Positive solutions for singular third-order nonhomogeneous boundary value problems with nonlocal boundary conditions

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Abstract: Under various weaker conditions, we establish various results on the existence and nonexistence of positive solutions for singular third-order nonhomogeneous boundary value problems with nonlocal boundary conditions. The arguments are based upon the fixed point theorem of cone expansion and compression. Finally, we give two examples to demonstrate our results.

Key–Words: Positive solutions, Fixed points, Boundary value problems, Nonhomogeneous, Ordinary differential equations.

1 Introduction

The world is nonlinear in essence. Because nonlinear phenomena is studied by nonlinear theories and methods, every field becomes nonlinear and then nonlinear mechanics, nonlinear optics and nonlinear mathematics appear. Since the development of physics and applied mathematics calls for the global and high level development of the mathematics ability of analyzing and controlling objective phenomena, nonlinear functional analysis which is one of the most important research fields in modern mathematics is formed by the continuously accumulation of nonlinear results. Until 1950's, nonlinear functional analysis has initially formed a theory system. In recent years, because nonlinear functional analysis has been an important tool for studying the nonlinear problem in mathematics, physics, aerospace engineering, biology engineering, it is greatly significant in the theory and application to study nonlinear functional analysis and its application.

Since the 20th century, the development of nonlinear functional analysis has achieved the great breakthrough. L. E. J. Brouwer had established the conception of topological degree for finite dimensional space in 1912. Then J. Leray and J. Schauder extended the conception to completely continuous field of Banach space in 1934, afterward E. Rothe, M. A. Krasnosel'skii, P. H. Rabinowitz, H. Amann, and K. Deimling carried on embedded research on topological degree and cone theory. Many well known mathematicians in China, for example, Guo Dajun, Zhang Gongqing, Chen Wenyuan, Ding Guanggui, Sun Jingxian etc., had proud works in various fields of nonlinear functional analysis(See [1-12]).

The method to research nonlinear problems mainly has topological degree method, critical point theory, partial order method, lower and upper solution method, fixed point theory, coincidence degree theory, monotone iterative technique, topological transversal degree and so on. The main questions to research are the existence of solution for nonlinear operator equation, uniqueness of solution, multi-solution, structure of solution, approximate solution, divergent theory of solution, iteration arithmetic, nonlinear operator theory as well as the application for partial differential equation, differential equation, integral equation and differential-integral equation. All these problems are among the most active domain in analyzing mathematics at present. Among them, firstly, singular boundary value problem of nonlinear differential equations. It has resulted from the applied disciplines of nuclear physics, hydromechanics, boundary layer theory, nonlinear optics and so on. It is an important research field of differential equations fields. Because it plays a very extensive and important role in the fields of physics, mathematics, aerospace engineering, biology engineering and so on, it has received high attention of numerous mathematicians. By applying the theories and methods of nonlinear functional analysis, the numerous famous mathematicians in the world have deeply studied the existence, uniqueness and multiplicity of solutions of singular boundary value problems and obtained lots of new results. However, because there are lots of difficulties in studying singular ordinary differential equations, at present it is still the advance orientation in the study of nonlinear analysis. Secondly, nonlocal boundary value problems for ordinary differential equations. The meaning of the nonlocal problems is that the definite condition of definite problem of ordinary differential equations not only depends on the value of solution in the end of interval, but also depends on the value of solution in some points of the interior of interval. Although lots of problems in theory and application can be reduced to nonlocal boundary value problems for ordinary differential equations, people started to fairly late study the nonlocal problems for the difficulties of nonlocal problems itself. Kiguradze, Lomtatidze(1984), Il'in and Moiseev(1987) began to discuss the existence of solutions of nonlinear multi-point boundary value problems for ordinary differential equations. Within the following ten years, the study on nonlocal boundary value problems for ordinary differential equations has been made great progress. However, it is not good enough and it is also a research topic to have a strong interest and maybe obtain some new significant achievements. Thirdly, system of nonlinear ordinary differential equations. Since lots of higher order differential-integral equations and implicit form equations can be reduced to the system of differential-integral equations by the appropriate variable substitution, the research of the system of equations plays a very important role in studying those equations.

The purpose of this paper is to establish the existence and nonexistence of positive solutions for the following singular third-order nonhomogeneous boundary value problems (BVPs for short) with nonlocal boundary conditions :

$$\begin{cases} u'''(t) + a(t)f(t, u(t), u'(t)) = 0, \\ u(0) = g_1\left(\int_0^1 u(s)d\alpha(s)\right), \\ u'(0) = g_2\left(\int_0^1 u'(s)d\beta(s)\right), \\ u'(1) = \lambda, \end{cases} \tag{1}$$

where $t \in (0,1), \lambda \in (0,\infty)$ is a parameter, $a \in C((0,1),[0,+\infty))$ and may be singular at $t = 0$ or $t = 1$; $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, $g_1, g_2 : [0,+\infty) \to [0,+\infty)$ are continuous; $\int_0^1 u(s)\mathrm{d}\alpha(s)$, and $\int_0^1 u'(s)\mathrm{d}\beta(s)$ denote the Riemann-Stieltjes integrals, α, β are increasing nonconstant functions defined on [0,1] with $\alpha(0) = \beta(0) = 0$. Here, we call a function u^* a positive solution of BVP(1) if u^* satisfies BVP(1) and $u^*(t) > 0$, for any $t \in (0,1)$.

In the last years, third-order ordinary differential equations with a two-point or multi-point boundary value problem have been studied widely in the literature(see [1-7] and [10-21]). For example, Guo et al. in [1] discussed the following nonlinear third-order three-point boundary value problem:

$$\begin{cases} u'''(t) + a(t)f(t, u(t), u'(t)) = 0, \\ u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta), \end{cases} \tag{2}$$

where $t \in (0,1), 0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. The authors established the existence of at least a positive solution for the above problem when f is superlinear or sublinear. Zhang et al. in [6] studied the following third-order eigenvalue problems:

$$\begin{cases} u'''(t) = \lambda f(t, u(t), u'(t)), \\ u(0) = u'(\eta) = u''(0) = 0, \end{cases} \tag{3}$$

where $t \in (0,1), \lambda > 0$ is a parameter and $\frac{1}{2} \le \eta < 1$ is a constant, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $\mathbb{R} = (-\infty,+\infty)$. By using Leray-Schauder nonlinear alternative, the authors obtain the existence and uniqueness of nontrivial solution of (1.3) when λ in some interval.

However, to our knowledge, the corresponding results for third-order nonhomogeneous boundary value problems, especially in the case that the BVPs with nonlocal boundary conditions , are rarely seen (see, for example, [7-8] and references therein). Sun et al. in [7] studied the existence and nonexistence of positive solutions of nonhomogeneous BVPs of third-order ordinary differential equations. Du et al. in [8] consider the following third-order nonlocal BVPs:

$$\begin{cases} u'''(t) = f(t, u(t), u'(t), u''(t)), \\ u(0) = 0, \quad u'(0) = 0, \\ u'(1) = \int_0^1 u'(s)\mathrm{d}g(s), \end{cases} \tag{4}$$

where $t \in (0,1), f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $g : [0,1] \to [0,\infty)$ is a nondecreasing function with $g(0) = 0$. Under the resonance condition $g(1) = 1$, an existence results is given by using the coincidence degree theory.

Obviously, what we consider is more different from those in [1-8]. Firstly, we will consider the boundary conditions which is nonlocal. Secondly, f and $g_i (i = 1,2)$ satisfy the limit conditions which are more extensive than the superlinear and sublinear conditions, and the nonexistence of positive solutions of BVP (1) is also studied.

The paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, various conditions on the existence and nonexistence of positive solutions for the BVP (1) are discussed. In Section 4, we give two examples to demonstrate our results.

2 Preliminaries and Lemmas

In this Section, we present some lemmas that will be used in the proof of our main results.

Throughout this paper, we assume that:
(H_0) $a \in C((0,1),[0,+\infty))$ may be singular at $t=0$ or $t=1$ and $a(t)$ does not vanish identically on any subinterval of $(0,1)$ with $\int_0^1 s(1-s)a(s)\mathrm{d}s < +\infty$;
(H_1) $f:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ and $g_i(i=1,2):[0,+\infty)\to[0,+\infty)$ are continuous.

Lemma 1 *Let $h \in C((0,1),[0,+\infty))$ with $\int_0^1 s(1-s)h(s)\mathrm{d}s < +\infty$, and $u \in C^1[0,1]$ be the function from the set $\{u : u(t)\ge 0, u'(t)\ge 0, 0\le t\le 1\}$, with*

$$u(t) = g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \int_0^1 G(t,s)h(s)\mathrm{d}s + \frac{t^2}{2}\lambda + \left(t-\frac{t^2}{2}\right)g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right), \tag{5}$$

then u is the unique solution of the following boundary value problem

$$\begin{cases} u'''(t)+h(t)=0, \quad 0<t<1, \\ u(0)=g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right), \\ u'(0)=g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right), \\ u'(1)=\lambda, \end{cases} \tag{6}$$

where

$$G(t,s)=\begin{cases} \frac{s(2t-s-t^2)}{2}, & 0\le s\le t\le 1, \\ \frac{t^2(1-s)}{2}, & 0\le t\le s\le 1. \end{cases} \tag{7}$$

Proof: First suppose that $u(t)$ is a solution of problem (6). Then we may suppose that

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 h(s)\mathrm{d}s + At^2 + Bt + C. \tag{8}$$

By the boundary condition (6), we get

$$\begin{aligned} A &= \frac{1}{2}\left(\lambda - g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right)\right) \\ &\quad -\frac{1}{2}\left(\int_0^1 (s-1)h(s)\mathrm{d}s\right), \\ B &= g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right), \\ C &= g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right). \end{aligned} \tag{9}$$

Substituting (9) into (8), we obtain

$$\begin{aligned} u(t) = {} & g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right)t + g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) \\ & -\frac{1}{2}\int_0^t (t-s)^2 h(s)\mathrm{d}s + \frac{t^2}{2}\lambda \\ & -\frac{t^2}{2}g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) - \frac{t^2}{2}\int_0^1 (s-1)h(s)\mathrm{d}s \\ = {} & g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + tg_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ & -\frac{t^2}{2}g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) + \frac{t^2}{2}\lambda \\ & -\frac{t^2}{2}\int_0^1 (s-1)h(s)\mathrm{d}s - \frac{1}{2}\int_0^t (t-s)^2 h(s)\mathrm{d}s \\ = {} & g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \frac{(2t-t^2)}{2}g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ & +\frac{t^2}{2}\lambda + \int_0^1 G(t,s)h(s)\mathrm{d}s. \end{aligned}$$

Conversely, directing differentiation of (5), we can obtain (6). This completes our proof. □

From (7), we can prove that $G(t,s)$ have the following properties.

Lemma 2 *For all $(t,s)\in[0,1]\times[0,1]$, we have*

$$0 \le G_t(t,s) \le (1-s)s.$$

where

$$G_t(t,s)=\begin{cases} s(1-t), 0\le s\le t\le 1, \\ t(1-s), 0\le t\le s\le 1. \end{cases}$$

Lemma 3 *For all $(t,s)\in[1/3,2/3]\times[0,1]$, we have*

$$\frac{1}{18}s(1-s) \le G(t,s) \le \frac{1}{2}(1-s)s,$$

$$G_t(t,s) \ge \frac{1}{3}(1-s)s.$$

Proof: For $\forall\ \frac{1}{3}\le t\le\frac{2}{3}$,

$$\begin{aligned} \frac{G_t(t,s)}{G_s(s,s)} &= \frac{s(1-t)}{s(1-s)} = \frac{1-t}{1-s} \\ &\ge 1-t \ge \frac{1}{3},\ 0\le s\le 1, \end{aligned}$$

$$\begin{aligned} \frac{G_t(t,s)}{G_s(s,s)} &= \frac{t(1-s)}{s(1-s)} = \frac{t}{s} \\ &\ge t \ge \frac{1}{3},\ 0\le s\le 1, \end{aligned}$$

So,

$$G_t(t,s) \ge \frac{1}{3}G_s(s,s) = \frac{1}{3}s(1-s)$$

for $\frac{1}{3}\le t\le\frac{2}{3}$,$0\le s\le 1$. □

Lemma 4 *In Lemma 1, the solution $u(t)$ of boundary value problem (6) and $u'(t)$ are nonnegative and satisfy*

$$\min_{\frac{1}{3}\le t\le\frac{2}{3}} u(t) \ge \frac{1}{9}\|u\|, \quad \min_{\frac{1}{3}\le t\le\frac{2}{3}} u'(t) \ge \frac{1}{3}\|u'\|.$$

Proof: It is obvious that $u(t)$ and $u'(t)$ are nonnegative.

Step 1. We will prove that

$$\min_{\frac{1}{3}\le t\le\frac{2}{3}} u(t) \ge \frac{1}{9}\|u\|.$$

For any $t \in [0,1]$, by (5) and Lemma 3, it follows that

$$\begin{aligned} u(t) &= g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) \\ &\quad + \tfrac{(2t-t^2)}{2} g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + \tfrac{t^2}{2}\lambda + \int_0^1 G(t,s)h(s)\mathrm{d}s \\ \le &\quad g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \tfrac{1}{2}g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + \tfrac{\lambda}{2} + \tfrac{1}{2}\int_0^1 s(1-s)h(s)\mathrm{d}s, \end{aligned}$$

thus

$$\begin{aligned} \|u\| &\le g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \tfrac{1}{2}g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + \tfrac{\lambda}{2} + \tfrac{1}{2}\int_0^1 s(1-s)h(s)\mathrm{d}s. \end{aligned}$$

On the other hand, (5) and Lemma 3 imply that, for any $t \in [\frac{1}{3}, \frac{2}{3}]$,

$$\begin{aligned} u(t) &= g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \int_0^1 G(t,s)h(s)\mathrm{d}s \\ &\quad + \left(t - \tfrac{t^2}{2}\right) g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) + \tfrac{t^2}{2}\lambda \\ \ge &\quad g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \tfrac{5}{18}g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + \tfrac{1}{18}\lambda + \tfrac{1}{18}\int_0^1 s(1-s)h(s)\mathrm{d}s \\ \ge &\quad \tfrac{1}{9}\left[g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \tfrac{1}{2}g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right)\right] \\ &\quad + \tfrac{1}{9}\left[\tfrac{\lambda}{2} + \tfrac{1}{2}\int_0^1 s(1-s)h(s)\mathrm{d}s\right] \\ \ge &\quad \tfrac{1}{9}\|u\|. \end{aligned}$$

Therefore,

$$\min_{\frac{1}{3}\le t\le\frac{2}{3}} u(t) \ge \frac{1}{9}\|u\|.$$

Step 2. We will prove that

$$\min_{\frac{1}{3}\le t\le\frac{2}{3}} u'(t) \ge \frac{1}{3}\|u'\|.$$

For any $t \in [0,1]$, by (5) and Lemma 3, it follows that

$$\begin{aligned} u'(t) &= (1-t)\, g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + t\lambda + \int_0^1 G_t(t,s)h(s)\mathrm{d}s \\ &\le g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + \lambda + \int_0^1 s(1-s)h(s)\mathrm{d}s, \end{aligned}$$

thus

$$\begin{aligned} \|u'\| &\le g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + \lambda + \int_0^1 s(1-s)h(s)\mathrm{d}s. \end{aligned}$$

On the other hand, (5) and Lemma 3 imply that, for any $t \in [\frac{1}{3}, \frac{2}{3}]$,

$$\begin{aligned} u'(t) &= (1-t)\, g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + t\lambda + \int_0^1 G_t(t,s)h(s)\mathrm{d}s \\ &\ge \tfrac{1}{3}g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) + \tfrac{1}{3}\lambda \\ &\quad + \tfrac{1}{3}\int_0^1 s(1-s)h(s)\mathrm{d}s \\ &\ge \tfrac{1}{3}\|u'\|. \end{aligned}$$

Therefore,

$$\min_{\frac{1}{3}\le t\le\frac{2}{3}} u'(t) \ge \frac{1}{3}\|u'\|.$$

This completes our proof. □

We shall discuss the existence of a positive solution of the BVP(1) by using the following fixed-point theorem of cone expansion and compression.

Lemma 5 *[9] Let K be a cone of the real Banach space E, Ω_1, $\Omega_2 \subset E$ be bounded open sets of E. $\theta \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, $A : K\bigcap(\overline{\Omega_2}\backslash\Omega_1) \to K$ is a completely continuous mapping such that one of the following two conditions is satisfied:*

1) $\|Au\| \le \|u\|, \forall\, u \in K\cap\partial\Omega_1; \|Au\| \ge \|u\|, \forall\, u \in K\cap\partial\Omega_2$.

2) $\|Au\| \ge \|u\|, \forall\, u \in K\cap\partial\Omega_1; \|Au\| \le \|u\|, \forall\, u \in K\cap\partial\Omega_2$.

Then A has a fixed point in $K\cap(\overline{\Omega_2}\backslash\Omega_1)$.

We shall consider the Banach space $C^1[0,1]$ equipped with the standard norm

$$\|u\|_1 = \|u\| + \|u'\|$$

where $||u|| = \max_{0\le t\le 1} |u(t)|$.

Define the cone P by

$$P = \left\{ u \in C^1[0,1] \;\middle|\; \begin{array}{c} u(t)\ge 0, u'(t)\ge 0, \\ \min_{\frac{1}{3}\le t\le\frac{2}{3}} u(t) \ge \frac{1}{9}\|u\|, \\ \min_{\frac{1}{3}\le t\le\frac{2}{3}} u'(t) \ge \frac{1}{3}\|u'\| \end{array} \right\}.$$

Define an operator T by

$$\begin{aligned} (Tu)(t) &= g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \tfrac{t^2}{2}\lambda \\ &\quad + \left(t - \tfrac{t^2}{2}\right) g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + \int_0^1 G(t,s)a(s)f(s,u(s),u'(s))\mathrm{d}s. \end{aligned} \tag{10}$$

By Lemma 1, it is clear that the existence of a positive solution of BVP (1) is equivalent to the existence of a nontrivial fixed point of T.

Lemma 6 *Assume that* (H_0), *and* (H_1) *hold. Then* $T : P \to P$ *is completely continuous.*

Proof: Since the proof of the completed continuity is standard, we need only to prove $T(P) \subset P$. In fact, for $(x, y) \in P,\ t \in J$, by $(\mathrm{H}_0), (\mathrm{H}_1)$ and (10), it is obvious that $(Tu)(t)$ and $(Tu)'(t)$ are nonnegative.
Step 1. We will prove that

$$\min_{\frac{1}{3} \le t \le \frac{2}{3}} (Tu)(t) \ge \frac{1}{9}\|u\|.$$

For any $t \in [0, 1]$, by (10) and Lemma 3, it follows that

$$\begin{aligned}(Tu)(t) &= g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) \\ &\quad + \left(t - \tfrac{t^2}{2}\right) g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + \tfrac{t^2}{2}\lambda + \int_0^1 G(t, s)h(s)\mathrm{d}s \\ &\le g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \tfrac{1}{2} g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + \tfrac{\lambda}{2} + \tfrac{1}{2}\int_0^1 s(1 - s)h(s)\mathrm{d}s,\end{aligned}$$

thus

$$\begin{aligned}\|u\| &\le g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \tfrac{1}{2} g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + \frac{\lambda}{2} + \frac{1}{2}\int_0^1 s(1 - s)h(s)\mathrm{d}s.\end{aligned}$$

On the other hand, (10) and Lemma 3 imply that, for any $t \in [\frac{1}{3}, \frac{2}{3}]$,

$$\begin{aligned}&(Tu)(t) \\ &= g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \tfrac{(2t-t^2)}{2} g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &+ \tfrac{t^2}{2}\lambda + \int_0^1 G(t, s)h(s)\mathrm{d}s \\ &\ge g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \tfrac{5}{18} g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &+ \tfrac{1}{18}\lambda + \tfrac{1}{18}\int_0^1 s(1 - s)h(s)\mathrm{d}s \\ &\ge \tfrac{1}{9}\left[g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) + \tfrac{1}{2} g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right)\right] \\ &+ \tfrac{1}{9}\left[\tfrac{\lambda}{2} + \tfrac{1}{2}\int_0^1 s(1 - s)h(s)\mathrm{d}s\right] \\ &\ge \tfrac{1}{9}\|Tu\|.\end{aligned}$$

Therefore,

$$\min_{\frac{1}{3} \le t \le \frac{2}{3}} (Tu)(t) \ge \frac{1}{9}\|Tu\|.$$

Step 2. We will prove that

$$\min_{\frac{1}{3} \le t \le \frac{2}{3}} (Tu)'(t) \ge \frac{1}{3}\|(Tu)'\|.$$

For any $t \in [0, 1]$, by (10) and Lemma 3, it follows that

$$\begin{aligned}(Tu)'(t) &= (1 - t)\, g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + t\lambda + \int_0^1 G_t(t, s)h(s)\mathrm{d}s \\ &\le g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) + \lambda + \int_0^1 s(1 - s)h(s)\mathrm{d}s,\end{aligned}$$

thus

$$\|u'\| \le g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) + \lambda + \int_0^1 s(1 - s)h(s)\mathrm{d}s.$$

On the other hand, (10) and Lemma 3 imply that, for any $t \in [\frac{1}{3}, \frac{2}{3}]$,

$$\begin{aligned}(Tu)'(t) &= (1 - t)\, g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ &\quad + t\lambda + \int_0^1 G_t(t, s)h(s)\mathrm{d}s \\ &\ge \tfrac{1}{3}\left[g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) + \lambda + \int_0^1 s(1 - s)h(s)\mathrm{d}s\right] \\ &\ge \tfrac{1}{3}\|(Tu)'\|.\end{aligned}$$

Therefore,

$$\min_{\frac{1}{3} \le t \le \frac{2}{3}} (Tu)'(t) \ge \frac{1}{3}\|(Tu)'\|.$$

From the above discussion, we assert that $T(x, y) \in P$. Therefore, $T : P \to P$ is completely continuous. This completes our proof. □

3 Main results

In the following, for convenience, we denote $J = [0, 1], J_0 = [\frac{1}{3}, \frac{2}{3}]$ and set

$$f^\alpha = \limsup_{x+y \to \alpha} \max_{t \in J} \frac{f(t, x, y)}{x + y},$$

$$f_\alpha = \liminf_{x+y \to \alpha} \min_{t \in J_0} \frac{f(t, x, y)}{x + y},$$

$$g_i^\alpha = \limsup_{x \to \alpha} \frac{g_i(x)}{x},$$

where α is 0^+ or $+\infty,\ i = 1, 2$.

Throughout this section, we assume that $p_i, i = 1, 2, 3, 4$, are four positive numbers satisfying $\frac{1}{p_1} +$

$\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}\le 1$. And

$$A=\left(\frac{3p_3}{2}\int_0^1 s(1-s)a(s)\mathrm{d}s\right)^{-1},$$
$$B=\left(\frac{1}{162}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s)a(s)\mathrm{d}s\right)^{-1}.$$

In this section, we are concerned with the existence and nonexistence of positive solutions of BVP (1). We obtain the following theorems.

Theorem 7 *Assume that* (H_0) *and* (H_1) *hold. In addition, assume that* $f,\ g_1,\ g_2$ *satisfy:*

(1) $0\le f^0<A,\ f_\infty>B$,

(2) $0\le g_1^0<\dfrac{1}{p_1\alpha(1)},\ 0\le g_2^0<\dfrac{2}{3p_2\beta(1)}$.

Then BVP (1) has at least one positive solution for λ *small enough and has no positive solution for* λ *large enough.*

Proof: Take a sufficiently small positive number ϵ : $0<\epsilon<\min\{\frac{1}{p_1\alpha(1)},\frac{2}{3p_2\beta(1)}\}$ such that $g_1^0<\frac{1}{p_1\alpha(1)}-\epsilon,\ g_2^0<\frac{2}{3p_2\beta(1)}-\epsilon$. Then there exists an $0<\eta_1<1$ such that

$$g_1(x)\le\left(\frac{1}{p_1\alpha(1)}-\epsilon\right)(x),$$

$$g_2(x)\le\left(\frac{2}{3p_2\beta(1)}-\epsilon\right)(x),\ 0<x\le\eta_1.$$

By $0\le f^0<A$, we have that there exists $\eta_2>0$ such that

$$f(t,x,y)\le A(x+y),\ t\in J,\ x+y\le\eta_2. \tag{11}$$

Let $r_1=\min\left\{\eta_2,\frac{\eta_1}{\alpha(1)},\frac{\eta_1}{\beta(1)}\right\}$, and $\Omega_1=\{u\in P:\|u\|_1<r_1\}$. Then for any $u\in\partial\Omega_1\cap P$,

$$\begin{aligned}&g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right)\le\left(\tfrac{1}{p_1\alpha(1)}-\epsilon\right)\int_0^1 u(s)\mathrm{d}\alpha(s)\\&\le\tfrac{1}{p_1\alpha(1)}\alpha(1)r_1=\tfrac{r_1}{p_1},\\&g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right)\le\left(\tfrac{2}{3p_2\beta(1)}-\epsilon\right)\int_0^1 u'(s)\mathrm{d}\beta(s)\\&\le\tfrac{2}{3p_2\beta(1)}\beta(1)r_1=\tfrac{2r_1}{3p_2}.\end{aligned}\tag{12}$$

Therefore, for any $u\in\partial\Omega_1\cap P$, let $0<\lambda\le\frac{2r_1}{3p_4}$, we have

$$\begin{aligned}&\|Tu\|+\|(Tu)'\|\\\le\ &g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right)+\tfrac{3}{2}\lambda+\tfrac{3}{2}g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right)\\+\ &\tfrac{3}{2}\int_0^1 s(1-s)a(s)f(s,u(s),u'(s))\mathrm{d}s\\\le\ &\tfrac{r_1}{p_1}+\tfrac{r_1}{p_2}+\tfrac{3\lambda}{2}+\tfrac{3}{2}\int_0^1 s(1-s)a(s)A\mathrm{d}sr_1\\\le\ &(\tfrac{1}{p_1}+\tfrac{1}{p_2}+\tfrac{1}{p_3}+\frac{1}{p_4})r_1\le r_1,\end{aligned}$$

Consequently,

$$\|Tu\|_1\le\|u\|_1,\ \forall\, u\in\partial\Omega_1\cap P. \tag{13}$$

On the other hand, by $f_\infty>B$, there exists $r_2>r_1$ such that

$$f(t,x,y)\ge B(x+y),\ t\in J_0,\ x+y\ge\frac{1}{9}r_2. \tag{14}$$

Let $\Omega_2=\{u\in P:\|u\|_1<r_2\}$. Then for any $u\in\partial\Omega_2\cap P$, we have

$$\begin{aligned}&(Tu)(\tfrac{1}{3})=g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right)\\&+\tfrac{5}{18}g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right)+\tfrac{1}{18}\lambda\\&+\int_0^1 G(\tfrac{1}{3},s)a(s)f(s,u(s),u'(s))\mathrm{d}s\\&\ge\tfrac{1}{18}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s))a(s)f(s,u(s),u'(s))\mathrm{d}s\\&\ge\tfrac{1}{18}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s))a(s)B(u(s)+u'(s))\mathrm{d}s\\&\ge\tfrac{1}{18}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s))a(s)B\frac{1}{9}r_2\mathrm{d}s\\&\ge\tfrac{1}{162}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s))a(s)Br_2\mathrm{d}s=r_2,\end{aligned}$$

Consequently,

$$\|Tu\|_1\ge\|u\|_1,\ \forall\, u\in\partial\Omega_2\cap P. \tag{15}$$

Thus, (13), (15) and Lemma 5 imply T has a fixed point $u^*\in P$ such that $0<r_1<\|u^*\|_1<r_2$ and hence u^* is a positive solution of the BVP (1).

Next, we prove that problem (1) has no positive solution for λ large enough. Otherwise, there exist $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$, with $\lim\limits_{n\to\infty}\lambda_n=+\infty$, such that for any positive integer n, the BVP$(0<t<1)$

$$\begin{cases}u'''(t)+a(t)f(t,u(t),u'(t))=0,\\u(0)=g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right),\\u'(0)=g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right),\\u'(1)=\lambda_n,\end{cases}\tag{16}$$

has a positive solution $u_n(t)$. By (10), we have

$$\begin{aligned} & u_n(1) \\ = \ & g_1\left(\int_0^1 u_n(s)\mathrm{d}\alpha(s)\right)+\tfrac{1}{2}\lambda_n \\ + \ & \tfrac{1}{2}g_2\left(\int_0^1 u_n'(s)\mathrm{d}\beta(s)\right) \\ + \ & \int_0^1 G(1,s)a(s)f(s,u_n(s),u_n'(s))\mathrm{d}s \\ \geq \ & \tfrac{1}{2}\lambda_n \to \infty, (n\to\infty). \end{aligned} \tag{17}$$

Thus,

$$\|u_n\| \to \infty, (n\to\infty).$$

By $f_\infty > B$, there exists $R>0$ such that

$$f(t,x,y)\geq B(x+y),\ t\in J_0,\ x+y\geq \frac{1}{9}R. \tag{18}$$

Let n large enough such that $\|u_n\|\geq R$. Then

$$\begin{aligned} \|u_n\|_1 > \ & \int_0^1 G(\tfrac{1}{3},s)a(s)f(s,u_n(s),u_n'(s))\mathrm{d}s \\ \geq \ & \tfrac{B}{18}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s))a(s)(u_n(s)+u_n'(s))\mathrm{d}s \\ \geq \ & \tfrac{B}{18}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s))a(s)\frac{1}{9}\|u_n\|_1\mathrm{d}s \\ \geq \ & \tfrac{B}{162}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s))a(s)\|u_n\|_1\mathrm{d}s \\ = \ & \|u_n\|_1, \end{aligned}$$

which is a contradiction. The proof of Theorem 7 is completed. □

Similar to the proof of Theorem 7, we can also prove the following results.

Corollary 8 *Assume that* (H_0) *and* (H_1) *hold. In addition, assume that* $f,\ g_1,\ g_2$ *satisfy:*

(1) $f^0=0,\ f_\infty=\infty,$

(2) $g_1^0=0,\ g_2^0=0.$

Then BVP (1) has at least one positive solution for λ *small enough and has no positive solution for* λ *large enough.*

Theorem 9 *Assume that* (H_0) *and* (H_1) *hold. In addition, assume that* $f,\ g_1,\ g_2$ *satisfy:*

(1) $f_0>B,\ 0\leq f^\infty<A,$

(2) $0\leq g_1^\infty<\frac{1}{p_1\alpha(1)},\ 0\leq g_2^\infty<\frac{2}{3p_2\beta(1)}.$

Then BVP (1) has at least one positive solution for any $\lambda\in(0,\infty)$.

Proof: By $f_0>B$, there exists $R_1>0$, such that

$$f(t,x,y)\geq B(x+y),\ t\in J_0,\ x+y\leq R_1. \tag{19}$$

Let $\Omega_3=\{u\in P:\|u\|_1<R_1\}$. Then for any $u\in\partial\Omega_3\cap P$, we have

$$\begin{aligned} & (Tu)(\tfrac{1}{3}) = g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right)+\tfrac{1}{18}\lambda \\ & +\tfrac{5}{18}g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right) \\ & +\int_0^1 G(\tfrac{1}{3},s)a(s)f(s,u(s),u'(s))\mathrm{d}s \\ & \geq \tfrac{1}{18}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s))a(s)f(s,u(s),u'(s))\mathrm{d}s \\ & \geq \tfrac{B}{18}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s))a(s)(u(s)+u'(s))\mathrm{d}s \\ & \geq \tfrac{B}{18}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s))a(s)\frac{1}{9}R_1\mathrm{d}s \\ & \geq \tfrac{B}{162}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s))a(s)R_1\mathrm{d}s \\ & \geq R_1, \end{aligned}$$

Consequently,

$$\|Tu\|_1\geq\|u\|_1,\ \forall\, u\in\partial\Omega_3\cap P. \tag{20}$$

Take a sufficiently small positive numberϵ :

$$0<\epsilon<\min\{\frac{1}{p_1\alpha(1)},\frac{2}{3p_2\beta(1)}\}$$

such that

$$g_1^\infty<\frac{1}{p_1\alpha(1)}-\epsilon,\quad g_2^\infty<\frac{2}{3p_2\beta(1)}-\epsilon.$$

This together with $0\leq f^\infty<A$ implies that there exists an $C>0$ such that for $\forall\, x,\ y\in[0,+\infty)$,

$$\begin{aligned} & f(t,x,y)\leq A(x+y)+C,\ t\in J, \\ & g_1(x)\leq\left(\tfrac{1}{p_1\alpha(1)}-\epsilon\right)(x)+C, \\ & g_2(x)\leq\left(\tfrac{2}{3p_2\beta(1)}-\epsilon\right)(x)+C. \end{aligned} \tag{21}$$

Let

$$\begin{aligned} R_2 > \max\Big\{ & R_1, \big(\tfrac{5C}{2}+\tfrac{3\lambda}{2} \\ & +\tfrac{3C}{2}\int_0^1 s(1-s)a(s)\mathrm{d}s\big)\,p_4\Big\}, \end{aligned}$$

and $\Omega_4=\{u\in P:\|u\|_1<R_2\}$. Then for any $u\in\partial\Omega_4\cap P$,

$$\begin{aligned} & g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right) \\ & \leq\left(\tfrac{1}{p_1\alpha(1)}-\epsilon\right)\int_0^1 u(s)\mathrm{d}\alpha(s)+C \\ & \leq\tfrac{1}{p_1\alpha(1)}\alpha(1)R_2+C \\ & =\tfrac{R_2}{p_1}+C, \end{aligned} \tag{22}$$

$$\begin{aligned}
&g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right)\\
&\le \left(\tfrac{2}{3p_2\beta(1)}-\epsilon\right)\int_0^1 u'(s)\mathrm{d}\beta(s)+C\\
&\le \tfrac{2}{3p_2\beta(1)}\beta(1)R_2+C\\
&= \tfrac{2R_2}{3p_2}+C.
\end{aligned} \tag{22}$$

Therefore, for any $u \in \partial\Omega_4 \cap P$, we have

$$\begin{aligned}
&\|Tu\|+\|(Tu)'\|\\
&\le g_1\left(\int_0^1 u(s)\mathrm{d}\alpha(s)\right)\\
&+\tfrac{3}{2}g_2\left(\int_0^1 u'(s)\mathrm{d}\beta(s)\right)+\tfrac{3}{2}\lambda\\
&+\tfrac{3}{2}\int_0^1 s(1-s)a(s)f(s,u(s),u'(s))\mathrm{d}s\\
&\le \tfrac{R_2}{p_1}+\tfrac{R_2}{p_2}+\tfrac{3\lambda}{2}+\tfrac{5}{2}C\\
&+\tfrac{3}{2}\int_0^1 s(1-s)a(s)(AR_2+C)\mathrm{d}s\\
\le\ &(\tfrac{1}{p_1}+\tfrac{1}{p_2}+\tfrac{1}{p_3}+\tfrac{1}{p_4})R_2 \le R_2,
\end{aligned}$$

Consequently,

$$\|Tu\|_1 \le \|u\|_1,\ \forall\, u \in \partial\Omega_4 \cap P. \tag{23}$$

Thus, (20), (23) and Lemma 5 imply T has a fixed point $u^* \in P$ such that $0 < R_1 < \|u^*\|_1 < R_2$ and hence u^* is a positive solution of BVP (1). The proof of Theorem 9 is completed. □

Similar to the proof of Theorem 3.1, we can also prove the following results.

Corollary 10 *Assume that* (H_0) *and* (H_1) *hold. In addition, assume that* $f,\ g_1,\ g_2$ *satisfy:*

(1) $f_0=\infty,\ f^\infty=0$,

(2) $g_1^\infty=0,\ g_2^\infty=0$.

Then BVP (1) has at least one positive solution for any $\lambda \in (0,\infty)$.

4 Example

Example 11 *Consider the following singular third-order nonhomogeneous boundary value problems:*

$$\begin{cases}
u'''(t)+a(t)f(t,u(t),u'(t))=0,\\
u(0)=g_1\left(\displaystyle\int_0^1 u(s)\mathrm{d}\alpha(s)\right),\\
u'(0)=g_2\left(\displaystyle\int_0^1 u'(s)\mathrm{d}\beta(s)\right),\\
u'(1)=\lambda,
\end{cases} \tag{24}$$

where $t \in (0,1), a(t)=\dfrac{1}{t},\ \alpha(s)=s,\ \beta(s)=2s,\ g_1(t)=t^2,\ g_2(t)=t^3$, *and* $f(t,x,y)=t(x+y)^2,\ x,\ y\ge 0,\ 0\le t\le 1$.

Then BVP (24) has at least one positive solution for λ *small enough and has no positive solution for* λ *large enough.*

It is easy to check that (H_0) and (H_1) hold. In addition, assume that $f,\ g_1,\ g_2$ satisfy: $f^0=0,\ f_\infty=\infty$, and $g_1^0=0,\ g_2^0=0$, By Corollary 8, our conclusion follows.

Example 12 *Consider the following singular third-order nonhomogeneous boundary value problems:*

$$\begin{cases}
u'''(t)+a(t)f(t,u(t),u'(t))=0,\\
u(0)=g_1\left(\displaystyle\int_0^1 u(s)\mathrm{d}\alpha(s)\right),\\
u'(0)=g_2\left(\displaystyle\int_0^1 u'(s)\mathrm{d}\beta(s)\right),\\
u'(1)=\lambda,
\end{cases} \tag{25}$$

where $0<t<1, a(t)=\frac{1}{1-t},\ \alpha(s)=s-\frac{s^2}{2},\ \beta(s)=s,\ g_1(t)=t^{\frac{1}{2}},\ g_2(t)=t^{\frac{1}{3}}$, *and* $f(t,x,y)=(x+y)^{\frac{1}{2}}+t^2(x+y)^{\frac{1}{3}},\ x,\ y\ge 0,\ 0\le t\le 1$.

Then BVP (25) has at least one positive solution for any $\lambda \in (0,\infty)$.

It is easy to check that (H_0) and (H_1) hold. In addition, assume that $f,\ g_1,\ g_2$ satisfy: $f_0=\infty,\ f^\infty=0$, and $g_1^\infty=0,\ g_2^\infty=0$. By Corollary 10, our conclusion follows.

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