in Z \mid x < a\} \cup \{-\infty\}$ (resp. $V = (a, +\infty] = \{x \in Z \mid x > a\} \cup \{+\infty\}$).

Definition 2.1. 1)*(Pareto minimum)*
$MIN\ A = \{a \in A \mid a \not> b,\ \forall b \in A\}$
($MAX\ A = \{a \in A \mid a \not< b,\ \forall b \in A\}$*);*
2)*[21]* $INF\ A = \{y \in Z \mid y \leq a,\ \forall a \in A,$ and if $y' \leq a\ \forall a \in A \Rightarrow y' \leq y\}$
($SUP\ A = \{y \in Z \mid y \geq a,\ \forall a \in A,$ and if $y' \geq a\ \forall a \in A \Rightarrow y' \geq y\}$*);*
3)*[10]* $INF\ A = MIN\ \text{cl}\ A$
($SUP\ A = MAX\ \text{cl}\ A$*);*
4)*[4][17]* $INF\ A = MIN\ \text{cl}\ (A + \mathbb{R}^p_+)$
($SUP\ A = MAX\ \text{cl}\ (A + \mathbb{R}^p_+)$*);*
5)*[7][18]* $INF\ A = wMIN\ \text{cl}\ (A + \text{Int}\ Z_+)$
($SUP\ A = wMAX\ \text{cl}\ (A + \text{Int}\ Z_+)$*);*
6)*[8]* $INF\ A = \{y \in Z \mid y - a \notin \text{Int}\ Z_+,\ \forall a \in A,$ and if $y' - y \in \text{Int}\ Z_+ \Rightarrow a \in A,\ y' - a \in \text{Int}\ Z_+\}$
($SUP\ A = \{y \in Z \mid y - a \notin -\text{Int}\ Z_+,\ \forall a \in A,$ and if $y' - y \in -\text{Int}\ Z_+ \Rightarrow a \in A,\ y' - a \in -\text{Int}\ Z_+\}$*);*
7)*[10]* $INF\ A = \{y \in \bar{Z} \mid y - a \notin Z_+ \setminus \{0\},\ \forall a \in A,$ and if $y' - y \in Z_+ \setminus \{0\} \Rightarrow \exists a \in A,\ y' - a \in Z_+ \setminus \{0\}\}$
($SUP\ A = \{y \in \bar{Z} \mid y - a \notin -Z_+ \setminus \{0\},\ \forall a \in A,$ and if $y' - y \in -Z_+ \setminus \{0\} \Rightarrow \exists a \in A,\ y' - a \in -Z_+ \setminus \{0\}\}$*).*

Remark 2.2. a) *Definition 2.1 5) may be given in a general form* $INF\ A = MIN\ \text{cl}\ (A + Z_+)$.

b) *The author denote the infimal points set given in Definition 2.1 7) by* $INF_1 A$ *and it is called the proximal infimum points set. An other infimal points set may be found in [9], the infimum set given by* $\widetilde{INF}\ A = \{y \in \bar{Z} \mid y \not> a\ \forall a \in A\} = (A + Z_+ \setminus \{0\})^c \cup \{-\infty\}$. *Using this notations, let remark that* $INF_1 A = MAX\ \widetilde{INF}\ A$ *and the set given in Definition 2.1 6) is* $wINF_1 A$.

c) *In [8], the author proves that the infimal set given in Definition 2.1 6) has the property that* $INF\ A = \text{Fr}\ (A + Z_+)$ *(see Theorem I-17).*

d) *Obviously, the notions presented in Definition 2.1 1)-6) may be given in the same form for sets from* $\bar{Z}$.

In what follows we are interested about the relationships between these kinds of infimal points, more exactly between
(1) $INF\ A = \text{Fr}\ (A + Z_+)$
(2) $INF\ A = MAX\ \widetilde{INF}\ A$ $(w2)$ $INF\ A = wMAX\ w\widetilde{INF}\ A$
(3) $INF\ A = MIN\ \text{cl}\ (A + Z_+)$ $(w3)$ $INF\ A = wMIN\ \text{cl}\ (A + Z_+)$.

Proposition 2.3. *Let* $A \subset \bar{Z}$. *Under the previous notations, we have:* $(2) \Rightarrow (1)$, $(3) \Rightarrow (1)$ *and if* $\text{Int}\ Z_+ \neq \emptyset$ *then* $(1) \iff (w2) \iff (w3)$

Remark 2.4. In general, the implications $(1) \Rightarrow (2)$, $(1) \Rightarrow (3)$, $(2) \Rightarrow (3)$ and $(3) \Rightarrow (2)$ are not true. Indeed, let consider $Z = \mathbb{R}^2$, $Z_+ = \mathbb{R}^2_+$ and $A = \{(x, y) \mid (x-1)^2 + (y-1)^2 < 1\}$. The point $a = (0, 1) \in \text{Fr}\ (A + Z_+)$, $a \in \widetilde{INF}\ A$ but $a < b = (0, 2) \in \widetilde{INF}\ A$, so $(1) \not\Rightarrow (2)$. Also, $b \in \text{Fr}\ (A + Z_+)$ but $b \notin MIN\ \text{cl}\ (A + Z_+)$ since $a \in \text{cl}\ (A + Z_+)$ and $a < b$. Thus $(1) \not\Rightarrow (3)$. In the same time, we can see that $a \in MIN\ \text{cl}\ (A + Z_+)$ but $a \notin MAX\ \widetilde{INF}\ A$ since $b > a$ and $\not\exists a' \in A$ such that $a' < b$. By this way, $(3) \not\Rightarrow (2)$. Now, if we take $B = A \cup \{b\}$, we'll have that $b \in MAX\ \widetilde{INF}\ B$ but $b \notin MIN\ \text{cl}\ (B + Z_+)$ and thus $(2) \not\Rightarrow (3)$.

Proof. (of the proposition) For proving $(2) \Rightarrow (1)$, let $x \in MAX\ \widetilde{INF}\ A$ and suppose that $x \notin \text{Fr}\ (A + Z_+)$. Since $\text{Fr}\ (A + Z_+) = \text{Fr}\ (A + Z_+ \setminus \{0\}) = \text{Fr}\ (A + Z_+ \setminus \{0\})^c = \text{Fr}\ \widetilde{INF}\ A$, we have $x \notin \text{Fr}\ \widetilde{INF}\ A$. Since $x \in \widetilde{INF}\ A$, we must have $x \in \text{Int}\ (\widetilde{INF}\ A)$ and consequently there exists $V \in \mathcal{V}(x)$, $V \subset \widetilde{INF}\ A$. Since $x \in \text{cl}\ \ (x + Z_+ \setminus \{0\})$, we find $y \in V \cap (x + Z_+ \setminus \{0\})$. Thus, $y \in \widetilde{INF}\ A$ and $y > x$, which contradict the fact that $x \in MAX\ \widetilde{INF}\ A$.

Now, let $x \in MIN\ \ \text{cl}\,(A + Z_+)$ and suppose that $x \in \text{Int}\,(A + Z_+)$. This implies that $\exists V \in \mathcal{V}(x)$, $V \subset A + Z_+$. Consequently, we find $\varepsilon \in Z_+ \setminus \{0\}$ such that $x - \varepsilon \in V \subset A + Z_+ \subset \text{cl}\,(A + Z_+)$ which contradict the choices of $x \in MIN\ \text{cl}\,(A + Z_+)$. Thus, $(3) \Rightarrow (1)$. If the interior of the cone Z_+ is nonempty, then we consider $\varepsilon \in \text{Int}\ Z_+$ and this will yields to $(w3) \Rightarrow (1)$. Following Remark 2.2c) we get $(1) \iff (w2)$. Similarly to implication $(2) \Rightarrow (1)$, we get $(w2) \Rightarrow (1)$ and thus, $(w2) \iff (1)$. We'll prove in what follows that $(1) \Rightarrow (w3)$. Let $x \in \text{Fr}\ (A + Z_+)$ and suppose that $x \notin wMIN\text{cl}\ (A + Z_+)$. This means that there exists $y \in \text{cl}\ (A+Z_+), y-x \in -\text{Int}\ Z_+$. Thus, $x-\text{Int}\ Z_+ \in \mathcal{V}(y)$ and so $x-\text{Int}\ Z_+ \cap (A+Z_+) \neq \emptyset$ which gives that $x \in A + \text{Int}\ Z_+ \subset \text{Int}\ (A + Z_+)$. But initially, we chose $x \in \text{Fr}\ (A + Z_+)$ and this contradiction shows that our implication is valid. Finally, the proposition's equivalencies are shown. ⋄

It is a known fact that the existence of the Pareto minimal points involves some special conditions concerning the set or the cone. A necessary and sufficient condition for this fact appears in [6], Theorem 3.4:

Theorem 2.5. *Assume that Z_+ is a convex cone and $A \subset Z$, $A \neq \emptyset$. $MIN\ A \neq \emptyset$ if and only if A has a nonempty strongly Z_+-complete section.*

Let recall that a set $U \subset Z$ is K-complete (resp., strongly K-complete) if it has no cover of the form $\{(x_\alpha - \text{cl}\ K)^c,\ \alpha \in I\}$ (resp., $\{(x_\alpha - K)^c,\ \alpha \in I\}$) with $\{x_\alpha,\ \alpha \in I\}$ being a decreasing net in A. A section of A is a set $A_x = A \cap (x - K)$ (see [6]).

Following this, we deduce necessary and sufficient conditions for the existence of the infimal (supremal) points given in Proposition 2.3.

Theorem 2.6. *a) A nonempty set $A \subset Z$ has a nonempty infimal points set (2) if and only if $-(A + Z_+)^c$ has a nonempty strongly Z_+-complete section.*
b) A nonempty set $A \subset Z$ has a nonempty infimal points set (3) if and only if $\text{cl}\ (A+Z_+)$ *has a nonempty strongly Z_+-complete section.*

For the weak infimal or supremal points we can prove a more explicit result. Following Proposition 2.3, the infimal points sets (1), (w2) and (w3) are coincident with the general notion of weak infimum points set presented in Definition 2.1 7). By this reason, in what follows we'll use the notations presented in Remark 2.2b).

Theorem 2.7. *Let $A \subset \bar{Z}$ be a nonempty set of a locally convex space Z ordered by a convex, pointed, closed cone with nonempty interior. Then, $\emptyset \neq wINF_1A \subset Z$ if and only if $w\widetilde{INF}\ A \neq \{-\infty\}$. In this case, the following "domination" properties (wDP) does holds:*

$$A \subseteq (wINF_1A + K) \cup \{+\infty\};$$
$$w\widetilde{INF}\ A = (wINF_1A - K) \cup \{-\infty\}$$

Proof. Obviously, if $\emptyset \neq wINF_1A \neq\subset Z$, then $w\widetilde{INF}\ A \neq \{-\infty\}$ since $wINF_1A \subset w\widetilde{INF}\ A$. Now, let $a \in A,\ a \neq \infty$ and $\tilde{y} \in w\widetilde{INF}\ A \setminus \{-\infty\}$. Since K is a generating cone, we find $z \neq a, \neq \tilde{y}$ such that $z - a \in -K$ and $z - \tilde{y} \in -K$ which implies $z \in w\widetilde{INF}\ A$. Let denote $z(t) = z + t(a - z), t > 0$ and we remark that for $t > 1$, $z(t) \in A + K \setminus \{0\}$. If we consider $\inf\{t > 0 \mid z(t) \in A + K \setminus \{0\}\} = \bar{t}$, we have $\bar{t} \leq 1$ and $\bar{z} = z + \bar{t}(a - z) \notin A + K \setminus \{0\}$ which implies $\bar{z} \in w\widetilde{INF}\ A$. We'll prove in what follows that $\bar{z} \in wINF_1A$. Let consider $\mu \in \text{Int}\ Z_+$; we can find $\varepsilon > 0$ such that $\mu - \varepsilon'(a - z) \in \text{Int}\ Z_+$, for all $0 < \varepsilon' < \varepsilon$ and for this ε there exists ε' such that $z + (\bar{t} + \varepsilon')(a - z) \in A + K \setminus \{0\}$. Thus, $\bar{z} + \mu = z + \bar{t}(a - z) + \mu = z + (\bar{t} + \varepsilon')(a - z) - \varepsilon'(a - z) + \mu \in A + K \setminus \{0\}$.

Consequently, $\bar{z} \in wINF_1A \neq \emptyset$ and $\bar{z} - a = (\bar{t} - 1)(a - z) \in K$ which gives that $A \subset (wINF_1A + K) \cup \{+\infty\}$.

Similarly, if we consider $z \in w\widetilde{INF}\ A \setminus \{-\infty\}$ and $a \in A$, we can find $a' \in A + K \setminus \{0\}$, $a' - z \in \text{Int}\ Z_+$ and the element $\bar{z} = z + \bar{t}(a' - z)$ with $\bar{t} = \inf\{t > 0 \mid z + t(a' - z) \in A + K \setminus \{0\}\}$ will be an element from $wINF_1A$ which have the property that $\bar{z} - z \in K$ and thus the equality $wINF\ A = (wINF_1A - K) \cup \{-\infty\}$ does holds. ⋄

Remark 2.8. *If $A \subset Z$, the equivalence "$wINF_1A \neq \emptyset$ if and only if $w\widetilde{INF}\ A \neq \emptyset$" is Theorem I-18 [8].*

The following proposition will be useful for the study of the solution properties in the next section.

Proposition 2.9. *Let $A \subset Z$, $x \in A + \text{Int}\ Z_+$ and $y \in wINF\ A$. Then, $[x, y] \cap wINF_1A \neq \emptyset$.*

Proof. Since $Z = (A + K \setminus \{0\}) \cup (A + K \setminus \{0\})^c$, following Proposition 2.3 we get $Z = (A + K \setminus \{0\}) \cup wINF\ A = (A + K \setminus \{0\}) \cup \text{Int}\ wINF\ A \cup \text{Fr}\ \ wINF\ A = (A + K \setminus \{0\}) \cup \text{Int}\ wINF\ A \cup wINF_1A$. Thus $[x, y] = ([x, y] \cap (A + K \setminus \{0\})) \cup ([x, y] \cap \text{Int}\ wINF\ A) \cup ([x, y] \cap wINF_1A)$. If we suppose $[x,y] \cap wINF_1 A = \emptyset$ then $[x,y] = ([x,y] \cap (A + \text{Int } Z_+)) \cup ([x,y] \cap \text{Int } wINF\ A)$. Following the hypothesis, $y \notin A + \text{Int } Z_+$ and thus the both sets $[x,y] \cap (A + \text{Int } Z_+)$ and $[x,y] \cap \text{Int } wINF\ A$ will be nonempty, open and disjoint sets. But $[x,y]$ is a point wise set and then, this contradiction prove the proposition. ⋄

Remark 2.10. *The properties of the weak infimal and the weak supremal points made possible the definition of a vector fuzzy integral for multifunction with some properties similar to the real fuzzy integral as we can see in [15], [16], [1].*

3 Solutions for (VP_0)

In what follows we'll consider Z be a locally convex space, $\mathcal{V}(X)$ be a fundamental system of neighborhoods forms by symmetric, barrelled, convex neighborhoods and $F : X \rightrightarrows \bar{Z}$, $C \subset X$. The vector optimization problems

$$(VP_0)\ INF_1 \bigcup_{x \in C} F(x)$$

$$(wVP_0)\ wINF_1 \bigcup_{x \in C} F(x)$$

As usually, an element from $INF_1 \bigcup_{x \in C} F(x)$ ($wINF_1 \bigcup_{x \in C} F(x)$) will be called a value of (VP_0) (respectively (wVP_0)) and following Theorem 2.7, the set of values for (wVP_0) is identically $\{-\infty\}$ or is a nonempty subset of Z which satisfies (wDP) properties.

Generally, for a vector optimization problem (VP), a solution for the problem is a point $x_0 \in C$ with the property $F(x_0) \cap Eff \bigcup_{x \in C} F(x) \neq \emptyset$. For the problem (VP_0), this definition is not suitable since if $F(x_0) \cap INF_1 \bigcup_{x \in C} F(x) \neq \emptyset$ then $F(x_0) \cap MIN \bigcup_{x \in C} F(x) \neq \emptyset$. Thus, using this definition, we will study in fact the existence of solutions for the problem $MIN \bigcup_{x \in C} F(x)$ which may not exist even the values set for the problem (VP_0) is nonempty.

In [20] we can find an approximative vector problem i.e. $(VP^{\varepsilon})\ {}^{\varepsilon}MIN \cup_{x \in C} F(x)$. As usually, a solution for this problem will be a point $x_0 \in C$ with the property $F(x_0) \cap^{\varepsilon} MIN \bigcup_{x \in C} F(x) \neq \emptyset$.

Let recall that for $\varepsilon > 0$, ${}^{\varepsilon}MIN\ A = \{a \in A \mid a \not< b - \varepsilon,\ \forall b \in A\}$. We remark that if $INF_1 A \neq \emptyset$ then for all $\varepsilon > 0$, ${}^{\varepsilon}MIN\ A \neq \emptyset$.

Taking into account this remark, we may think to define the solution of the problem (VP_0) using the solutions for the approximative problems, (VP_{ε}). Thus we can consider that $x_0 \in C$ is a solution for (VP_0) if $F(x_0) \cap {}^{\varepsilon}MIN \bigcup_{x \in C} F(x) \neq \emptyset$ for all $\varepsilon > 0$. A problem occur now if F is a single valued map since in this case, the previous condition $F(x_0) \cap {}^{\varepsilon}MIN \bigcup_{x \in C} F(x) \neq \emptyset$ for all $\varepsilon > 0$ conclude to the fact that $F(x_0) \cap MIn \bigcup_{x \in C} F(x) \neq \emptyset$ and we reduce our study again to the problem $MIN \cup_{x \in C} F(x)$.

By this reasons, we will consider in Definition 3.4 the solution of our problem as a net of approximative efficient points. Firstly, we'll present a general notion of approximative efficient points.

Definition 3.1. *Let $V \subset \mathcal{V}(0)$ and $A \subset Z$. The V-minimum points set of A will be denoted ${}^{V}MIN\ A$ and it will be given by*

$${}^{V}MIN\ A = \{y \in A \mid\ (y + V) \cap INF\ A \neq \emptyset\}$$

Remark 3.2. *If Z_+ is a generating cone then*

$$\bigcup_{\varepsilon > 0}{}^{\varepsilon}\ MIN\ A = \cup_{V \in \mathcal{V}(0)}{}^{V}\ MIN\ A$$

.

Indeed, if $y \in^{\varepsilon} MIN\ A$ then $y \in^{V_{\varepsilon}} MIN\ A$ where $V_{\varepsilon} = V \cup \{\varepsilon\}$ with $V \in \mathcal{V}(0)$. Now, if $y \in^{V} MIN\ A$ where $V \in \mathcal{V}(0)$ and $v \in V$ such that $y + v \in INF\ A$ we can find $\varepsilon > 0$ with $v > -\varepsilon$ (Z_+ is a generating cone). Thus, $y - \varepsilon \in INF\ A$ which implies $y \in^{\varepsilon} MIN\ A$ and the equality follows.

Definition 3.3. *a) A generalized sequence $(a_{\alpha})_{\alpha \in I} \subset Z$ ($(I, \succeq)$ is a directed set) is convergent to $y \in \bar{Z}$ if for each $V \in \mathcal{V}(0)$, $\exists \alpha' \in I$ such that $a_{\alpha} \in y + V$, $\forall \alpha \succeq \alpha'$. In this case we denote $a_{\alpha} \to_{\alpha} Y$*

b) Let $(A_{\alpha})_{\alpha \in I} \subset Z$. $\underset{\alpha}{Liminf}\ \ A_{\alpha} = \{y \in Z \mid \exists a_{\alpha} \in A_{\alpha},\ a_{\alpha} \to_{\alpha} y\}$.

In what follows we'll consider $\mathcal{V}(0)$ a fundamental system of convex, symmetric and barrelled neighborhoods. This is a directed set if we consider the preorder given by $V \succeq V'$ if and only if $V \subset V'$.

Proposition 3.4. *Let $A \subset Z$ be a nonempty set. Then,*

$$\underset{V}{Liminf}\ \ w^{V}MIN\ A = wMIN\ \bar{A}$$

.

Proof. Let $y \in wMIN\ \bar{A} = \bar{A} \cap wINF_1\ \bar{A}$. For all $V \in \mathcal{V}(0)$ we'll find $\mu_V \in (\alpha + V) \cap A$. Thus $\alpha \in (\mu_V - V) \cap wInf_1\bar{A} = (\mu_V + V) \cap wINF_1 A$, which say that $\mu_V \in w^V MIN\ A$ and $\mu_V \Rightarrow_V y$. We obtain that $y \in Liminf\ w^V MIN\ A$ and finally $wMIN\ \bar{A} \subseteq Liminf\ w^V MIN\ A$.

Now, let consider $y \in Liminf\ w^V MIN\ A$. We find $a_V \in w^V MIN\ A$, $a_V \Rightarrow_V y$ and since $a_V \in A$ we get $y \in \bar{A}$. Let suppose that $y \notin wINF\ A$; in this case we find $a \in (y - \text{Int}\ Z_+) \cap A$. Since $x = 1/2(a + y) + \text{Int}\ Z_+ \in \mathcal{V}(y)$ we find $V' \in \mathcal{V}(0)$ such that for all $V \subset V'$, $a_V \in x + \text{Int}\ Z_+$. Also, there exists $U \in \mathcal{V}(0)$ such that $x + U \subset a + \text{Int}\ Z_+$. For $V = V' \cap U$ we have $x + V \subset a + \text{Int}\ Z_+$ and $a_V + V \subset x + V + \text{Int}\ Z_+ \subset a + \text{Int}\ Z_+$.
But $a_V + V \cap wINF\ A \neq \emptyset$ and this contradiction proves the inclusion $\underset{V}{Liminf}\ \ w^V MIN\ A \subseteq wMIN\ \bar{A}$ and finally we get the equality. ⋄

Definition 3.5. *We'll call solution for the problem* (VP_0) *a net* $(x_V)_{V\in\mathcal{V}(0)}$ *from* X *such that for all* $V \in \mathcal{V}(0)$, $x_V \in^V MIN \bigcup_{x\in C} F(x)$. *Shortly, we'll write* (x_V).

Remark 3.6. *If the interior of the ordering cone* Z_+ *is nonempty, then* $(x_V)_{V\in\mathcal{V}(0)} \subset X$ *is a solution for* (wVP_0) *if and only if for all* $V \in \mathcal{V}(0)$, $(F(x_V) + V) \cap wINF_1 \bigcup_{x\in C} F(x) \neq \emptyset$.

Indeed, following Definition 3.1, if for all $V \in \mathcal{V}(0)$, $(F(x_V) + V) \cap wINF_1 \bigcup_{x\in C} F(x) \neq \emptyset$, then $(x_V)_{V\in\mathcal{V}(0)} \subset X$ is a solution for (wVP_0). Now, let $V \in \mathcal{V}(0)$, $\alpha \in F(x_V)$ and $v \in V$ such that $\alpha + v \in wINF \bigcup_{x\in C} F(x)$. If $\alpha \in wINF \bigcup_{x\in C} F(x)$, then $\alpha \in wINF_1 \bigcup_{x\in C} F(x)$ and $(F(x_V) + V) \cap^w INF_1 \bigcup_{x\in C} F(x) \neq \emptyset$. If $\alpha \notin wINF \bigcup_{x\in C} F(x)$, following Proposition 2.9 we get $[\alpha, \alpha + V] \cap wINF_1 \bigcup_{x\in C} F(x) \neq \emptyset$. Since V is a barrelled set we have $[\alpha, \alpha + v] \subset F(x_V) + V$and thus $(F(x_V) + V) \cap wINF_1 \bigcup_{x\in C} F(x) \neq \emptyset$. ⋄

In [2] we find the notion of "asymptotically weakly Pareto optimizing" (a.w.p.) sequence used for the characterization of the vector convex functions having the property that approximate necessary first order weakly-efficiency condition implies approximate weakly efficiency, a generalization of the asymptotically well-behaved functions from the scalar case. Thus, if X, Z are Banach spaces and $F : X \rightrightarrows \bar{Z}$ is a set-valued map, a sequence (x_n) in $\text{dom}\ F$ will be called *asymptotically weakly Pareto optimizing* (a.w.p) if $dist(F(x_n), wMIN \overline{\bigcup_{x\in X} F(x)}) \to 0$ when $-\infty \notin \bigcup_{x\in X} F(x)$ and $F(x_n) \to -\infty$, else.

It is not difficult to see that a generalization for this notion in a locally convex space is the following:

Definition 3.7. (x_V) *is a asymptotically weakly Pareto optimizing sequence if for all* $V \in \mathcal{V}(0)$, *there exists* $V' \in \mathcal{V}(0)$ *such that for all* $V'' \succeq V'$ *we have* $F(x_{V''}) \cap wMIN \overline{\bigcup_{x\in X} F(x)} + V \neq \emptyset$ *when* $-\infty \notin \bigcup_{x\in X} F(x)$ *and* $F(x_V) \to -\infty$, *else.*

Now we'll present the links between the solution of a vector optimization problem and an a.w.p. sequence.

Proposition 3.8. *Let* X, Z *be locally convex spaces and let* $F : X \rightrightarrows \bar{Z}$ *be a set-valued map. If* (x_V) *is a solution for the* (VP_0) *problem, then there exists a subsequence* $(x_{V'})$ *of* (x_V) *which is an a.w.p. sequence. Conversely, if* (x_V) *is an a.w.p. sequence and* (VP_0) *has a solution, then there exists a subsequence* $(x_{V'})$ *of* (x_V) *which is a solution for the* (VP_0) *problem.*

Proof. Let (x_V) be a solution for the (VP_0) problem. Obviously, $-\infty \notin \bigcup_{x\in X} F(x)$. Thus, $F(x_V) \in^V MIN \bigcup_{x\in X} F(x)$. Following Proposition3.4, for all $V \in \mathcal{V}(0)$, there exists a subsequence $(x_{V'})$ of (x_V) such that for all $V'' \succeq V'$, $F(x_{V''}) \cap wMIN \overline{\bigcup_{x\in X} F(x)} + V \neq \emptyset$. Thus, (x_V) is an a.w.p. sequence.

Now, let (x_V) be an a.w.p. sequence. If (VP_0) has a solution, then $-\infty \notin \bigcup_{x\in X} F(x)$ and thus for all $V \in \mathcal{V}(0)$, there exists $V' \in \mathcal{V}(0)$ such that for all $V'' \succeq V'$ we have $F(x''_V) \cap wMIN \overline{\bigcup_{x\in X} F(x)} + V \neq \emptyset$. Thus,for all $V \in \mathcal{V}(0)$, there exists $V' \in \mathcal{V}(0)$ such that for all $V'' \succeq V'$ we have $F(x''_V) \cap wINF \bigcup_{x\in X} F(x) + V \neq \emptyset$. Consequently, $F(x_{V'}) \in^V MIN \bigcup_{x\in X} F(x)$ and thus, $(x_{V'})$ is a subsequence of (x_V) which is a solution for (VP_0). ⋄

To conclude this section, let's give a characterization for the (VP_0)'s solution using the Pareto approximative subdifferentials. Let recall that for a multifunction $F : X \to Z$ and $V \subset Z$, the Pareto V-subdifferential of F at x_0 denoted by $\partial^V_{\not\succ} F(x_0)$ is

$\partial^V_{\not>} F(x_0) = \{T \in \mathcal{L}(X,Z) \mid \exists y_0 \in F(x_0)$ and $v \in V$ $T(x-x_0) \not> y - y_0 + v\}$.

We remark that $T \in \partial^V_{\not>} F(x_0)$ if and only if

$$(T(x-x_0) + F(x_0) + V) \cap INF \bigcup_{x\in X} F(x) \neq \emptyset$$

It is not difficult to give now the proposed characterization.

Proposition 3.9. *(x_V) is a solution for (VP_0) if and only if $0 \in \partial^V_{\not>} F(x_V)$ for all $V \in \mathcal{V}(0)$.*

4 Vector and scalar optimization problems

We study in this section the vector optimization problem defined with the efficient points presented in the first section in connection with a scalar optimization problem. The vector optimization problems will be

$$(VP_0)\ INF_1 \bigcup_{x\in C} F(x)$$

$$(wVP_0)\ wINF_1 \bigcup_{x\in C} F(x)$$

As usually, an element from $INF_1 \bigcup_{x\in C} F(x)$ ($wINF_1 \bigcup_{x\in C} F(x)$) will be called a value of (VP_0) (respectively (wVP_0)) and following Theorem 2.7, the set of values for (wVP_0) is identically $\{-\infty\}$ or is a nonempty subset of Z which satisfies (wDP) properties. For $V \in \mathcal{V}(0)$, we say that $x_V \in C$ is an approximative solution for (VP_0) if $(F(x_V) + V) \cap INF \bigcup_{x\in C} F(x) \neq \emptyset$.

For $x^* \in Z^*_+ \setminus \{0\}$ we'll consider the scalar problems:

$(P_{x^*}) :\ \inf \bigcup_{x\in C} x^* \circ F(x)$

$(P^*) :\ \bigcup_{y^*\in Z^*_+\setminus\{0\}} \inf \bigcup_{x\in C} x^* \circ F(x)$

$(SP) :\ \inf \bigcup_{x\in C} \inf x^* \circ F(x)$. Similarly, for $V \in \mathcal{V}(0)$, an approximative solution for (P_{x^*}) will be an element $x_V \in C$ such that $x^*(F(x_V)+V) \cap INF \bigcup_{x\in C} x^* \circ F(x) \neq \emptyset$. For the problem (P^*), an approximative solution will be a pair (x_V, x^*_V) satisfying $x^*_V(F(x_V)+V) \cap INF \bigcup_{x\in C} x^*_V \circ F(x) \neq \emptyset$.

Remark 4.1. *Following Theorem 2.7 and the definition of the "INF_1" set we have $wINF_1 \bigcup_{x\in C} F(x) + \mathrm{Int}\, Z_+ = \bigcup_{x\in C} F(x) + \mathrm{Int}\, Z_+$ and thus $x^* \circ v(wVP_0) + (0,\infty) = x^* \circ (\bigcup_{x\in C} F(x)) + (0,\infty)$. Consequently, $v(P_{x^*}) = \inf x^* \circ v(wVP_0)$.*

Remark 4.2. *It is not difficult to see that our scalar multivalued problem (P_{x^*}) is equivalent with the scalar single valued problem (SP) i.e. the problems (SMP) and (SP) have the same approximative solutions and the same values. Indeed, since $\inf \bigcup_{i\in I} A_i = \inf \bigcup_{i\in I} \inf A_i$ for $A_i \subset \mathbb{R}$, we deduce that the both problems have the same values. Now, for $V = (-\varepsilon, \varepsilon)$, if $x_V = x_\varepsilon)$ is an approximative solution for (SP) we have $m(x_\varepsilon) < \inf \bigcup_{x\in X} m(x) + \varepsilon = \inf \bigcup_{x\in X} \inf x^* \circ F(x) + \varepsilon = \inf \bigcup_{x\in X} x^* \circ F(x) + \varepsilon$. We can find $\varepsilon' > 0$ and $y_\varepsilon \in F(x_\varepsilon)$ such that $x^*(y_\varepsilon) < m(x_\varepsilon) + \varepsilon' < \inf \bigcup_{x\in X} x^* \circ F(x) + \varepsilon$. Thus, x_ε is an approximative solution for (P_{x^*}). Conversely, let x_ε be an approximative solution for (P_{x^*}). Thus, there exists $y_\varepsilon \in x^* \circ F(x_\varepsilon)$ such that $\inf \bigcup_{x\in X} m(x) + \varepsilon = \inf \bigcup_{x\in X} x^* \circ F(x) + \varepsilon > y_\varepsilon \geq m(x_\varepsilon)$. We conclude that x_ε is an approximative solution for (SP).*
We remark also that F is convex if and only if m is convex.

Proposition 4.3. *Let $V \in \mathcal{V}(0)$. If (x_V, x^*_V) is an approximative solution for (P^*) then x_V is an approximative solution for (wVP_0).*
If x_V is an approximative solution for (P_{x^*}) for some $x^* \in Z^*_+ \setminus \{0\}$ then x_V is a solution for (wVP_0).
If $\bigcup_{x\in C} F(x) + Z_+ \setminus \{0\}$ is a convex set and x_V is an approximative solution for (VP_0) then there exists $x^*_V \in Z^*_+ \setminus \{0\}$ such that (x_V, x^*_V) is an approximative solution for (P^*).

Proof. Let (x_V, x^*_V) be an approximative solution for (P^*). This means that there exists $v \in V$ and $y_V \in F(x_V)$ such that $x^*_V(y_V + v) \leq x^*_V(y)$ for all $y \in F(x)$, $x \in C$. Thus $y_V + v \notin \bigcup_{x\in C} F(x) + \mathrm{Int}\, Z_+$ which implies that $(F(x_V)+V) \cap wINF \bigcup_{x\in C} F(x) \neq \emptyset$ i.e. x_V is a solution for (wVP_0).

The proof is similar for the second assertion. The last part follows using the Hahn-Banach separation theorem. ◇

In what follows we'll prove that, although u_V is not an approximative solution for our problem, we can find an approximative solution x_V in a "neighborhood". More exactly, we have the following result.

Proposition 4.4. *Let $V \in \mathcal{V}(0)$ and let $u_V \in C$ which is not an approximative solution for (VP_0). Then there exists an approximative solution $x_V \in C$ such that $F(x_V) \cap (F(u_V) + V - Z_+ \setminus \{0\}) \neq \emptyset$.*

Proof. Let consider $K(u) = \{v \mid F(v) \cap (F(u) + V - Z_+ \setminus \{0\}) \neq \emptyset\}$. Since u_V is not an approximative solution , $K(u_V) \neq \emptyset$ and we denote $Y(v) = F(v) \cap (F(u) + V - Z_+ \setminus \{0\})$. For $x^* \in Z_+^* \setminus \{0\}$, there exists $x_V \in K(u_V)$ such that

$$\inf_{v \in K(u_V)} x^* \circ Y(v) + \sup x^*(V) > x^*(y_V)$$

where $y_V \in Y(x_V)$. Consequently, we can find $\tilde{v} \in V$ such that

$$\inf_{v \in K(u_V)} x^* \circ Y(v) + x^*(\tilde{v}) > x^*(y_V).$$

We'll prove that x_V is an approximative solution. Let suppose that this is not true and thus

$$F(x_V) + V \subseteq \bigcup_{x \in X} F(x) + Z_+ \setminus \{0\}.$$

This inclusion implies that for al $v \in V$ there exists z_V such that $y_V + v \in F(z_V) + Z_+\{0\}$. Since $y_V \in Y(x_V)$ we get $F(z_V) \cap (F(u_V) + V - Z_+ \setminus \{0\}) \neq \emptyset$ which say that $z_V \in K(u_V)$ and $y_V + v \in Y(z_V) + Z_+ \setminus \{0\}$. Let consider now $v = -\tilde{v}$; we get

$$\inf_{v \in K(u_V)} x^* \circ Y(v) > x^*(y_V) + x^*(-\tilde{v}) \geq$$

$$\geq \inf x^* \circ Y(z_V) \geq \inf_{v \in K(u_V)} x^* \circ Y(v)$$

which is a contradiction. Thus x_V is a solution and $F(x_V) \cap (F(u_V) + V - Z_+ \setminus \{0\}) \neq \emptyset$. $\diamond$

Another simplified problem which allows to us some information concerning the (wVP_0) problem is the problem (wVP_a) considered for the case when F is a subdifferentiable multifunction and given by

$$(wVP_a): \ wINF_1 \bigcup_{x \in X} S(x)$$

where $S(x) = \{Tx \mid T \in \partial_{\leq} F(x)\}$.

Proposition 4.5. *Let $V \in \mathcal{V}(0)$. If x_V is an approximative solution for (wVP_a) then x_V is an approximative solution for (wVP_0).*

Proof. Let suppose that x_V is an approximative solution for (wVP_a) and is not an approximative solution for (wVP_0). In this case, $F(x_V) + V \subseteq \bigcup_{x \in X} F(x) + \text{Int } Z_+$. Let consider $T \in \partial_{\leq} F(x_V)$. There exists $y_V \in F(x_V)$ such that $T(x - x_V) \leq y - y_V, \ \forall y \in F(x), \ \forall x \in X$. Let $v \in V$ and following our assumption, we can find $x' \in X$ such that $y_V + v \in F(x') + \text{Int } Z_+$. Thus, $T(x' - x_V) \leq y - y_V - v + v, \ \forall y \in F(x')$ which give to us that $T(x' - x_V) \leq v$. Consequently, $T(x_V) + V \subseteq \bigcup_{x \in X} T(x) + \text{Int } Z_+$ for all $T \in \partial_{\leq} F(x_V)$ which contradict the fact that x_V is an approximative solution for (wVP_a). $\diamond$

5 The INFSUP problem.

Let Y be a locally convex space, $D \subset Y$, $0 \in D$ and $\Phi : X \times Y \rightrightarrows Z$ such that $\Phi(x, 0) = F(x)$. For $y \in D$, let consider the following problem called the *perturbed problem*:

$$(P_y): \ \ INF_1 \bigcup_{x \in C} \Phi(x, y)$$

We denote $H(y) = INF_1 \bigcup_{x \in C} \Phi(x, y)$ and we'll say that (P_y) is stable if $\partial^V_{\not>} H(0) \neq \emptyset$ for all $V \in \mathcal{V}(0)$ where $\partial^V_{\not>} H(0) = \{S \in \mathcal{L}(Y, Z) \mid H(0) + V \not\subset \bigcup_{y \in Y} H(y) - S(y) + Z_+ \setminus \{0\}\}$. Let remark that if $(-\varepsilon, \varepsilon)$ is a neighborhood for 0 for $\varepsilon \in Z_+ \setminus \{0\}$ the $\partial^{(-\varepsilon,\varepsilon)}_{\not>} H(x_0)$ is the Pareto ε subdifferential for H at x_0 given in [5].

Definition 5.1. *We say that (x_V, y_V) is a solution for the perturbed problem if $(\Phi(x_V, y_V) + V) \cap INF \bigcup_{x \in C} \Phi(x, y_V) \neq \emptyset$.*

Obviously, this definition is equivalent with the following

Definition 5.2. *We say that (x_V, y_V) is a solution for the perturbed problem if $\Phi(x_V, y_V) \cap INF \bigcup_{x \in C} (\Phi(x, y_V) + V) \neq \emptyset$.*

As the problem (P_y), we can consider the problem $(P_x) : INF_1 \bigcup_{y \in D} \Phi(x, y)$ and the "complementary" problems $(Q_x) : SUP_1 \bigcup_{y \in D} \Phi(x, y)$ and $(Q_y) : SUP_1 \bigcup_{x \in C} \Phi(x, Y)$.

As expected, (x_V, y_V) is a solution for the problem (Q_x) if $(\Phi(x_V, y_V) + V) \cap SUP \bigcup_{x \in C} \Phi(x, y_V) \neq \emptyset$.

The dual problem for (P_x) (respectively (P_y)) will be

$$(D) : SUP_1 \bigcup_{y\in D} INF_1 \bigcup_{x\in C} \Phi(x,y)$$

(respectively $(D) : SUP_1 \bigcup_{x\in C} INF_1 \bigcup_{y\in D} \Phi(x,y)$) and the dual problem for (Q_x) (respectively (Q_y)) will be

$$(D') : INF_1 \bigcup_{x\in C} SUP_1 \bigcup_{y\in D} \Phi(x,y)$$

(respectively $INF_1 \bigcup_{y\in D} SUP_1 \bigcup_{x\in C} \Phi(x,y)$.)

Since our problem (D) may be expressed as $SUP_1 \bigcup_{y\in D} H(y)$, similarly to definition 3.5, a solution for this problem will be a net $(y_V)_V$ such that

$$H(y_V) \cap (SUP \bigcup_{y\in D} H(y) + V) \neq \emptyset.$$

The problems (D) and (D') are similar to the MINMAX problems and will be called the INFSUP problems. The saddle points for the MINMAX problems will be replaced by the saddle solutions given in the following definition.

Definition 5.3. *A net $(x_V, y_V)_V$ will be called a saddle solution for (D) if for all $V \in \mathcal{V}(o)$ we have $\Phi(x_V, y_V) \bigcap (INF \bigcup_{x\in C} \Phi(x, y_V) + V) \bigcap (SUP \bigcup_{y\in D} \Phi(x_V, y) + V) \neq \emptyset$.*

Remark 5.4. *If (x_V, y_V) is a saddle solution for (D) then (x_V, y_V) is a solution for (P_y) and (Q_x).*

Using theorem 2.7, the following proposition follows easily.

Proposition 5.5. *Let (x_V, y_V) be a saddle solution for (D). Then $\Phi(x_V, y_V) \bigcap (SUP_1 \bigcup_{y\in D} INF_1 \bigcup_{x\in C} \Phi(x,y) + V - Z_+) \bigcap (INF_1 \bigcup_{x\in C} SUP_1 \bigcup_{y\in D} \Phi(x,y) + V + Z_+) \neq \emptyset$.*

Now we are interested to see the links between the saddle solutions for (D) and the solutions for (VP_0) and (D). For this, we'll prove firstly a weak duality theorem.

Theorem 5.6. *(weak duality theorem) Let suppose that the following hypothesis does hold:*
$(H1)$: $SUP_1 \bigcup_{y\in D} \Phi(x,y) \bigcap (INF_1 \bigcup_{x\in C} \Phi(x,y) - \text{Int } Z_+) = \emptyset, \forall x \in C,\ y \in D$, then $INF_1 \bigcup_x SUP_1 \bigcup_y \Phi(x,y) \subset SUP \bigcup_y INF_1 \bigcup_x \Phi(x,y)$, $SUP_1 \bigcup_y INF_1 \bigcup_x \Phi(x,y) \subset INF \bigcup_x SUP_1 \bigcup_y \Phi(x,y)$, and $v(P_0) \bigcap (v(D) - \text{Int } Z_+) = \emptyset$.

Proof. Let $\alpha \in INF_1 \bigcup_x SUP_1 \bigcup_y \Phi(x,y)$ and suppose that there exists y_0 and $\mu \in INF_1 \bigcup_x \Phi(x,y_0)$ such that $\mu > \alpha$. In this case will exists x_0 and $\beta \in SUP_1 \bigcup_y \Phi(x_0,y)$, $\beta - \mu \in -\text{Int } Z_+$. This contradict our hypothesis, and thus the first inclusion id proved. Similarly, we get the second inclusion. Now, following the hypothesis and Theorem 2.7, we obtain

$$\bigcup_x \Phi(x,0) \bigcap (v(D) - \text{Int } Z_+) = \emptyset$$

and finally $v(P_0) \bigcap (v(D) - \text{Int } Z_+) = \emptyset$. ◇

Remark 5.7. *The conclusions from this theorem may be rewritten as follows:*

$$v(P_0) \subset (v(D) + \text{Int } Z_+) \cup \{+\infty\}$$

$$v(D) \subset (v(P_0) - \text{Int } Z_+) \cup \{-\infty\}$$

$$v(P_0) \bigcap (v(D) - \text{Int } Z_+) = \emptyset$$

Using the notations from [12], this is equivalent with $(D) \underset{\leftrightarrow}{<<} (P_0)$.

Theorem 5.8. *(strong duality theorem) Let suppose that the following hypothesis does hold:*
$(H1)$: $SUP_1 \bigcup_{y\in D} \Phi(x,y) \bigcap (INF_1 \bigcup_{x\in C} \Phi(x,y) - \text{Int } Z_+) = \emptyset, \forall x \in C,\ y \in D$
$(H2)$: $SUP_1 \bigcup_{y\in D} \Phi(x,y) = \Phi(x,0)$.

If (x_V, y_V) is a saddle point for Φ then :
i) (x_V) is a solution for (VP_0)
ii) (y_V) is a solution for (D) and
iii) $(\Phi(x_V, 0) + V + K) \bigcap (H(y_V) + V - K) \neq \emptyset$.

Proof. Following the Definition 5.3, if (x_V, y_V) is a saddle solution for (D), then for all $V \in \mathcal{V}(o)$ there exists
$\alpha_V \in \Phi(x_V, y_V) \bigcap (INF \bigcup_{x\in C} \Phi(x, y_V) + V) \bigcap (SUP \bigcup_{y\in D} \Phi(x_V, y) + V)$. $(*)$
Following Theorem 2.7 there exists $v_1 \in V$ such that $\alpha_V - v_1 \in (SUP_1 \bigcup_{y\in D} INF_1 \bigcup_{x\in C} \Phi(x,y) - K)$.(1)

Also, following the same theorem and the hypothesis (H_2), we have

$$\alpha_V \in \Phi(x_V, 0) + V + K.$$

Let suppose that $(\Phi(x_V, 0) + V) \bigcap INF \bigcup\limits_x \Phi(x, 0) = \emptyset$. Thus,

$$(\Phi(x_V, 0) + V) \subset \bigcup_x \Phi(x, 0) + \text{Int } Z_+.$$

Following hypothesis (H_2), $(\Phi(x_V, 0) + V) \subset \bigcup\limits_x SUP_1 \bigcup\limits_y \Phi(x, y) + \text{Int } Z_+ = INF_1 \bigcup\limits_x SUP_1 \bigcup\limits_y \Phi(x, y) + \text{Int } Z_+$.

By these reasons, $\alpha_V - v_1 \in \Phi(x_V, 0) + V - v_1 + K \subseteq \Phi(x_V, 0) + V + K \subseteq INF_1 \bigcup\limits_x SUP_1 \bigcup\limits_y \Phi(x, y) + \text{Int } Z_+$. (2)

From (1) and (2) we deduce that there exists $\alpha \in SUP_1 \bigcup\limits_{y \in D} INF_1 \bigcup\limits_{x \in C} \Phi(x, y)$ and $\alpha' \in INF_1 \bigcup\limits_x SUP_1 \bigcup\limits_y \Phi(x, y) = INF_1 \bigcup\limits_x \Phi(x, 0)$ such that $\alpha' - \alpha \in -\text{Int } Z_+$. Using the weak duality theorem we have $\alpha' \in SUP \bigcup\limits_y INF_1 \bigcup\limits_x \Phi(x, y)$. In this case, we can't have $\alpha' - \alpha \in -\text{Int } Z_+$ and the contradiction shows that (x_V) is a solution for (wP_0). Similarly we get that (y_V) is a solution for (D) and iii) follows obviously from (*). ◇

Remark 5.9. *In the hypothesis of the strong duality theorem, if iii) does hold, then i) and ii) follows .*

Indeed if $(\Phi(x_V, 0) + V + K) \bigcap (H(y_V) + V - K) \neq \emptyset$, then $(SUP_1 \bigcup\limits_y INF_1 \bigcup\limits_x \Phi(x, y) + V - K) \bigcap (INF_1 \bigcup\limits_x SUP_1 \bigcup\limits_y \Phi(x, y) + V + K) \neq \emptyset$. Following the same idea as in the theorem's proof, will exists α_V and v_1 such that $\alpha_V - v_1 \in (SUP_1 \bigcup\limits_{y \in D} INF_1 \bigcup\limits_{x \in C} \Phi(x, y) - K)$ and $\alpha_V - v_1 \in (INF_1 \bigcup\limits_{x \in C} SUP_1 \bigcup\limits_{y \in D} \Phi(x, y) + K)$. Using the same argument as in the theorem's proof, if we suppose that i) or ii) does not hold, we obtain a contradiction and the conclusion follows.

Proposition 5.10. *In the hypothesis of the strong duality theorem, if (y_V) is a solution for (D), then there exists (x_V) a solution for (VP_0) and iii)does hold.*

Proof. Following the definition, if (y_V) is a solution of (D), we have $H(y_V) \bigcap (SUP \bigcup\limits_y H(y) + V) \neq \emptyset$. In this case, there exists $v \in V$ such that $H(y_V) - v \not\subset H(0) - \text{Int } Z_+ = INF_1 \bigcup\limits_x \Phi(x, 0) - \text{Int } Z_+$ and consequently, $(H(y_V) - v) \bigcap \bigcup\limits_x \Phi(x, 0) + \text{Int } Z_+ \neq \emptyset$ or $H(y_V) - v \bigcup H(0) \neq \emptyset$. In the first case we get (x_V) such that $(H(y_V) - v) \bigcup \Phi(x_V, 0) + K \neq \emptyset$ which implies iii). For the second case, we can find $v_+ \in V \bigcap \text{Int } Z_+$ such that $v_+ - v \in V$ and $H(y_V) - v + v_+ \bigcap \bigcup\limits_x \Phi(x, 0) + K \neq \emptyset$ and the conclusion follows now similarly to the first case. Using the precedent remark, (x_V) will be a solution for (VP_0). ◇

6 The lagrangean duality.

In this section we present a particular case for the duality problems presented in the previous section. In the following let consider U be a locally convex space ordered by a convex pointed cone U_+ and $G : X \rightrightarrows U$ be a set valued map. The constraint set C is usually considered on form

$$C = \{x \mid G(x) \subset -U_+\} \quad (1)$$

or

$$C = \{x \mid G(x) \bigcap -U_+ \neq \emptyset\} \quad (2)$$

. Y will be $Y = \mathcal{L}(U, Z)$ and $D = \mathcal{L}_+(U, Z) = T \in \mathcal{L}(U, Z)$, $T(U_+) \subset Z_+$. If the interior of the cone U_+ is nonempty and there exists x such that $G(x) \bigcap -\text{Int } U_+ \neq \emptyset$ we'll say that G satisfies the Slater condition. In this case, $\Phi(x, T) = F(x) + T \circ G(x)$ and it is called the lagrangean. Usually, it is denoted $L(x, T)$ and the duality in this case is called the lagrangean duality. A saddle solution for the problem (D) will be called the saddle point for the lagrangean. The perturbed problem is

$$(P_T): \ INF_1 \bigcup_{x \in C} (F(x) + T \circ G(x)).$$

The dual problem is

$$(D) SUP_1 \bigcup_{T \in D} INF_1 \bigcup_{x \in C} L(X, T).$$

In this conditions, it is not difficult to observe that $SUP_1 \bigcup\limits_{T \in D} L(x, T) = SUP_1 F(x)$ and $(SUP_1 F(x)) \bigcap (INF_1 \bigcup\limits_{x \in C} L(x, T) - \text{Int } Z_+) = \emptyset$. Thus, (H_1) is satisfied and the weak duality theorem does hold. If $SUP_1 F(x) = F(x)$ (in particular if F is a single valued map), (H_2) is also satisfied and thus the strong duality theorem does hold.

Proposition 6.1. *Let suppose that C is on form (2), the Slater condition is satisfied for the G and $F \times G(C) + \text{Int } Z_+ \times \text{Int } U_+$ is a convex set (this say that $F \times G$ is a $Z_+ \times U_+$- subconvexlike multifunction) . If (x_V) is a solution for (VP_0), then there exist $(H_V) \subset \mathcal{L}_+(U, Z)$ such that $H_V(G(x_V) \cap -U_+) \subset V$ and (x_V, H_V) is a solution for (P_H). The converse is true if C is on form (1).*

Proof. Let (x_V) be a solution for (VP_0) and $(\alpha'_V) \in F(x_V) + V$ such that $\alpha'_V \in INF \bigcup_{x\in C} F(x)$. Since $\alpha'_V = y_V + v$ where $y_V \in F(x_V)$, $v = v'_+ - v_+ \in V$ $(v_+, v'_+ \in V \cap Z_+)$, we remark that $\alpha_V = y_V - v_+ \in INF \bigcup_{x\in C} F(x)$. Thus $((F - \alpha_V) \times G)(C) + \text{Int } Z_+ \times \text{Int } U_+) \bigcap (-(\text{Int } Z_+ \times \text{Int } U_+)) = \emptyset$.
In this case will exist $z^* \in Z^*_+$ and $u^* \in U^*_+$ such that

$$z^*(y - \alpha_V) + u^*(z) \geq 0$$

for all $y \in F(x),\ z \in G(x),\ x \in C$. Let $x = x_V,\ y = y_V,\ z = z_V \in G(x_V) \bigcap (-U_+)$. Thus, $z^*(v_+) + u^*(z_V) \geq 0$, $u^*(z_V) \leq 0$ and consequently, $z^*(v_+) \geq -u^*(z_V)$. Necessarily, $z^* \neq 0$ (else, $u^*(z) \geq 0$ for all $z \in G(C)$ and the Slater condition assure the existence of $z_0 \in \text{Int } U_+$ with $u^*(z_0) < 0$). Let $H_V(z) = u^*(z)\frac{v_+}{z^*(v_+)}$. It is not difficult to see that $H_V(z_V) \in -V = V$ and thus $H_V(G(x_V) \cap -U_+)) \subset V$. Also, $H_V(U_+) \subset Z_+$ and $z^* \circ H_V = u^*$ which gives that $z^*(y - \alpha_V) + z^* \circ H_V(z) \geq 0$, for all $y \in F(C),\ z \in G(C)$. This implies that $y - y_V + v_+ + H(z) \notin -\text{Int } Z_+,\ \forall z \in G(x),\ y \in F(x),\ x \in C$ or equivalently, $y - y_V + v_+ - H_V(z_V) + H(z) + H_V(z_V) \notin -\text{Int } Z_+,\ \forall z \in G(x),\ y \in F(x),\ x \in C$.
Since $H_V(z_V) \in V$ and $V + V = V$, we deduce $(F(x_V) + V + H_V(G(x_V) \bigcap -U_+)) \bigcap INF \bigcup_{x\in C} F(x) + H_V \circ G(x) \neq \emptyset$ which say that (x_V, H_V) is a solution for (P_H).

For the converse, let (x_V, H_V) be a solution for (P_H) and $H_V \circ G(x_V) \bigcap -U_+ \subset V$. Since C is on form (1), this means that $H_V \circ G(x_V) \subset V$. Thus there exists $y_V \in F(x_V),\ z_V \in G(x_V),\ v \in V$ such that $\beta_V = y_V + H_V(z_V) + v \in INF \bigcup_{x\in C} (F(x) + H_V \circ G(x))$. Let observe that following our hypothesis, $H_V(z_V) \in V$ and thus $\beta_V \in F(x_V) + V$. Now, let suppose that $\beta_V \notin INF \bigcup_{C} F(x)$. In this case, there exists $x \in C$ and $y \in F(x)$ such that $y < \beta_V$. Since the Slater condition does hold and $H_V \in \mathcal{L}_+(U, Z)$, there exists $z \in G(x) \bigcap -U_+$ and thus $H_V(z) + y < \beta_V$ which contradict the fact that $\beta_V \in INF \bigcup_{x\in C} (F(x) + H_V \circ G(x))$. Thus, $(F(x_V) + V) \bigcap INF \bigcup_{x\in c} F(x) \neq \emptyset$ and (x_V) will be a solution for (VP_0). $\diamond$

Remark 6.2. *We may observe that for the second part of the previous theorem we don't use the fact that $F \times G$ is a $Z_+ \times U_+$-subconvexlike multifunction so, this part rest valid without this condition.*

Corollary 6.3. *Let suppose that C is on form (1), the Slater condition is satisfied for G and $F \times G(C) + \text{Int } Z_+ \times \text{Int } U_+$ is a convex set. Also, let suppose that $SUP_1 F(x) = F(x)$. Then, (x_V) is a solution for (VP_0) if and only if there exists (H_V) solution for (D) such that $(F(x_V) + V + Z_+) \bigcap (\Phi(H_V) + V - Z_+) \neq \emptyset$.*

Proof. Following the precedent theorem, if (x_V) is a solution for (VP_0), there exists $(H_V) \subset D$ such that $H_V(G(x_V) \bigcap -U_+) \subset V$ and $(F(x_V) + H_V \circ G(x_V)) \bigcap INF \bigcup_{x\in C} (F(x) + H_V \circ G(x) + V) \neq \emptyset$. Thus, $(F(x_V) + V) \bigcap (\Phi(H_V) + V - Z_+) \neq \emptyset$. Following Remark 5.9, (H_V) will be a solution for (D). The same remark proves the converse. $\diamond$

Theorem 6.4. *The characterization of the saddle points for the lagrangean.*

A set of ordered pairs $(x_V, T_V) \in C \times D$ is a saddle point for the lagrangean L if and only if for all $V \in \mathcal{V}(0)$, there exists $y_V \in F(x_V),\ z_V \in G(x_V),\ v \in V$ such that:
a) $y_V + T_V(z_V) \in^V MIN\ L(C, T_V)$
b) $G(x_V) \subset -Z_+$
c) $-T(G(x_V)) \subset Z_+ \setminus (v + \text{Int } Z_+),\ \forall T \in D$
d)$(F(x_V) - y_V - T_V(z_V) - v) \bigcap \text{Int } Z_+ = \emptyset$

Remark 6.5. *If $(-\varepsilon, \varepsilon)$ is a fundamental system of neighborhoods for 0, these results leads to the theorems about the $\varepsilon-$duality, ε-Lagrangean multiplicators and ε-saddle points given in [11].*

Proof. (of the theorem) Let (x_V, T_V) be a saddle point for the lagrangean. Following the definition, there exists $y_V \in F(x_V)$, $z_V \in G(x_V)$, $v \in V$ such that

$$(y_V + T_V(z_V) + V) \bigcap INF \bigcup_{x\in C} L(x, T_V) \quad (1)$$

and

$$y_V + T_V(z_V) + v - \alpha - Tz' \notin -\text{Int } Z_+ \quad (2)$$

$\forall \alpha \in F(x_V),\ z' \in G(x_V),\ T \in D$. Obviously, (1) is equivalent with a).
Let put in (2), $\alpha = y_V,\ z' = z_V$. We get $T_V(z_V) -$

$T(z_V) + v_2 \notin \text{Int } Z_+,\ \forall T \in D$. let suppose that b) does not holds, i.e. there exists $z_V \in G(x_V)$ such that $z_V \notin -Z_+$. Thus, there exists $z^* \in Z_+^* \setminus \{0\}$ with $z^*(z_V) > 0$. Let consider $\bar{T}(z) = \frac{z^*(z)}{z^*(z_V)}(\bar{k} + v) + T_V(z), \bar{k} \in \text{Int } Z_+,\ \bar{k} > -v$. We observe that $\bar{T} \in D$ and if we take $\bar{T}$ in (2), we get $\bar{k} \notin \text{Int } Z_+$, false. Thus, b) does holds and $-T_V \circ G(x_V) \subset Z_+$.If we take $T = 0$ in (2), we get $-T_V \circ G(x_V) - v \notin \text{Int } Z_+$ and c) follows. For d), it suffice to take $T = 0$ in (2).

For the converse, using d), b), c) we obtain

$$y_V + T_V(z_V) = v - y \notin -\text{Int } Z_+,\ \forall y \in F(x_V)$$

Let suppose that there exists $T \in D$, $z \in G(x_V)$ such that

$$y_V + T_V(z_V) + v - y - T(z) \in -\text{Int } Z_+$$

This gives that $y_V + T_V(z_V) = v - y \in T(z) - \text{Int } Z_+ \subset -Z_+ - \text{Int } Z_+ = -\text{Int } Z_+$, false. We conclude that $y_V + T(z_V) \in V + SUP \bigcup_{T \in D} (F(x_V) + T \circ G(x_V))$ and following a) we get also that $y_V + T_V(z_V) \in V + INF \bigcup_{x \in C} (F(x) + T_V \circ G(x))$. Thus, (x_V, T_V) will be a saddle point for the lagrangean. $\diamond$

7 The conjugate duality

In this section we present another kind of vector duality, the conjugate duality. The name comes from the use of the conjugate multifunction to defining the dual problem. We recall that the conjugate of a multifunction $G : V \rightrightarrows Z \cup \{+\infty\}$ is a multifunction $G^* : \mathcal{L}(V, Z) \rightrightarrows \bar{Z}$ given by

$$G^*(S) = wSUP_1\{Sv - u,\ u \in G(v),\ v \in \text{Dom } G\}$$

where $\text{Dom } G = \{v \in V \mid \emptyset \neq G(v) \subset Z$.
Our constrained vector problem considered until now $(P_y) :\ INF_1 \bigcup_{x \in C} \Phi(x, y)$ will be restrained to an unconstrained problem

$$(\tilde{P}_y) :\ INF_1 \bigcup_{x \in X} \tilde{\Phi}(x, y)$$

by replacing the multifunction Φ with $\tilde{\Phi}$, $\tilde{\Phi}(x, y) = \Phi(x, y)$ if $x \in C, y \in D$ and $\tilde{\Phi}(x, y) = +\infty$ for the complementary cases. The conjugate dual problem associated to $(\tilde{P}_0)$ is

$$(D_C) :\ SUP_1 \bigcup_{S \in \mathcal{L}(Y,Z)} -\tilde{\Phi}^*(0, S).$$

These problems are presented in [14] and a weak duality theorem can be deduced from more general results presented there. Firstly, we recall some notations which will be used introduced in [12]. For two nonempty subsets of $\bar{Z}$

- $A << B$ if there is no $a \in A$ and $b \in B$ such that $a > b$
- $A \underset{\rightarrow}{<<} B$ if $A << B$ and for all $a \in A$ there exists $b \in B$ such that $a \leq b$.
- $A \underset{\leftarrow}{<<} B$ if $-B \underset{\rightarrow}{<<} -A$.
- $A \underset{\leftrightarrow}{<<} B$ if $A \underset{\rightarrow}{<<} B$ and $A \underset{\leftarrow}{<<} B$.

Theorem 7.1. *[14]*

$$(D_C) \underset{\leftrightarrow}{<<} (\tilde{P}_0).$$

Proof. Following theorem 3.1 [10] we have

$$wINF_1 \bigcup_{x \in X} \tilde{\Phi}(x, 0) \subseteq wSUP \bigcup_{T \in \mathcal{L}(Y,Z)} (-w\Phi^*(0, T))$$

$$wSUP_1 \bigcup_{T \in \mathcal{L}(Y,Z)} (-w\Phi^*(0, T)) \subseteq wINF \bigcup_{x \in X} \tilde{\Phi}(x, 0).$$

Now, following theorem 2.7 and the precedent notations, the theorem follows. $\diamond$

Remark 7.2. *The condition $(D_C) << (\tilde{P}_0)$ means that $v(\tilde{P}_0) \bigcap (v(D_C) - \text{Int } Z_+) = \emptyset$, a conclusion presented in the weak duality theorem 5.6. We remark also that the condition (H_1) imposed in theorem 5.6 is equivalent in this case with $\Phi(x, 0) \bigcap (-\tilde{\Phi}^*(0, S) - \text{Int } Z_+) = \emptyset$ which is satisfied because the conjugate definition.*

A lagrangean is also considered in this case, $\Phi'(x, S) = -SUP_1 \bigcup_y (S(y) - \tilde{\Phi}(x, y))$. Let remark that $INF_1 \bigcup_x \Phi'(x, S) = -\Phi^*(0, S)$ and thus the dual problem may be written as

$$(D_C) :\ SUP_1 \bigcup_{S \in \mathcal{L}(Y,Z)} INF_1 \bigcup_{x \in X} \Phi'(x, S).$$

A saddle point for the lagrangean Φ' is a saddle solution for the problem (D_C), i.e. a net of pairs $(x_V, S_V)_V$ such that $\Phi'(x_V, S_V) \bigcap (INF \bigcup_x \Phi'(x, S_V) + V) \bigcap (SUP \bigcup_S \Phi'(x_V, S) + V) \neq \emptyset$. A strong duality theorem may also be given, in the same conditions as in theorem 5.8. More exactly, the condition (H_1) is automatically satisfied as we already have seen in the precedent remark and the condition $(H2)$ means that $SUP_1 \bigcup_S \Phi'(x, S) = \Phi(x, 0)$.

Theorem 7.3. *Let suppose that* $SUP_1 \bigcup_S \Phi'(x,S) = \Phi(x,0)$ *and* (x_V, S_V) *is a saddle point for the lagrangean. Then,* (x_V) *is a solution for* (P_0)*,* (S_V) *is a solution for* (D_C) *and* $-\Phi^*(0,S_V) \bigcap \Phi(x_V,0) + K + V \neq \emptyset$.

Proof. Let consider (x_V, S_V) a saddle point for the lagrangean. Thus, there exists $\alpha_V \in \Phi'(x_V,S_V) \bigcap INF(\bigcup_x \Phi'(x,S_V) + V) \bigcap SUP(\bigcup_S \Phi'(x_V,S) + V)$. Thus, $\alpha_V \in -\Phi^*(0,S_V) - K + V$ and $\alpha_V \in \Phi(x_V,0) + K + V$. In this case, we can find $v \in V$ such that $\alpha_V \in -\Phi^*(0,S_V) - \text{Int } Z_+ \subset SUP_1 \bigcup_S -\Phi^*(0,S) - \text{Int } Z_+ = v(D_C) - \text{Int } Z_+$. If we suppose that (x_V) is not a solution for (P_0), we'll have $\Phi(x_V,0) + V \subset \bigcup_x \Phi(x,0) + \text{Int } Z_+$ and consequently, we have $\alpha_V - v \in \Phi(x_V,0) + K + V - v \subset \Phi(x_V,0) + K + V \subset \bigcup_x \Phi(x,0) + \text{Int } Z_+ = INF_1 \bigcup_x \Phi(x,0) + \text{Int } Z_+ = v(P) + \text{Int } Z_+$. We conclude that $\alpha_- v \in V(D_C) \bigcap v(P) + \text{Int } Z_+$ which contradict the weak duality theorem. Similarly, (S_V) is a solution for the dual problem (D_C) and the intersection does hold obviously from the saddle point condition. ◇

Remark 7.4. *As we can see from the theorem proof, the condition* $-\Phi^*(0,S_V) \bigcap \Phi(x_V,0) + K + V \neq \emptyset$ *implies that* (x_V) *is a solution for* (P_0) *and* (S_V) *is a solution for* (D_C).

Remark 7.5. *It is not difficult to see that if we denote* $\Phi_x(y) = \Phi(x,y)$*, the condition* (H_2) *is equivalent with* $\Phi_x(0) = \Phi_x^{**}(0)$.

Conclusions. We studied here a general vector optimization problem (wVP_0) with a nonempty valued set in some general conditions concerning the multifunction values set. The results proved show that we may obtain informations concerning the values and the approximative solutions for our vector optimization problem (wVP_0) by studying a more simplified problem, the scalar optimization problem. Also, we define a dual problem and we present the lagrangean and the conjugate duality. Some stability results as well as some results for the case when the cone has an empty interior will be given in a future paper.

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