Rich dynamical behaviors of a predator-prey system with state feedback control and a general functional responses

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Abstract: - In this paper, we study dynamics of a logistical predator-prey system with state feedback control and a general functional responses. By using the Poincare map, some conditions for the existence and stability of semi-trivial solution and positive periodic solution are obtained. Numerical results are carried out to illustrate the feasibility of our main results, and it is shown that a chaotic solution is generated via a cascade of period-doubling bifurcations, which implies that the presence of pulses makes the dynamic behavior more complex.

Key-Words: - Prey-predator system; State feedback control; Periodic solution; Extinction; Bifurcation

1 Introduction

Recently, many authors have investigated the predator-prey models concerning the impulsive pest control [1-10]. Two important methods for pest control are biological control and chemical control. Pest controls involve two mathematical ways. One is the fixed moment pulse which often is applied to describe spraying pesticides or releasing natural enemies at fixed time [1, 2, 3, 5, 6, 10]. Another is the state pulse control which often is employed to represent the pest control when the number of insect pests in the field reaches the economic injury level [4, 7, 8, and 9].

In the fixed moments pulse control, authors often assume that the pests grow with Logistic law and the predation effects follow different Holling functional responses which made the standard Lotka-Volterra system more realistic[2,11,12]. In these literatures, pest-eradication periodic solutions and the permanence of systems are investigated. In the state pulse control, scholars often suppose that the pests grow with linear law and the predation effects follow bilinear law (such βxy, x and y are the numbers of pest and natural enemy, respectively.). In these literatures, the existence and the stability of positive periodic solution are studied. In recently, the state pulse control has been widely applied in a microorganism continuous culture system such as literatures [13], [14] and [15]. In practical ecological systems, the control measures (by catching, poisoning the prey or releasing the natural enemy, etc.) are taken only when the amount of species reaches a threshold value, rather than the usual impulsive fixed-time control strategy for the former is more economical and beneficial to ecological equilibrium than the latter.

As well known, the class of functional responses, how much "attention" the predator is pay to the prey, is undifferentiated among different predator-prey relationship. Pei has given detailed summary in literature [2]. But in the realistic situation, there are other functional responses such as Beddington-DeAngelis, Arditi-cakaya, and AkArditi-Ginzburg which are modified by predator density [16]. Another hand, the Logistic model is applied widely which was proposed in 1837 by Holland biologist Verhulst [17] in his research on population development process. He considered that, for the population model, a stable population would consequently have a saturation level characteristic: this is typically called the carrying capacity K, and forms a numerical upper bound on the growth size. To incorporate this limiting form he introduced the logistic growth equation which is shown later to provide an extension to the exponential model. Morgan [18] ingeniously used the equation to describe herding behaviors of African elephants. Krebs [19] also used the logistic equation to fit to population data for Peruvian anchovies. The logistic function model is also being used to describe microbial growth [20]. More analysis of logistic growth models were proposed in 2001 by Solaris [21]. Some good results are also obtained [22-24]. Based on their study, it is more appropriate to add the logistic growth term to our models. In this paper, we will incorporate a general functional responses and logistic growth law into a prey-predator model and formulate a state impulsive control system.

The paper is arranged like this. In Section 2, a logistical predator-prey system with impulsive state feedback control and a general of functional responses is given. In Section 3, a semi-trivial periodic solution obtained and the stability is analyzed by making use of the analogue of the Poincare criterion. Theoretical results of dynamical behaviors are presented, including the transcritical and flip bifurcations. In Section 4, numerical results, such as phase portraits of period solutions, chaotic solutions and bifurcation diagrams, are illustrated by computer graphics.

2 Problem Formulation

The model we considered is based on the following plausible predator-prey interaction model:

$$\begin{cases} \dfrac{du}{dt} = ru(1-\dfrac{u}{K}) - h(u)v \\ \dfrac{dv}{dt} = -dv - \delta v^2 + h(u)v \end{cases} \qquad (1)$$

where r, K, e, δ, d are positive constants. The variables u and v denote the density of prey species and that of predator species, respectively. We assume that the prey is a dangerous pest, and that the predator was introduced to suppress its density.

It also is assumed that the prey (pest) grows logistically to its environmental capacity K, with an intrinsic birth rate constant r. The function $h(u)$ is the functional response which means that the per capita rate at which predator u is represented by the term $h(u)$. We can also understand it as follow: $h(u)$ represents how much "attention" the predator is pay to the prey. The constant d is the death rate of predator u and δ is the density restraint rate. It is the assumption that the equation $h(0) = 0$. Here the term of $h(u)$ indicates that the predator species is nonlinear density-dependent.

It is convenient at the outset to rescue the aforementioned system by writing:

$$\frac{u}{K} = x,\ rs = x,\ \frac{v}{rk} = y,\ h(xK) = f(x),\ x \in \{0.1).$$

Then it becomes

$$\begin{cases} \dot{x} = x(1-x) - f(x)y \\ \dot{y} = \theta y(-1 + f(x) - \lambda y) \end{cases} \qquad (2)$$

It follows: where x and y respectively, represent densities or biomass of the prey-species and the predator-species; $\theta = \dfrac{d}{r}$ and $\lambda = \delta rK$ are resulting coefficients of the new system (2); $f(x)$ and θ is the so-called functional response and the death rate of the predator in the new system, respectively. A direct calculation shows that (0, 0) are saddle.

In (2), we will use the following notations and assumptions:

(A1): x_i, y_i are positive equilibrium points of the system in the first quadrant with $x_i > 0, y_i > 0$, $i = 1,2\cdots n.$.

(A2): Set $x^* = \min\{x_i\}$, and A (x^*, y^*) is a stable equilibrium point.

(A3) $f(x) > 0$ for $x \in (0,1)$ and $x(1+x)/f(x) > [-1 + f(x)]/\lambda$ for $x \in (0, x^*)$.

Based on the above assumptions, for $x \in (0, x^*)$. It follows that

$$H(x) = \lambda x(1-x) - f^2(x) + f(x) > 0. \qquad (3)$$

Now we consider the prey-predator model (2) by introducing a state feedback control strategy, rather than the usual fixed-time control strategy. The controlled system is modeled by the following equations:

$$\begin{cases} \left.\begin{aligned} \dot{x} &= x(1-x) - f(x)y \\ \dot{y} &= \theta y(-1 + f(x) - \lambda y) \end{aligned}\right\} x \neq h \\ \left.\begin{aligned} \Delta x &= -px \\ \Delta y &= qy + \tau \end{aligned}\right\} x = h, \end{cases} \qquad (4)$$

where the parameters $p \in (0,1)$, $h \in (0,1)$ and $x^* > h > 0$, $q > 0$, $\tau \ge 0$, $\Delta x(t) = x(t^+) - x(t)$, $\Delta y(t) = y(t^+) - y(t)$. The biological significance of the other coefficients is the same as in the model (2). When the amount of the prey-species reaches the threshold value h at the time $t_i(h)$, controlling measures (catching or poisoning the prey species and releasing the predator) are taken and the amounts of prey-species and predator-species abruptly turn to $(1-p)h$ and $(1+q)y(t_i)(h) + \tau$, respectively. In this paper, $X(t) = (x(t), 0)$ is called a semi-trivial solution for $y = 0$.

3 Dynamical properties

3.1 Poincare map

To discuss the dynamics of system (4), we define three cross-sections to the vector field (4) by

$\sum_1 = \{(x,y): x = h, y \ge 0$

$\sum_0 = \{(x,y): x = (1-p)h, y \ge 0\}$.

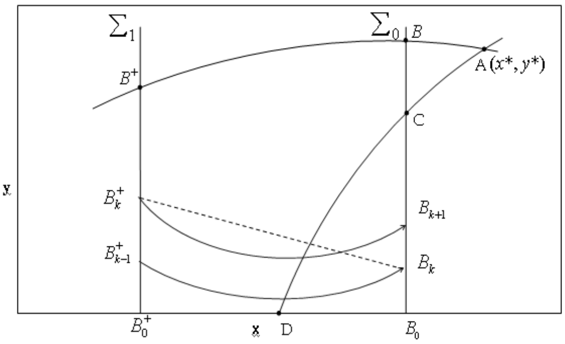


Figure 1: Poincare map of system (3)

As shown in Fig.1, The curve $\widehat{BB^+}$, and $\widehat{AD}$ is respectively represented by $y = \dfrac{x(1-x)}{f(x)}$ *and* $y = \dfrac{-1+f(x)}{\lambda}$, where $(1-p)h < x < h$,

$B^+ = ((1-p)h, \dfrac{(1-p)h(1-(1-p)h)}{f((1-p)h)}),$

$B = (h, \dfrac{h(1-h)}{f(h)}), B_0 = (h,0), B_0^+ = ((1-p)h, 0),$

$C(h, \dfrac{-1+f(h)}{\lambda}), D(\overline{x}, 0)$, where $f(\overline{x}) = 1,$

$B_k = (h, y_k), B_{k-1}^+ = ((1-p)h, y_{k-1}).$ Denote

$\Pi = \{(x,y) \mid 0 < y < \dfrac{x(1-x)}{f(x)}, (1-p)h < x < h\}$

and $\Pi_1 = \{(x,y) \mid 0 < y < \dfrac{-1+f(x)}{\lambda}, \overline{x} < x < h\}$.

It is obvious that $dx > 0$ and $dx > 0, dy > 0$, are satisfied at the point $(x,y) \in \Pi$ and at the point $(x,y) \in \Pi_1$, at the same time that $dx > 0, dy < 0$ are satisfied at the point $(x,y) \in \{\Pi - \Pi_1\}$. The case that $dx = 0, dy < 0$ are satisfied at the point $(x,y) \in \widehat{BB^+}$. Any orbit initialing at the point $(\hat{x}, 0) \in \overline{B_0^+ B_0}$ keeps $y(t) = 0$ and tends to ∞.

Now, we construct two Poincare maps.

First, we choose section $\sum_1$ as a Poincare$'$ section. In view of the vector field of system (4), the trajectory with the initial point B_{k-1}^+ intersects the section $\sum_0$ at the point $B_k(h, y_k)$, where y_k is determined by y_{k-1}^+, which can be expressed by $y_k = g(y_{k-1}^+)$. At the point B_k , the trajectory of (4) is subjected by impulsive effects to jumps to the $B_k^+((1-p)h, (1+q)y_k + \tau)$, and it can be obtained the following Poincare map P :

$$y_k^+ = (1+q)g(y_{k-1}^+). \qquad (5)$$

Secondly, we consider another type of the Poincare map. Choose section $\sum_0$ as another Poincare section. In view of the vector field of system (4), the point B_k jumps to point B_k^+ on the section $\sum_1$ due to the impulsive effects, and the trajectory with the initial point B_k^+ intersects the section $\sum_0$ at the point $B_{k+1}(h, y_{k+1})$, where y_{k+1} is determined by y_{k+1} , which can be expressed by:

$$y_{k+1} = g((1+q)y_+ + \tau) \equiv F(q, \tau, y_k). \qquad (6)$$

The Poincare map P_1 is constructed as (6).

3.2 Existence and stability of positive periodic solution with the case of $\tau = 0$

Definition 5. *A solution* $z(t) = (x(t), y(t))$ *of system (4) is called a semi-trivial solution if one of its components is zero and the other is nonzero.*

It is easy to see that the semi-trivial periodic solution with $y = 0$ of system (4) exists if and only if $\tau = 0$. Hence we can begin our study by setting $\tau = 0$.

When $\tau = 0$, system (4) has the following special form:

$$\begin{cases} \left.\begin{aligned} &\dot{x} = x(1-x) - f(x)y, \\ &\dot{y} = \theta(-y + f(x)y - \lambda y^2), \end{aligned}\right\} x \ne h, \\ \left.\begin{aligned} &\Delta x = -px \\ &\Delta y = qy \end{aligned}\right\} x = h. \end{cases} \qquad (7)$$

3.2.1. Semi-trivial periodic solution

In this section, the semi-trivial solution of system (7) will be considered. Let $y(t) = 0$ for $t \in (0, \infty)$, then from system (7), there have

$$\begin{cases} \dot{x} = x(1-x), x \ne h, \\ \Delta x = -px, x = h. \end{cases}$$

(8)

Setting $x_0 = x(0) = (1-p)h$ leads to the solution $x(t)=\dfrac{c_0\exp(t)}{1+c_0\exp(t)}$ for system (8), where

$c_0=\dfrac{(1-p)h}{1-(1-p)h}$. Denote $T=\ln\dfrac{h}{(1-h)c_0}$, then there have $x(T)=h$ and $x(T^+)=(1-p)h$. This means that system (7) has the semi-trivial periodic solution as follows:

$$\begin{cases} x(t)=\dfrac{c_0\exp(t-(k-1)T)}{1+c_0\exp(t-(k-1)T)},(k-1)T<t\le kT,k\in N \\ y(t)=0. \end{cases} \tag{9}$$

which is denoted by $(\xi(t),0)$.

We now discuss the stability of the semi-trivial periodic solution (9). In order to present our results in a straight forward manner, we introduce the following technical lemma.

Lemma 3.1 *(see[25]) The T-periodic solution $(x,y)=(\xi(t),\eta(t))$ of the system*

$$\begin{cases} \dfrac{dx}{dt}=P(x,y),\dfrac{dy}{dt}=Q(x,y),\text{if } \phi(x,y)\ne 0, \\ \Delta x=\alpha(x,y),\Delta y=\beta(x,y),\text{if } \phi(x,y)=0. \end{cases}$$

is orbitally asymptotically stable if the Floquet multiplier satisfies the condition $|\mu_2|<1$, where

$$|\mu_2|=\prod_{k=1}^{q}\Delta_k\exp\left(\int_0^T\frac{\partial P}{\partial x}(\xi(t),\eta(t))+\frac{\partial Q}{\partial y}(\xi(t),\eta(t))dt\right),$$

$$\Delta_k=\frac{P_+(\frac{\partial\beta}{\partial y}\frac{\partial\phi}{\partial x}-\frac{\partial\beta}{\partial x}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial x})+Q_+(\frac{\partial\alpha}{\partial x}\frac{\partial\phi}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y})}{P\frac{\partial\phi}{\partial x}+Q\frac{\partial\phi}{\partial y}},$$

and $P,Q,\dfrac{\partial\alpha}{\partial x},\dfrac{\partial\alpha}{\partial y},\dfrac{\partial\beta}{\partial x},\dfrac{\partial\beta}{\partial y},\dfrac{\partial\phi}{\partial x}$ *and* $\dfrac{\partial\phi}{\partial y}$ *are calculated at the point* $(\xi(t_k),\eta(t_k))$, $P_+=P(\xi(t_k^+),\eta(t_k^+))$ *and* $Q_+=Q(\xi(t_k^+),\eta(t_k^+))$. *Here $\phi(x,y)$ is a sufficiently smooth function such that grad $\phi(x,y)\ne 0$, and $t_k(k\in N)$ is the time of the k-th jump.*

By means of this lemma, we can derive the following result about the existence and the stability of the semi-trivial period solution of system (7):

Theorem 3.1 The semi-trivial periodic solution $(\xi(t),0)$ of system (7) is stable if and only if the following condition holds:

$$0<q<\left(\frac{1-(1-p)h}{(1-h)(1-p)}\right)^{\theta}\exp\left(-\theta\int_0^T f(\xi(t))dt\right)-1. \tag{10}$$

Proof. In our case, we have

$$P(x,y)=x(1-x)-f(x)y,$$
$$Q(x,y)=\theta(-y+f(x)y-\lambda y^2),$$
$$\alpha(x,y)=-px,\ \beta(x,y)=qy,$$
$$\phi(x,y)=x-h,(\xi(T),\eta(T))=(h,0),$$
$$(\xi(T^+),\eta(T^+))=((1-p)h,0).$$

Then, by virtue of Lemma 3.1, a simple calculation gives

$$\frac{\partial P}{\partial x}=1-2x-f'(x)y,\frac{\partial Q}{\partial y}=\theta(-1+f(x)-2\lambda y).$$

$$\frac{\partial\alpha}{\partial x}=-p,\frac{\partial\alpha}{\partial y}=0,\frac{\partial\beta}{\partial x}=0,\frac{\partial\beta}{\partial x}=q,\frac{\partial\phi}{\partial x}=1,\frac{\partial\phi}{\partial y}=0,$$

and Δ_1 =

$$\frac{P_+(\frac{\partial\beta}{\partial y}\frac{\partial\phi}{\partial x}-\frac{\partial\beta}{\partial x}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial x})+Q_+(\frac{\partial\alpha}{\partial x}\frac{\partial\phi}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y})}{P\frac{\partial\phi}{\partial x}+Q\frac{\partial\phi}{\partial y}}$$

$$=\frac{P_+(q+1)+Q_+(0)}{h(1-h)}=\frac{1-p}{1-h}(q+1)(1-(1-p)h).$$

Moreover, we find

$$exp\left(\int_0^T\frac{\partial P}{\partial x}(\xi(t),\eta(t))+\frac{\partial Q}{\partial y}(\xi(t),\eta(t)))dt\right)$$

$$=exp\left(\int_0^T\left(1-2\frac{c_0e^t}{1+c_0e^t}+\theta(-1+f(\xi(t))\right)dt\right)$$

$$=\left(\frac{1-(1-p)h}{(1-h)(1-p)}\right)^{1-\theta}\left(\frac{1-h}{1-(1-p)h}\right)^2$$

$$\cdot\exp\left(\theta\int_0^T f(\xi(t))dt\right).$$

Thus, the Floquet multiplier can be calculated directly as follows:

$$\mu_2=\Delta_1\exp\left(\int_0^T\frac{\partial P}{\partial x}(\xi(t),\eta(t))+\frac{\partial Q}{\partial y}(\xi(t),\eta(t)))dt\right)$$

$$=(1+q)\left(\frac{1-(1-p)h}{(1-h)(1-p)}\right)^{-\theta}\exp\left(\theta\int_0^T f(\xi(t))dt\right).$$

It is easy to see that $|\mu|<1$ if and only if (10) holds. So we complete the proof of Theorem 3.1.

Remark 3.1. *It is not able that if we set*

$$q^* = \left(\frac{1-(1-p)h}{(1-h)(1-p)}\right)^{\theta} \exp\left(-\theta\int_0^T f(\xi(t))dt\right) - 1 \tag{11}$$

a bifurcation may occur at $q = q^*$ for $|\mu_2| = 1$.
As a result, a positive periodic solution may appear when $q > q^*$. The detailed research will be brought forth in the next subsection.

3.2.2 Transcritical bifurcation analysis

In this subsection, we deal with the problem of the bifurcation of nontrivial periodic solution of system (7) near the semi-trivial one $(\xi(t), 0)$. Consider the Poincare map (5) with $\tau = 0$. Set $u = y_k^+$ and $u \geq 0$ to be sufficiently small. In terms of the new variable, the map can be written as

$$u \mapsto (1+q)g(u) \equiv G(u,q). \tag{12}$$

As a result of the uniqueness of solution, we have $g(0) = 0$. Hence, the semi-trivial periodic solution discussed in Section 6.1 is associated with the fixed point of zero of this map. Due to the dependence of solutions on the initial conditions, the function $G(u,q)$ is continuously differentiable with respect to both u and q, then we have $\lim_{t\to\infty} g(u) = g(0) = 0.$

In order to discuss the bifurcation of the map (12), the following lemma is introduced:

Lemma 3.2. *(see [26]) Let $F : R \times R \to R$ be a one-parameter family of C^2 map satisfying*

$$(i) F(0,\mu) = 0, (ii) \frac{\partial F}{\partial x}(0,0) > 0,$$

$$(iii) \frac{\partial^2 F}{(\partial x)\partial \mu}(0,0) > 0, (iv) \frac{\partial^2 F}{\partial x^2}(0,0) < 0.$$

Then F has two branches of fixed points for μ near zero. The first branch is $x_1(\mu) = 0$ for all μ. The second bifurcating branch $x_2(\mu)$ changes its value from negative to positive as μ increases through $\mu = 0$ with $x_2(0) = 0$. The fixed points of the first branch are stable if $\mu < 0$ and unstable if $\mu > 0$, while those of the bifurcating branch have the opposite stability.

To apply Lemma 3.2, the values of $g'(u)$ and $g''(u)$ are needed to be calculated at $u = 0$. Denote $u_0 \equiv \frac{(1-p)h(1-(1-p)h)}{f((1-p)h)}$. According to the above discussions, for $0 \leq u \leq u_0$, any trajectory through the initial point $((1-p)h, u)$ always stays in Π as t increases and intersects with the curve $\sum_0$ at the point $(h, g(u))$. Thus, orbits in Π are concerned, any point in which satisfies $\dot{x} > 0$ as indicated in Section 5, then system(2) can be transformed as follows:

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}. \tag{13}$$

where

P(x,y) = x(1-x)-f(x)y,

$Q(x,y) = \theta(-y + f(x)y - \lambda y^2).$

Let $(x, y(x; x_0, y_0))$ be an orbit of system (13), and set $x_0 = (1-p)h, y_0 = u, 0 \leq u \leq u_0$, then there have

$$\begin{cases} y(x;(1-p)h,u) \equiv y(x,u), \\ 0 \leq u \leq u_0, (1-p)h \leq x \leq h. \end{cases} \tag{14}$$

Using (14), we find

$$\frac{\partial y(x,u)}{\partial u} = \exp\left(\int_{(1-p)h}^{x} \frac{\partial}{\partial y}\left(\frac{Q(s,y(s,u))}{P(s,y(s,u))}\right)ds\right).$$

and

$$\frac{\partial^2 y(x,u)}{\partial u^2} = \frac{\partial y(x,u)}{\partial u} \bullet A,$$

where

$$A = \exp\left(\int_{(1-p)h}^{x} \frac{\partial^2}{\partial y^2}\left(\frac{Q(s,y(s,u))}{P(s,y(s,u))}\right)\frac{\partial y(s,u)}{u}ds\right).$$

Clearly, it can be deduced that $\frac{\partial y(x,u)}{\partial u} > 0$, and

$$\begin{aligned} g'(0) &= \frac{\partial y(x,0)}{\partial u} \\ &= \exp\left(\int_{(1-p)h}^{h} \frac{\partial}{\partial y}\left(\frac{Q(s,y(s,0))}{P(s,y(s,0))}\right)ds\right) \\ &= \exp\left(\int_{(1-p)h}^{h} \frac{\theta(-1+f(s))}{s(1-s)}ds\right) \\ &= \exp\left(\int_0^T \theta(-1+f(s))dt\right) \\ &= \left(\frac{1-(1-p)h}{(1-h)(1-p)}\right)^{-\theta} \exp\left(\theta\int_0^T f(\xi(t))dt\right) \end{aligned} \tag{15}$$

Furthermore, there have

$$g''(0) = g'(0)\int_{(1-p)h}^{h} m(s)\frac{\partial y(s,0)}{\partial u}ds,$$

where ,

$$m(s)=\frac{\partial^2}{\partial y^2}\left(\frac{Q(s,y(s,0))}{P(s,y(s,0))}\right)$$
$$=\frac{\partial}{\partial y}\left(\frac{K+\theta f(s)(-y+f(s)y-\lambda y^2}{s(1-s)-f(s)y^2}\right)$$
$$=\frac{-2\theta(\lambda s(1-s)-f(s)(-1+f(s))}{s^2(1-s^2)}$$

$s\in[(1-p)h,h]$,

$K=\theta(-1+f(s)-2\lambda y)(s(1-s)-f(s)y)$

Using (3), $m(s)$ can be written as

$$m(s)=\frac{-2\theta H(s)}{s^2(1-s)^2}.$$

It is easy to be found that

$$m(s)<0, s\in[(1-p)h,h]. \tag{16}$$

Because the fact (3) is that $f(s)>0$ when $s\in(0,x^*)$, so we have

$$g''(0)<0. \tag{17}$$

Moreover, we can apply Lemma 3.2 to the single parameter map (12) and obtain the following theorem:

Theorem 3.2. *In concerning with the map (12), a transcritical bifurcation occurs when ever $q=q^*$, where q^* is given by (11). Therefore, a stable positive fixed point appears when the parameter q changes through q^* from left to right. Correspondingly, system (7) has a stable positive periodic solution if $q\in(q,q+\delta)$ with $\delta>0$.*

Proof. To prove that system (7) has a stable positive periodic solution, we need to check whether the following conditions are satisfied.

(i) It is easy to verify

$$G(0,q)=0, q\in(0,+\infty).$$

(ii) Using (15), we deduce

$$\frac{\partial G(0,q)}{\partial u}=(1+q)g'(0)$$
$$=(1+q)\left(\frac{1-(1-p)h}{(1-h)(1-p)}\right)^{-\theta}\exp\left(\theta\int_0^T f(\xi(t))dt\right).$$

which yields

$$\frac{\partial G(0,q^*)}{\partial u}=1.$$

This means that $(0,q^*)$ is a fixed point with the eigenvalue 1 of the map (12).

(iii) By means of (15), we have

$$\frac{\partial^2 G(0,q^*)}{\partial u\partial q}=g'(0)>0.$$

(iv) Finally, inequality (17) implies that

$$\frac{\partial^2 G(0,q^*)}{\partial u^2}=(1+q^*)g''(0)<0.$$

By virtue of Lemma 3.2, the proof of Theorem 3.2 is completed.

3.3. The case of $\tau>0$.

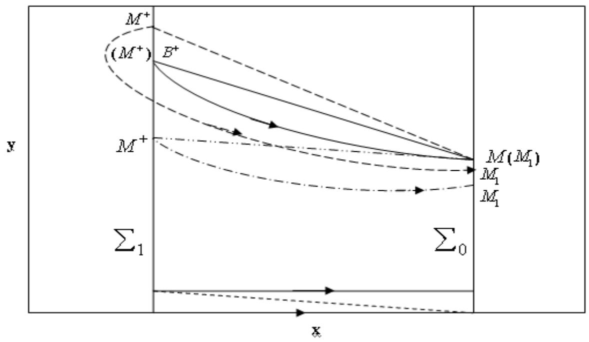


Figure 2: Location of positive periodic solution of system (3)

In this section, we discuss the existence of positive periodic solution with $\tau>0$ by using the Poincare map (6). As shown in Fig.2, set $y_k=0$, then

$y_k^+=(1+q)y_k+\tau=\tau$, $F(q,\tau,0)=g(\tau)>0$ and

$$0-F(q,\tau,0)<0. \tag{18}$$

Consider the point

$B^+((1-p)h,\dfrac{(1-p)h(1-(1-p)h)}{f((1-p)h)})$ where $dy<0$, and $dx=0$, the trajectory originated at the initial point B^+ is tangent to the curve $\sum_1$ and intersects the Poincare´ section $q=36\tau\tau\approx0.06\tau\approx0.13t$ at the point $M=(h,m_0)$, and then jumps to the point $M^+=((1-p)h,(1+q)m_0+\tau)$ and reaches the point $M_1=(h,\overline{m}_0)$ on $\sum_0$ again. To seek the location of the point M^+, we suppose that there exists a number $\overline{q}$ such that

$$(1+\overline{q})m_0+\tau=\frac{(1-p)h(1-(1-p)h)}{f((1-p)h)}.$$

Then the point M coincides with the point B^+ for $q=\overline{q}$. Accordingly the point M^+ is above the B^+ for $q>\overline{q}$ while under the point B^+ for $q<\overline{q}$, and the trajectory of which is plotted with dashed and dotted points, respectively (see

Fig.2). However, for any $q>0$, the point M_1 is not above the point M in view of the vector field of system (2.1). As a result we have that $\overline{m}_0 \le m_0$. That is,

(1) If $\overline{m}_0 = m_0(q=\overline{q})$, then system (3) has a periodic solution $\overrightarrow{MB^+M}$

(2) If $\overline{m}_0 < m_0(q \ne \overline{q})$, then

$$m_0 - F(q,\tau,m_0) = m_0 - \overline{m}_0 > 0. \qquad (19)$$

It follows from (18) and (19) that the Poincare map (6) has a fixed point $y_k \in (0,m)$, which corresponds to a positive periodic solution of system (3). The result is summarized in the following theorem:

Theorem 3.3. *System (3) always has a positive periodic solution under the condition* $\tau>0$ *and* $q>0$.

3.4. Flip bifurcation

According to Sections 6 and 7, a positive periodic solution exists when $\tau=0$, $q \ge q^*$ or $\tau>0$, $q>0$. In what follows, we suppose that the periodic solution with period T passes through the points $E^+((1-p)h,(1+q)\eta_0+\tau)$ and $E(h,\eta_0)$ in which $\eta_0 \le m_0$ holds because of the reason of the vector field of system (2) discussed in the preceding subsection.

As the expression and the period T of the solution is not known, we discuss the stability of this positive periodic solution by using Lemma 3.1. The difference between this case and that of Theorem 3.1 lies in that

$$(\xi(T),\eta(T)) = (h,\eta_0),$$

$$(\xi(T^+),\eta(T^+)) = ((1-p)h,(1+q)\eta_0+\tau),$$

while the others are just the same. Then we have

$$\Delta_1 = \frac{P_+\left(\frac{\partial\beta}{\partial y}\frac{\partial\phi}{\partial x}-\frac{\partial\beta}{\partial x}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial x}\right)+Q_+\left(\frac{\partial\alpha}{\partial x}\frac{\partial\phi}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\right)}{P\frac{\partial\phi}{\partial x}+Q\frac{\partial\phi}{\partial y}}$$

$$=\frac{P_+(q+1)+Q_+(0)}{h(1-h)} = \frac{1-p}{1-h}(q+1)(1-(1-p)h)$$

$$=\frac{[(1-p)h(1-(1-p)h)-f(\xi(T^+)((1+q)\eta_0+\tau)](1+q)}{h(1-h)-f(h}\eta_0.$$

Set $G(t)=\frac{\partial P}{\partial x}(\xi(t),\eta(t))+\frac{\partial Q}{\partial y}(\xi(t),\eta(t)))$,

then we have

$$\mu_2 = \Delta_1 \exp\left(\int_0^T \frac{\partial P}{\partial x}(\xi(t),\eta(t))+\frac{\partial Q}{\partial y}(\xi(t),\eta(t)))dt\right)$$

$$=(1+q)\frac{[(1-p)h(1-(1-p)h)-f(\xi(T^+)((1+q)\eta_0+\tau)]}{h(1-h)-f(h)\eta_0}$$

$$\bullet \exp\int_0^T G(t)dt \qquad (20)$$

As mentioned above, the point E^+, $E^+=((1-p)h,(1+q)\eta_0+\tau)$ coincides with the point $A((1-p)h,\frac{(1-p)h(1-(1-p)h)}{f((1-p)h)})$ for $q=\overline{q}>0$, then we have

$$(1+\overline{q})\eta_0+\tau = \frac{(1-p)h(1-(1-p)h)}{f((1-p)h)},$$

that is $\overline{q} = \frac{(1-p)h(1-(1-p)h)}{f((1-p)h)\eta_0} - \frac{\tau}{\eta_0} - 1$.

Hence,

$$\mu_2|_{q=\overline{q}} = (1+\overline{q})\frac{A-f(\xi(T^+)((1+\overline{q})\eta_0+\tau)}{h(1-h)-f(h)\eta_0}$$

$$\bullet\exp\int_0^T G(t)dt = 0$$

where $A=(1-p)h(1-(1-p)h)$.

Thus, this periodic solution is stable since $|\mu_2|=0<1$.

For $0<q<\overline{q}$, the point E^+ is under the point A and $(1+q)\eta_0+\tau < \frac{(1-p)h(1-(1-p)h)}{f((1-p)h)}$ holds, which results in that $1-\frac{(1+q)\eta_0+\tau}{\frac{(1-p)h(1-(1-p)h)}{f((1-p)h)}}>0$

In view of $\exp(\int_0^T G(t)dt)>0$ and $h(1-h)-f(h)\eta_0>0$, we have that $\mu_2>0$ for $0<q<\overline{q}$.

For $q>\overline{q}$, the point E^+ is above the point A, which results in

$(1+q)\eta_0+\tau > \frac{(1-p)h(1-(1-p)h)}{f((1-p)h)}$. Then we have that $\mu_2<0$ for $q>\overline{q}$.

If $|\mu_2|<1$ or $q<\breve{q}$, where

$$\breve{q}=\left|\frac{h(1-h)-f(h)\eta_0}{A-B}\exp\left(-\int_0^T G(t)dt\right)\right|-1.$$
$$A=(1-p)h(1-(1-p)h),$$
$$B=f(\xi(T^+)((1+q)\eta_0+\tau).$$

Then the periodic solution is stable. Consequently, the results about the stability of this positive period-1 solution can be summarized as follows:

Theorem 3.4. *For any $\tau>0$, $q>0$ or $\tau=0$, $q\geq q^*$, system (3) has a positive period-1 solution. Furthermore, suppose that equality (20) holds, then this period-1 solution is stable.*

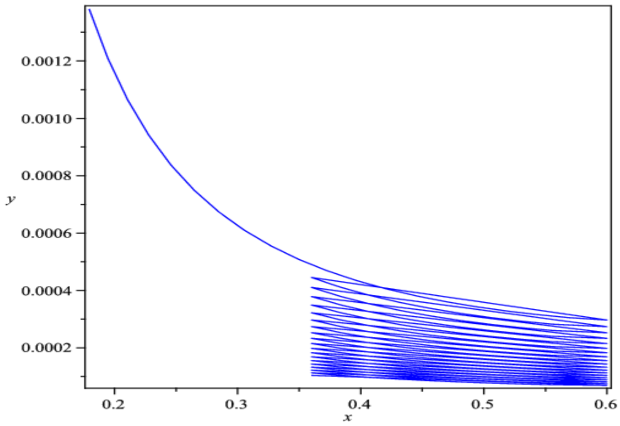
Remark 3.2. *From the above discussion, we know that $\mu_2<0$ for $q>\bar{q}$. So if there exists a $\hat{q}>\bar{q}$ such that $\mu_2=-1$, the positive period-1 solution loses its stability at $q=\hat{q}$ and a flip bifurcation may occur at $q=\hat{q}$. If a flip bifurcation occurs, there exists a stable positive period-2 solution of system (3) for $q>\hat{q}$, which may also lose its stability when q increases.*

4. Numerical simulations and discussion

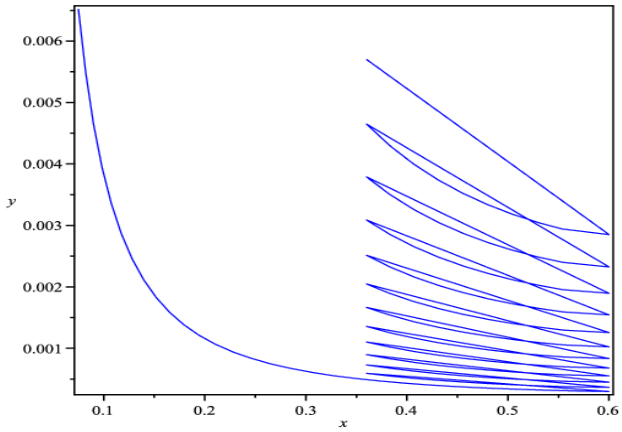
Our focus so far has been on the dynamic analysis of system (4). Now we study how the state impulsive perturbation affects the dynamical behavior of system (4). To facilitate the interpretation of our mathematical results in model (4), we proceed to investigate it by numerical simulations. Now consider the following example:

$$\begin{cases}\left.\begin{aligned}\dot{x}&=x(1-x)-\frac{2xy}{1+0.6x},\\ \dot{y}&=2(-y+\frac{2xy}{1+0.6x}-0.1y^2),\end{aligned}\right\}x\neq h,\\ \left.\begin{aligned}\Delta x&=-px\\ \Delta y&=qy+\tau\end{aligned}\right\}x=h.\end{cases}\tag{21}$$

In numerical simulation, let $f(x)=\dfrac{2x}{1+0.6x},\theta=2$. $\lambda=0.1$. System (21) has a stable positive equilibrium point (0.73, 0.19). Set $h\leq 0.73$. In the case of $\tau=0$, system (21) has a stable semi-trivial periodic solution and an unstable semi-trivial periodic solution. The solution from the initial point (0.03, 0.04) of system (21) with $q=2$ and $h=0.2$ tends to the stable semi-trivial periodic solution when *t* increases (Fig.3 (a)). Note that $\mu_2>1$ is always true for any $q>0$, and then the periodic semi-trivial solution is unstable (Fig.3 (b)). In the case of $\tau=0.1$, Fig.4 (a) and Fig.5 (a) illustrate a stable period-1 solution with $q=4$ and a stable period-2 solution with $q=17$, respectively. Fig.4 (b) and Fig.5 (b) are time-series of corresponding natural enemy y, respectively.

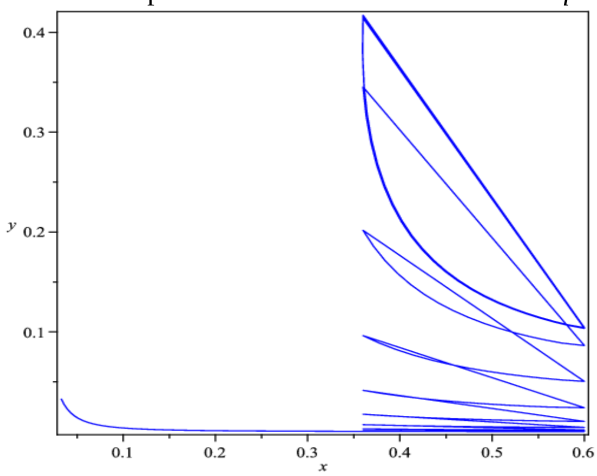


(3-a)

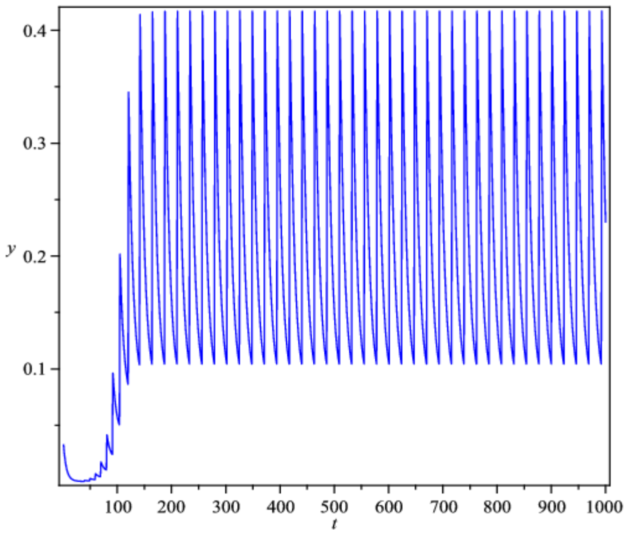


(3-b)

Figure 3: Dynamical behavior of the system (21) in the case of $\tau=0$ with p = 0.4, h = 0.6 and the initial point (0.03, 0.04). (3-a) the semi-trivial periodic solution is stable when q= 1.5. (3-b) The semi-trivial periodic solution is unstable when q = 3.

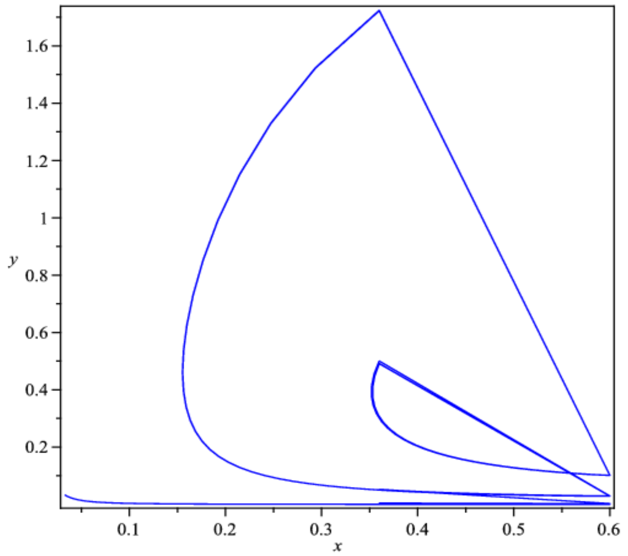


(4-a)

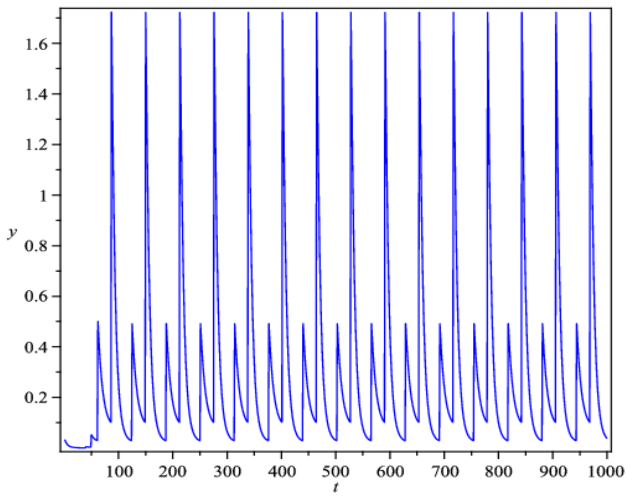


(4-b)

Figure 4: $P-1$ periodic solution of system (21) in the case of $\tau=0$ with the initial point (0.03, 0.04), h = 0.6, p = 0.4, q = 4. (4-a) Phase portrait. (4-b) Time-series.

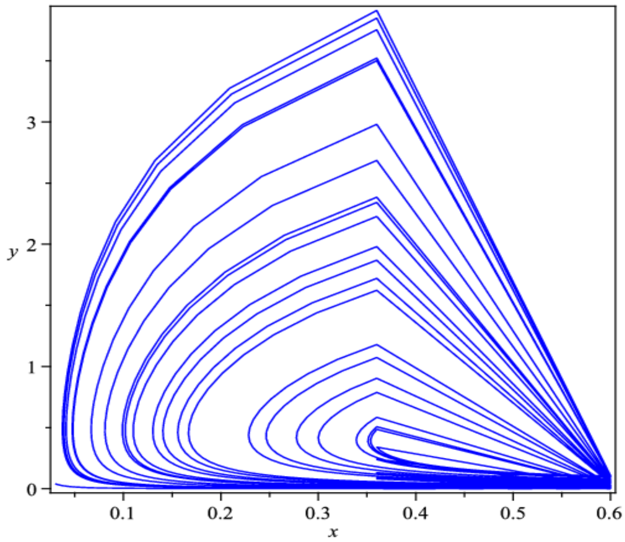


(5-a)

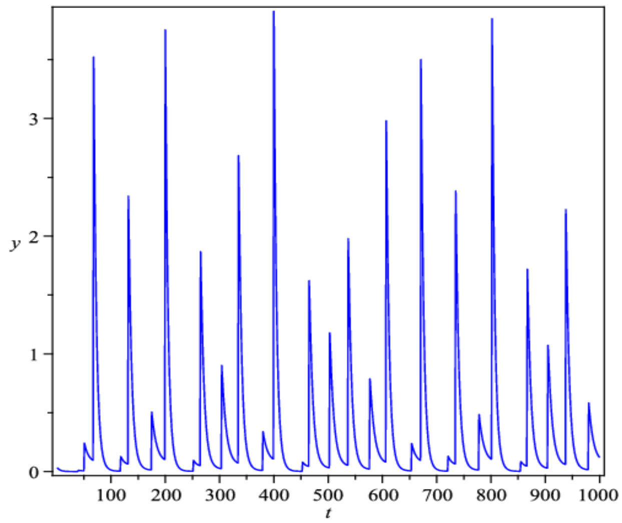


(5-b)

Figure 5: P-2 periodic solution of system (21) in the case of $\tau=0$ with the initial point (0.03, 0.04), h = 0.6, p = 0.4, q = 17. (5-a) Phase portrait. (5-b) Time-series.

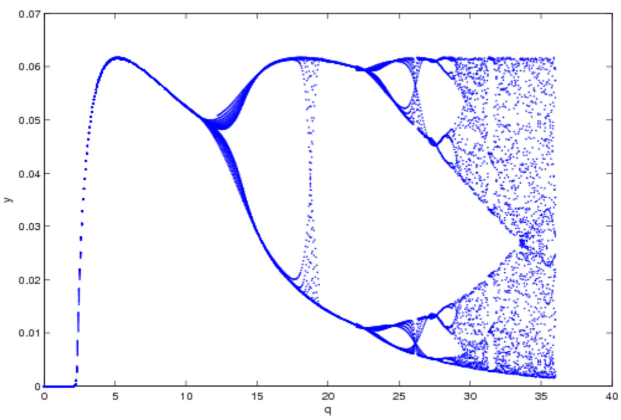


(6-a)

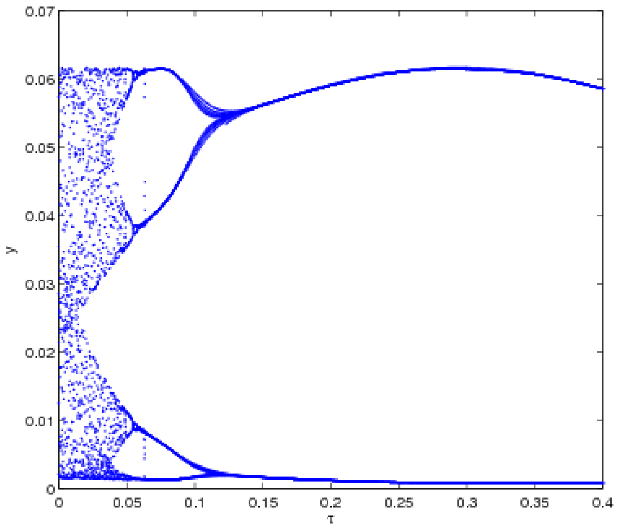


(6-b)

Figure 6: A strange attractor of system (21) in the case of $\tau=0$ with the initial point (0.03, 0.04), h = 0.6, p = 0.4, q = 35. (6-a) Phase portrait. (6-b) Time-series .



(7-a)



(7-b)

Figure 7: (7-a) Bifurcation diagrams of populations y with respect to parameter q with h = 0.6, p = 0.4, and $\tau = 0$ for $q \in$ (0, 36). (7-b) Bifurcation diagrams of populations y with respect to parameter τ with h = 0.6, p = 0.4, and q = 36 for $\tau \in$ *(0, 0.4).*

In numerical simulation, let $f(x) = \dfrac{2x}{1+0.6x}$, $\theta = 2$, $\lambda = 0.1$. System (21) has a stable positive equilibrium point (0.73, 0.19). Set $h \le 0.73$. In the case of $\tau = 0$, system (21) has a stable semi-trivial periodic solution and a unstable semi-trivial periodic solution. The solution from the initial point (0.03, 0.04) of system (21) with $q = 2$ and $h = 0.2$ tends to the stable semi-trivial periodic solution when t increases (Fig.3(a)). Note that $\mu_2 > 1$ is always true for any $q > 0$, and then the periodic semi-trivial solution is unstable (Fig.3(b)). In the case $\tau = 0.1$, Fig.4 (a) and Fig.5 (a) illustrate a stable period-1 solution with $q = 4$ and a stable period-2 solution with $q = 17$, respectively. Fig.4 (b) and Fig.5 (b) are time-series of corresponding natural enemy y, respectively.

As q increases, it can be observed that chaotic solutions appear. The phase portrait and time-series y of a chaotic solution with $q = 36$ are shown in Fig.6.

From Remark 5, we have

$$q^* = \left(\frac{1-(1-p)h}{(1-h)(1-p)}\right)^{\theta} \exp\left(-\theta\int_0^T f(\xi(t))dt\right) - 1$$
$$\approx 0.62$$

with $h = 0.6$. If we consider q as a parameter, the bifurcation diagram of the periodic solution of system (21)with $\tau = 0$ is presented in Fig. 7(a). It is seen from the bifurcation diagram that the semi-trivial periodic solution is stable for $q \in (0, 0.62)$ and unstable for $q \in (0.62, +\infty)$. A positive period-1 solution bifurcates from the semi-trivial periodic solution at $q \approx 0.62$ through a transcritical bifurcation. A positive period-2 solution bifurcates from the positive period-1 solution via a flip bifurcation at $q = \breve{q}$. It is seen from the bifurcation diagram that the stable positive period-2 solution of the system may also lose its stability when q increases in Fig.7(a). The period doubling bifurcation leads to chaos. On the other hand, if we take $q = 36$ and view τ as a bifurcation parameter in system (21), there is a route from chaos to stable periodic solutions via a cascade of reverse period-doubling bifurcation (see Fig. 7(b)). A positive period-4 solution bifurcates from chaos behavior via a flip bifurcation at $\tau \approx 0.06$. A positive period-2 solution bifurcates from the positive period-4 solution via a flip bifurcation at $\tau \approx 0.13$.

In order to find out the positive periodic solution, we transformed the problem into a fixed point problem. Theorem 3.2 was proved by using the bifurcation theory of dynamical systems. It was revealed that the positive periodic solution bifurcated from the trivial periodic solution through a transcritical bifurcation. As for other relationship between two species, analogous problems can be treated by using the same technique described herein.

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