A general iterative algorithm for equilibrium problems and strict pseudo-contractions in Hilbert spaces

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Abstract: In this paper an iterative scheme is presented for finding a common element of the set of solutions of the variational inequality, fixed points of strict pseudo-contraction and solutions of equilibrium problem in Hilbert spaces. Under suitable conditions, it is proved that implicit and explicit schemes are of strong convergence properties. Obtained results improve and extend the existed results.

Key–Words: Nonexpansive mapping, Fixed point, Equilibrium problem, Strict pseudo-contraction, Variational inequality, Iterative algorithm

1 Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Recall that a mapping T from C into itself is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$

The set of fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in D(T) : Tx = x\}$; If $C \subset H$ is nonempty, bounded, closed and convex and T is a nonexpansive self-mapping on C, then $F(T)$ is nonempty.

Let $A : C \to C$ be an operator, the variational inequality problem is to find $x^* \in C$ such that

$$VI : \ \langle Ax^*, x - x^* \rangle \geq 0, \ \forall x \in C. \tag{1}$$

The set of solutions of (1) is denoted by $VI(C, A)$.

Multiple iterative schemes have been proposed to approximating the fixed point of an operator, which also is a solution of the variational inequality.

In 1953, Mann [1] proposed the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{2}$$

where $x_0 \in C$ is an initial guess arbitrarily. If $\alpha_n \in [0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by (2) converges weakly to a fixed point of T.

In 2000, Moudafi [3] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H, starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \ \ n \geq 0, \tag{3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. Xu [4] proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (3) strongly converges to the unique solution $x^* \in C$ of the variational inequality

$$\langle (I - f) x^*, x - x^* \rangle \geq 0, \ \ \forall x \in F(T).$$

In 2001, Yamada [5] introduced the following hybrid iterative method:

$$x_{n+1} = T x_n - \mu \lambda_n F(T x_n), \ \ n \geq 0, \tag{4}$$

where F is a k-Lipschitzian and η-strongly monotone operator with $k > 0, \eta > 0, 0 < \mu < 2\eta/k^2$, and then he proved that if $\{\lambda_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (4) converges strongly to the unique solution $\tilde{x} \in F(T)$ of the variational inequality

$$\langle F\tilde{x}, x - \tilde{x} \rangle \geq 0, \forall x \in F(T).$$

In 2006, Marino and Xu [6] introduced the general iterative algorithm

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, n \geq 0, \tag{5}$$

where T is a self-nonexpansive mapping on H, f is a contraction of H into itself with coefficient $\rho \in (0, 1)$

satisfies certain condition, and A is a strongly positive bounded linear operator on H. He proved that $\{x_n\}$ generated by (5) converges strongly to a fixed point x^* of T, which is the unique solution to the following variational inequality:

$$\langle(\gamma f - A)x^*, x - x^*\rangle \le 0,\ \forall x \in F(T),$$

and is also the optimality condition for some minimization problem.

In 2010, M. Tian [7] introduced the general iterative algorithm

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, n \ge 0, \quad (6)$$

where $F : H \to H$ is an L-Lipschitzian and η-strongly monotone operator with $L, \eta > 0$. Under some mild assumptions, he proved that $\{x_n\}$ generated by (6) converges strongly to a point $x^* \in F(T)$, which is also the unique solution of the following variational inequality:

$$\langle(\mu F - \gamma f)x^*, x - x^*\rangle \ge 0,\ \forall x \in F(T).$$

Recently, many authors considered the iterative schemes to approximating the fixed point of a strictly pseudo-contraction. A mapping $S : C \to H$ is said to be k-strictly pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$\begin{aligned}\|Sx - Sy\|^2 \ \le\ & \|x - y\|^2 + k\|(I - S)x \\ & -(I - S)y\|^2,\ \forall x, y \in C.\end{aligned}$$

Note that the class of k-strict pseudo-contraction strictly includes the class of nonexpansive mapping, that is, S is nonexpansive if and only if S is 0-srtict pseudo-contractive; it is also said to be pseudo-contractive if $k = 1$. Clearly, the class of k-strict pseudo-contractions falls into the one between classes of nonexoansive mappings and pseudo-contractions.

For finding an element of $F(S) \cap VI(C, A)$, Takahashi and Toyoda [8] introduced the following iterative scheme: $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n),\ n \ge 1,$$

and obtained a weak convergence theorem in a Hilbert space, where $\{\alpha_n\}$ and $\{\lambda_n\}$ are sequences satisfied certain conditions.

For finding an element of $F(T)$, Qin et al [9] introduce a composite iterative scheme as follows with $x_1 = x \in H$:

$$\begin{cases} y_n = P_K[\beta_n x_n + (1 - \beta_n)Tx_n], \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n,\ \ n \ge 1, \end{cases}$$

where T is a non-self k-strict pseudo-contraction, f is a contraction and A is a strong positive linear bounded operator. Under certain appropriate assumptions on the sequence $\{\alpha_n\}$ and $\{\beta_n\}$, the iterative sequence $\{x_n\}$ converges strongly to a fixed point of the k-strict pseudo-contraction, which also solves some variational inequality.

Let ϕ be a bifunction of $C \times C$ into $\mathbb{R}$. The classical equilibrium problem for ϕ is to find $x \in C$ such that

$$EP : \phi(x, y) \ge 0, \quad \forall y \in C, \quad (7)$$

denoted the set of solutions by $EP(\phi)$. Given a mapping $T : C \to H$, let

$$\phi(x, y) = \langle Tx, y - x\rangle,\ \forall x, y \in C,$$

then $z \in EP(\phi)$ if and only if $\langle Tz, y - z\rangle \ge 0$ for all $y \in C$, that is, z is a solution of the variational inequality. Numerous problems in physics, optimizations and economics reduce to find a solution of (7). Some methods have been proposed to solve the equilibrium problem, see for instance, [2, 3] and the references therein.

By using the general approximation method, many authors constructed the compositive schemes to obtained the common element of fixed points of nonexpansive mapping and solutions of equilibrium problem. Next, we list their main contributions.

For finding an element of $EP(\phi) \cap F(S)$, Ceng et al [10] established the following iterative scheme:

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \forall y \in C, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)Su_n,\ \ \forall n \in \mathbb{N}, \end{cases}$$

under certain conditions, the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $EP(\phi) \cap F(S)$. Under the same conditions, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of $EP(\phi) \cap F(S)$ if and only if

$$\liminf_{n\to\infty} d(x_n, EP(\phi) \cap F(S)) = 0,$$

where $d(x_n, EP(\phi) \cap F(S))$ denotes the metric distance from the point x_n to $EP(\phi) \cap F(S)$.

To find an element of $EP(\phi) \cap F(S)$, Takahashi and Takahashi [11] introduced the following iterative scheme by the viscosity approximation method in a Hilbert space: $x_1 \in H$ and

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n,\ \ \forall n \in \mathbb{N}. \end{cases}$$

Under suitable conditions, some strong convergence theorems are obtained.

For finding an element of $EP(\phi) \cap VI(C, A) \cap F(S)$, recently Su, Shang and Qin [12] introduced the following iterative scheme: $x_1 \in H$ and

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(u_n - \lambda_n Au_n), \\ \quad \forall n \in \mathbb{N}. \end{cases}$$

Under suitable conditions, some strong convergence theorems are obtained, which are extend and improve the results of Iiduka et al [13] and Takahashi et al [11].

To find an element of $EP(\phi) \cap VI(C, A) \cap F(S)$, Plubtieng and Punpaeng [14] also introduced the following iterative scheme: $x_1 = u \in C$ and

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = P_C(u_n - \lambda_n Au_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n Ay_n), \\ \quad \forall n \in \mathbb{N}. \end{cases}$$

Under suitable conditions, some strong convergence theorems are obtained, which are extend some recent result of Yao and Yao [15].

In 2009, Y. Liu [16] introduced two iterative schemes by the general iterative method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a k-strictly pseudo-contractive non-self mapping in the setting of a real Hilbert space.

$$\begin{cases} \phi(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ y_n = \beta_n u_n + (1 - \beta_n)Su_n, \\ x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B)y_n, \ \ \forall n \in \mathbb{N}, \end{cases}$$

and $x_1 \in H$ arbitrarily,

$$\begin{cases} \phi(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ y_n = \beta_n u_n + (1 - \beta_n)Su_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)y_n, \ \ \forall n \in \mathbb{N} \end{cases}$$

where B is a strong positive bounded linear operator on H. Under some assumptions, the strong convergence theorems are obtained.

In 2011, M. Tian [17] adopted the hybrid steepest descent methods for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a strict pseudo-contraction mapping in the setting of real Hilbert spaces.

$$\begin{cases} \phi(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ y_n = \beta_n u_n + (1 - \beta_n)Su_n, \\ x_n = (I - \alpha_n \mu F)y_n, \ \ \forall n \in \mathbb{N}, \end{cases}$$

and $x_1 \in H$ arbitrarily,

$$\begin{cases} \phi(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ y_n = \beta_n u_n + (1 - \beta_n)Su_n, \\ x_{n+1} = (I - \alpha_n \mu F)y_n, \ \ \forall n \in \mathbb{N}, \end{cases}$$

where $F : H \to H$ be an L-Lipschitzian continuous and η-strongly monotone operator on H with $L, \eta > 0$. Under some assumptions, the strong convergence theorems are obtained.

Motivated and inspired by these facts, in this paper, we introduce two iteration methods, for finding an element of $EP(\phi) \cap F(S)$, where $S : C \to H$ is a k-strictly pseudo-contractive non-self mapping, which is also a solution of a variational inequality and then obtained two strong convergence theorems. Our results include Plubtieng and Punpaeng [18], S. Takahashi and W.Takahashi [11], Tada and W.Takahashi [19], Y. Liu [16] and M. Tian [17] as special cases. Furthermore, this paper will also be the development of the results of Ceng et al [10] in different directions.

2 Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C$$

Such a $P_C x$ is called the metric projection of H onto C. It is known that P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C x \Leftrightarrow \langle x - u, u - y\rangle \geq 0, \forall y \in C.$$

A mapping $F : C \to C$ is called L-Lipschitzian if there exists a positive constant L, such that

$$\|Fx - Fy\| \leq L\|x - y\|, \ \ \forall x, y \in C.$$

Obviously, F is nonexpansive if and only if $L = 1$. F is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|, \ \ \forall x, y \in C.$$

It is wildly known that H satisfies Opial's condition [13], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in H$ with $y \neq x$.

In order to solve the equilibrium problem for a bifunction $\phi : C \times C \to \mathbb{R}$, let us assume that ϕ satisfies the following conditions:

(A1) $\phi(x, x) = 0$, for all $x \in C$;

(A2) ϕ is monotone, that is, $\phi(x,y)+\phi(y,x) \le 0$, for all $x, y \in C$;

(A3) For all $x, y, z \in C, \lim_{t\downarrow 0} \phi(tz + (1-t)x, y) \le \phi(x,y)$;

(A4) For each fixed $x \in C$, the function $y \mapsto \phi(x,y)$ is convex and lower semicontinuous.

Let us recall the following lemmas which will be useful for our paper.

Lemma 1 *(see [20]).Let ϕ be a bifunction from $C \times C$ into $\mathbb{R}$ which satisfying (A1), (A2), (A3) and (A4), then, for any $r > 0$ and $x \in H$, there exists a point $z \in C$ such that*

$$\phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C.$$

Further, if

$$T_r x = \Big\{z \in C : \phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C\Big\},$$

then the followings hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, that is,

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle, \ \forall x, y \in H;$$

(3) $F(T_r) = EP(\phi)$;

(9) $EP(\phi)$ is nonempty, closed and convex.

Lemma 2 *(see [21]).If $S : C \to H$ is a k-strict pseudo-contraction, then the fixed point set $F(S)$ is closed convex, so that the projection $P_{F(S)}$ is well defined.*

Lemma 3 *(see [22]).Let $S : C \to H$ be a k-strict pseudo-contraction. Define $T : C \to H$ by*

$$Tx = \lambda x + (1-\lambda)Sx,$$

for each $x \in C$, then as $\lambda \in [k, 1)$, T is a nonexpansive mapping such that $F(T) = F(S)$.

Lemma 4 *(see [23]).In a Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \ \forall x, y \in H.$$

Lemma 5 *(see [4]).Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$s_{n+1} \le (1-\gamma_n)s_n + \gamma_n\delta_n, \quad \forall n \ge 0,$$

where $\{\gamma_n\}, \{\delta_n\}$ are sequences of real numbers such that:
(i)$\{\gamma_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty}\gamma_n = \infty$,
(ii) $\limsup_{n\to\infty}\delta_n \le 0$ or $\sum_{n=0}^{\infty}\gamma_n|\delta_n| < \infty$.
Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 6 *(see [7]).Let H be a Hilbert space, $f : H \to H$ is a contraction with coefficient $0 < \rho < 1$, and $F : H \to H$ is an L-Lipschitz continuous and η-strongly monotone operator with $L > 0,\ \eta > 0$. Then for $0 < \gamma < \mu\eta/\rho$,*

$$\begin{aligned} &\langle x - y, (\mu F - \gamma f)x - (\mu F - \gamma f)y\rangle \\ \ge\ &(\mu\eta - \gamma\rho)\|x - y\|^2, \ x, y \in H. \end{aligned}$$

That is, $(\mu F - \gamma f)$ is strongly monotone operator with coefficient $\mu\eta - \gamma\rho$.

3 Main Results

In the rest of this paper we always assume that F is an L-Lipschitzian continuous and η-strongly monotone operator with $L, \eta > 0$ and assume that $0 < \gamma < \frac{\tau}{\rho}$, $\tau = \mu(\eta - \mu L^2/2)$. Let $\{T_{\lambda_n}\}$ be mappings defined as Lemma 1. Define a mapping $S_n : C \to H$ by

$$S_n x = \beta_n x + (1-\beta_n)Sx, \ \forall x \in C,$$

where $\beta_n \in [k, 1)$, then, by Lemma 3, S_n is a nonexpansive. We consider the mapping G_n on H defined by

$$G_n x = [I - \alpha_n(\mu F - \gamma f)]S_n T_{\lambda_n} x, \ x \in H, n \in \mathbb{N},$$

where $\alpha_n \in (0,1)$. By Lemma 1 and 3, we have

$$\begin{aligned} &\|G_n x - G_n y\| \\ \le\ &[1 - \alpha_n(\tau - \gamma\rho)]\|T_{\lambda_n} x - T_{\lambda_n} y\| \\ \le\ &[1 - \alpha_n(\tau - \gamma\rho)]\|x - y\|. \end{aligned}$$

It is easy to see that G_n is a contraction. Therefore, by the Banach contraction principle, G_n has a unique fixed point $x_n^F \in H$ such that

$$x_n^F = [I - \alpha_n(\mu F - \gamma f)]S_n T_{\lambda_n} x_n^F.$$

For simplicity, we will write x_n for x_n^F provided no confusion occurs. Next, we prove that the sequence $\{x_n\}$ converges strongly to a point $q \in F(S) \cap EP(\phi)$ which solves the variational inequality

$$\langle(\mu F - \gamma f)q, p - q\rangle \ge 0, \ \forall p \in F(S) \cap EP(\phi). \quad (8)$$

Equivalently, $q = P_{F(S)\cap EP(\phi)}[I - (\mu F - \gamma f)]q$.

Theorem 7 *Let C be a nonempty closed convex subset of a real Hilbert space H and ϕ be a bifunction from $C\times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3) and (A4). Let $S : C \to H$ be a k-strictly pseudo-contractive nonself mapping such that $F(S) \cap EP(\phi) \neq \emptyset$. Let $F : H \to H$ be an L-Lipschitzian continuous and η-strongly monotone operator on H with $L, \eta > 0$ and $0 < \gamma < \frac{\tau}{\rho}$, $\tau = \mu(\eta - \mu L^2/2)$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} \phi(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \beta_n u_n + (1-\beta_n)Su_n, \\ x_n = [I - \alpha_n(\mu F - \gamma f)]y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $u_n = T_{\lambda_n}x_n$, $y_n = S_n u_n$, and $\{\lambda_n\} \subset (0, +\infty)$ satisfy $\liminf_{n\to\infty}\lambda_n > 0$, if $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1), \lim_{n\to\infty}\alpha_n = 0$;

(ii) $0 \leq k \leq \beta_n \leq \lambda < 1$ *and* $\lim_{n\to\infty}\beta_n = \lambda$,

then $\{x_n\}$ converges strongly to a point $q \in F(S) \cap EP(\phi)$ which solves the variational inequality (8).

Proof: First, take $p \in F(S) \cap EP(\phi)$. Since $u_n = T_{\lambda_n}x_n$ and $p = T_{\lambda_n}p$, from Lemma 1, for any $n \in \mathbb{N}$, we have

$$\|u_n - p\| = \|T_{\lambda_n}x_n - T_{\lambda_n}p\| \leq \|x_n - p\|. \quad (9)$$

Then, since $y_n = S_n u_n$ and $S_n p = p$, we obtain that

$$\begin{aligned} \|y_n - p\| &= \|S_n u_n - S_n p\| \\ &\leq \|u_n - p\| \leq \|x_n - p\|. \quad (10) \end{aligned}$$

Further, we have

$$\begin{aligned} \|x_n - p\| &= \|[I - \alpha_n(\mu F - \gamma f)]y_n - p\| \\ &= \|(I - \alpha_n\mu F)y_n - (I - \alpha_n\mu F)p \\ &\quad +\alpha_n\gamma f(y_n) - \alpha_n\gamma f(p) \\ &\quad +\alpha_n\gamma f(p) - \alpha_n\mu Fp\| \\ &\leq (1 - \alpha_n\tau)\|y_n - p\| + \alpha_n\gamma\rho\|y_n - p\| \\ &\quad +\alpha_n\|(\gamma f - \mu F)p\| \\ &\leq (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\gamma\rho\|x_n - p\| \\ &\quad +\alpha_n\|(\gamma f - \mu F)p\| \\ &\leq [1 - \alpha_n(\tau - \gamma\rho)]\|x_n - p\| \\ &\quad +\alpha_n\|(\gamma f - \mu F)p\|. \end{aligned}$$

It follows that $\|x_n - p\| \leq \frac{1}{\tau - \gamma\rho}\|(\gamma f - \mu F)p\|$. Hence, $\{x_n\}$ is bounded, and we also obtain that $\{u_n\}$ and $\{y_n\}$ are bounded.

Notice that

$$\begin{aligned} &\|u_n - y_n\| \leq \|u_n - x_n\| + \|x_n - y_n\| \\ =\ &\|u_n - x_n\| + \\ &\|[I - \alpha_n(\mu F - \gamma f)]y_n - y_n\| \\ =\ &\|u_n - x_n\| + \alpha_n\| - (\mu F - \gamma f)y_n\|. \quad (11) \end{aligned}$$

By Lemma 1, we have

$$\begin{aligned} &\|u_n - p\|^2 \leq \|T_{\lambda_n}x_n - T_{\lambda_n}p\|^2 \\ \leq\ &\langle x_n - p, u_n - p\rangle \\ =\ &\frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 \\ &-\|u_n - x_n\|^2). \end{aligned}$$

It follows that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2. \quad (12)$$

Thus, from Lemma 4, (10) and (12), we obtain

$$\begin{aligned} &\|x_n - p\|^2 = \|[I - \alpha_n(\mu F - \gamma f)]y_n - p\|^2 \\ =\ &\|[I - \alpha_n(\mu F - \gamma f)]y_n \\ &\quad -[I - \alpha_n(\mu F - \gamma f)]p \\ &-\alpha_n(\mu F - \gamma f)p\|^2 \\ \leq\ &\|[I - \alpha_n(\mu F - \gamma f)]y_n \\ &-[I - \alpha_n(\mu F - \gamma f)]p\|^2 \\ &+2\alpha_n\langle -(\mu F - \gamma f)p, x_n - p\rangle \\ \leq\ &[1 - \alpha_n(\tau - \gamma\rho)]^2\|y_n - p\|^2 \\ &+2\alpha_n\| - (\mu F - \gamma f)p\| \cdot \|x_n - p\| \\ \leq\ &[1 - \alpha_n(\tau - \gamma\rho)]^2\|u_n - p\|^2 \\ &+2\alpha_n\| - (\mu F - \gamma f)p\| \cdot \|x_n - p\| \\ \leq\ &[1 - \alpha_n(\tau - \gamma\rho)]^2(\|x_n - p\|^2 \\ &-\|x_n - u_n\|^2) \\ &+2\alpha_n\| - (\mu F - \gamma f)p\| \cdot \|x_n - p\| \\ =\ &[1 - 2\alpha_n(\tau - \gamma\rho) + \alpha_n^2(\tau - \gamma\rho)^2] \\ &\|x_n - p\|^2 - [1 - \alpha_n(\tau - \gamma\rho)]^2 \\ &\|x_n - u_n\|^2 + 2\alpha_n\| - (\mu F - \gamma f)p\| \\ &\cdot\|x_n - p\| \\ \leq\ &[1 + \alpha_n^2(\tau - \gamma\rho)^2]\|x_n - p\|^2 \\ &-[1 - \alpha_n(\tau - \gamma\rho)]^2\|x_n - u_n\|^2 \\ &+2\alpha_n\| - (\mu F - \gamma f)p\| \cdot \|x_n - p\|. \end{aligned}$$

It follows that

$$\begin{aligned} &[1 - \alpha_n(\tau - \gamma\rho)]^2\|x_n - u_n\|^2 \\ \leq\ &\alpha_n^2(\tau - \gamma\rho)^2]\|x_n - p\|^2 \\ &+2\alpha_n\| - (\mu F - \gamma f)p\| \cdot \|x_n - p\|. \end{aligned}$$

Since $\alpha_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0.$$

From (11), we derive

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \quad (13)$$

Define $T : C \to H$ by $Tx = \lambda x + (1-\lambda)Sx$. Then T is nonexpansive with $F(T) = F(S)$ by Lemma 3. Since

$$\begin{aligned} \|Tu_n - u_n\| &\leq \|Tu_n - y_n\| + \|y_n - u_n\| \\ &\leq |\lambda - \beta_n|\|u_n - Su_n\| + \|y_n - u_n\|, \end{aligned}$$

by using (13) and $\beta_n \to \lambda$ we obtain

$$\lim_{n\to\infty} \|Tu_n - u_n\| = 0.$$

Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ which converges weakly to q. We shall show that $q \in F(S) \cap EP(\phi)$. In fact, C is closed and convex, and hence C is weakly closed, we have $q \in C$. Let us show that $q \in F(S)$. Assume that $q \notin F(T)$, since $u_{n_i} \rightharpoonup q$ and $q \neq Tq$, it follows from the Opial's condition that

$$\begin{aligned} &\liminf_{n\to\infty} \|u_{n_i} - q\| < \liminf_{n\to\infty} \|u_{n_i} - Tq\| \\ \leq\ &\liminf_{n\to\infty} (\|u_{n_i} - Tu_{n_i}\| + \|Tu_{n_i} - Tq\|) \\ \leq\ &\liminf_{n\to\infty} \|u_{n_i} - q\|. \end{aligned}$$

This is a contraction. So we get $q \in F(T)$ and hence $q \in F(S)$.

Next, we show that $q \in EP(\phi)$. Since $u_n = T_{\lambda_n} x_n$, for any $y \in C$ we have

$$\phi(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0.$$

From (A2), it holds that

$$\frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq \phi(y, u_n).$$

Replacing n by n_i, we have

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \geq \phi(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup q$, it follows from (A4) that $0 \geq \phi(y, q)$ for all $y \in C$. Let

$$z_t = ty + (1-t)q, \quad \forall t \in (0,1], y \in C.$$

Then we have $z_t \in C$ and hence $\phi(z_t, q) \leq 0$. From (A1) and (A4) we get

$$\begin{aligned} 0 &= \phi(z_t, z_t) \\ &\leq t\phi(z_t, y) + (1-t)\phi(z_t, q) \\ &\leq t\phi(z_t, y), \end{aligned}$$

and hence $0 \leq \phi(z_t, y)$. From (A3) we get $0 \leq \phi(q, y)$ for all $y \in C$ and hence $q \in EP(\phi)$. Therefore, $q \in EP(\phi) \cap F(S)$.

On the other hand, since

$$\begin{aligned} &x_n - q \\ &= -\alpha_n(\mu F - \gamma f)q + [I - \alpha_n(\mu F - \gamma f)]y_n \\ &\quad -[I - \alpha_n(\mu F - \gamma f)]q \end{aligned}$$

we have

$$\begin{aligned} &\|x_n - q\|^2 \\ &= \langle -\alpha_n(\mu F - \gamma f)q, x_n - q\rangle + \langle [I - \alpha_n(\mu F - \gamma f)]y_n \\ &\quad -[I - \alpha_n(\mu F - \gamma f)]q, x_n - q\rangle \\ &\leq \alpha_n\langle -(\mu F - \gamma f)q, x_n - q\rangle \\ &\quad +[1 - \alpha_n(\tau - \gamma\rho)]\|x_n - q\|^2, \end{aligned}$$

which follows that

$$\|x_n - q\|^2 \leq \frac{1}{\tau - \gamma\rho}\langle -(\mu F - \gamma f)q, x_n - q\rangle,$$

in particular,

$$\|x_{n_i} - q\|^2 \leq \frac{1}{\tau - \gamma\rho}\langle -(\mu F - \gamma f)q, x_{n_i} - q\rangle.$$

Since $x_{n_i} \rightharpoonup q$, it follows from above that $x_{n_i} \to q$ as $i \to \infty$.

Next, we show that q is a solution of the variational inequality (8).

Since

$$\begin{aligned} x_n &= [I - \alpha_n(\mu F - \gamma f)]y_n \\ &= [I - \alpha_n(\mu F - \gamma f)]S_n T_{\lambda_n} x_n, \end{aligned}$$

we have

$$\begin{aligned} &(\mu F - \gamma f)x_n \\ =\ &-\frac{1}{\alpha_n}[(I - S_n T_{\lambda_n})x_n \\ &-\alpha_n(\mu F - \gamma f)(I - S_n T_{\lambda_n})x_n]. \end{aligned}$$

For any $p \in EP(\phi) \cap F(S)$,

$$\begin{aligned} &\langle(\mu F - \gamma f)x_n, x_n - p\rangle \\ =\ &-\frac{1}{\alpha_n}\langle(I - S_n T_{\lambda_n})x_n - \alpha_n(\mu F - \gamma f)\cdot \\ &(I - S_n T_{\lambda_n})x_n, x_n - q\rangle \\ =\ &-\frac{1}{\alpha_n}\langle(I - S_n T_{\lambda_n})x_n - (I - S_n T_{\lambda_n})q, \\ &x_n - q\rangle + \langle(\mu F - \gamma f)(I - S_n T_{\lambda_n})x_n, \\ &x_n - q\rangle. \end{aligned} \quad (14)$$

Note that $(I - S_n T_{\lambda_n})$ is monotone (i.e.,$\langle x - y, (I - S_n T_{\lambda_n})x - (I - S_n T_{\lambda_n})y\rangle \geq 0$, for all $x, y \in H$, due to the nonexpansivity of $S_nT_{\lambda_n}$). Replacing n in (14) by n_i and letting $i \to \infty$, we obtain

$$\begin{aligned} & \langle(\mu F-\gamma f)q, q-p\rangle \\ = & \lim_{i\to\infty}\langle(\mu F-\gamma f)x_{n_i}, x_{n_i}-p\rangle \\ \leq & \lim_{i\to\infty}\langle(\mu F-\gamma f)(I-S_{n_i}T_{\lambda_{n_i}})x_{n_i}, x_{n_i}-p\rangle \\ = & \ 0. \end{aligned} \quad (15)$$

That is, $q \in EP(\phi)\cap F(S)$ is a solution of (8).

To show that the sequence $\{x_n\}$ converges strongly to q, we assume that $x_{n_k} \to \hat{x}$. Similar to the proof above, we have $\hat{x} \in EP(\phi)\cap F(S)$. Moveover, it follows from the inequality (15) that

$$\langle(\mu F-\gamma f)q, q-\hat{x}\rangle \leq 0. \quad (16)$$

Interchange q and $\hat{x}$ to obtain

$$\langle(\mu F-\gamma f)\hat{x}, \hat{x}-q\rangle \leq 0. \quad (17)$$

From Lemma 6, adding up (16) and (17) yields

$$\begin{aligned} & (\mu\eta-\gamma\rho)\|q-\hat{x}\|^2 \\ \leq & \langle q-\hat{x}, (\mu F-\gamma f)q-(\mu F-\gamma f)\hat{x}\rangle \\ \leq & \ 0. \end{aligned}$$

Hence, $q = \hat{x}$, and therefore $x_n \to q$, as $n \to \infty$,

$$\langle[I-(\mu F-\gamma f)]q-q, q-p\rangle \geq 0 \quad \forall p \in EP(\phi)\cap F(S).$$

This is equivalent to the fixed point equation

$$P_{EP(\phi)\cap F(S)}[I-(\mu F-\gamma f)]q = q.$$

The desired result follows. □

Theorem 8 *Let C be a nonempty closed convex subset of a real Hilbert space H and ϕ be a bifunction from $C\times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3) and (A4). Let $S: C \to H$ be a k-strictly pseudo-contractive nonself mapping such that $F(S)\cap EP(\phi) \neq \emptyset$. Let $F: H \to H$ be an L-Lipschitzian continuous and η-strongly monotone operator on H with $L, \eta > 0$ and $0 < \gamma < \frac{\tau}{\rho}$, $\tau = \mu(\eta - \mu L^2/2)$. Let $\{x_n\}$ be a sequence generated by: $x_1 \in H$ arbitrarily,*

$$\begin{cases} \phi(u_n, y) + \frac{1}{\lambda_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \ \ \forall y \in C, \\ y_n = \beta_n u_n + (1-\beta_n)Su_n, \\ x_{n+1} = [I-\alpha_n(\mu F-\gamma f)]y_n, \ \ \forall n \in \mathbb{N}, \end{cases}$$

where $u_n = T_{\lambda_n}x_n, y_n = S_nu_n$, if $\{\alpha_n\}, \{\beta_n\}$ and $\{\lambda_n\}$satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1), \lim_{n\to\infty}\alpha_n$=0, $\sum_{n=1}^{\infty}\alpha_n = \infty, \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$;

(ii) $0 \leq k \leq \beta_n \leq \lambda < 1$ and $\lim_{n\to\infty}\beta_n = \lambda, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$;

(iii) $\{\lambda_n\} \subset (0,\infty), \lim_{n\to\infty}\lambda_n > 0, \sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty$;

then $\{x_n\}$ and $\{u_n\}$ converges strongly to a point $q \in F(S)\cap EP(\phi)$ which solves the variational inequality (8).

Proof: We first show that $\{x_n\}$ is bounded. Indeed, pick any $p \in F(S)\cap EP(\phi)$ to derive that

$$\begin{aligned} & \|x_{n+1}-p\| = \|[I-\alpha_n(\mu F-\gamma f)]y_n-p\| \\ = & \|(I-\alpha_n\mu F)y_n-(I-\alpha_n\mu F)p \\ & +\alpha_n\gamma f(y_n)-\alpha_n\gamma f(p)+\alpha_n\gamma f(p)-\alpha_n\mu Fp\| \\ \leq & (1-\alpha_n\tau)\|y_n-p\| \\ & +\alpha_n\gamma\rho\|y_n-p\|+\alpha_n\|(\gamma f-\mu F)p\| \\ \leq & [1-\alpha_n(\tau-\gamma\rho)]\|x_n-p\|+\alpha_n\|(\gamma f-\mu F)p\|. \end{aligned}$$

By induction, we have

$$\|x_n-p\| \leq \max\left\{\|x_1-p\|, \frac{1}{\tau-\gamma\rho}\|(\gamma f-\mu F)p\|\right\},$$

and hence $\{x_n\}$ is bounded. From (9) and (10), we also derive that $\{u_n\}$ and $\{y_n\}$ are bounded. Next, we show that $\|x_{n+1}-x_n\| \to 0$. We have

$$\begin{aligned} & \|x_{n+1}-x_n\| \\ = & \|[I-\alpha_n(\mu F-\gamma f)]y_n \\ & -[I-\alpha_{n-1}(\mu F-\gamma f)]y_{n-1}\| \\ \leq & [1-\alpha_n(\tau-\gamma\rho)]\|y_n-y_{n-1}\| \\ & +|\alpha_n-\alpha_{n-1}|\|(\gamma f-\mu F)y_{n-1}\| \\ \leq & [1-\alpha_n(\tau-\gamma\rho)]\|y_n-y_{n-1}\|+K|\alpha_n-\alpha_{n-1}|, \end{aligned} \quad (18)$$

where

$$K = sup\{\|(\gamma f-\mu F)y_n\| : n \in \mathbb{N}\} < \infty.$$

On the other hand, we have

$$\begin{aligned} & \|y_n-y_{n-1}\| = \|S_nu_n-S_{n-1}u_{n-1}\| \\ \leq & \|S_nu_n-S_nu_{n-1}\|+\|S_nu_{n-1}-S_{n-1}u_{n-1}\| \\ \leq & \|u_n-u_{n-1}\|+\|S_nu_{n-1}-S_{n-1}u_{n-1}\|. \end{aligned} \quad (19)$$

From $u_{n+1} = T_{\lambda_{n+1}}x_{n+1}$ and $u_n = T_{\lambda_n}x_n$, we get

$$\phi(u_{n+1}, y)+\tfrac{1}{\lambda_{n+1}}\langle y-u_{n+1}, u_{n+1}-x_{n+1}\rangle \geq 0 \quad \forall y \in C, \quad (20)$$

$$\phi(u_n, y)+\tfrac{1}{\lambda_n}\langle y-u_n, u_n-x_n\rangle \geq 0 \quad \forall y \in C. \quad (21)$$

Putting $y = u_n$ in (20) and $y = u_{n+1}$ in (21), we have

$$\phi(u_{n+1}, u_n) + \tfrac{1}{\lambda_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \quad \forall y \in C,$$
$$\phi(u_n, u_{n+1}) + \tfrac{1}{\lambda_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C.$$

From (A2) we get

$$\Big\langle u_{n+1} - u_n, \frac{u_n - x_n}{\lambda_n} - \frac{u_{n+1} - x_{n+1}}{\lambda_{n+1}}\Big\rangle \geq 0,$$

and hence

$$\Big\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{\lambda_n}{\lambda_{n+1}}(u_{n+1} - x_{n+1})\Big\rangle \geq 0.$$

Since $\lim_{n\to\infty} \lambda_n > 0$, without loss of generality, we can assume that there exists a real number a such that $\lambda_n > a > 0$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \Big\langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{\lambda_n}{\lambda_{n+1}})(u_{n+1} - x_{n+1})\Big\rangle \\ &\leq \|u_{n+1} - u_n\|\Big\{\|x_{n+1} - x_n\| + \Big|(1 - \frac{\lambda_n}{\lambda_{n+1}})\Big|\|u_{n+1} - x_{n+1}\|\Big\},\end{aligned}$$

and hence

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{a} M_0 \quad (22)$$

where $M_0 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$.

Now we estimate $\|S_n u_{n-1} - S_{n-1}u_{n-1}\|$. Notice that

$$\begin{aligned}&\|S_n u_{n-1} - S_{n-1}u_{n-1}\| \\ =\ & \|[\beta_n u_{n-1} + (1 - \beta_n)Su_{n-1}] - [\beta_{n-1}u_{n-1} + (1 - \beta_{n-1})Su_{n-1}]\| \\ \leq\ & |\beta_n - \beta_{n-1}|\|u_{n-1} - Su_{n-1}\|. \quad (23)\end{aligned}$$

From (22), (23) and (19), we obtain

$$\begin{aligned}&\|y_n - y_{n-1}\| \\ \leq\ & \|x_n - x_{n-1}\| + \frac{M_0}{a}|\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}|\|u_{n-1} - Su_{n-1}\| \\ \leq\ & \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1 + |\beta_n - \beta_{n-1}|M_1. \quad (24)\end{aligned}$$

where M_1 is an appropriate constant such that

$$M_1 \geq \frac{M_0}{a} + \|u_{n-1} - Su_{n-1}\|, \ \forall n \in \mathbb{N}.$$

From (18) and (24), we obtain

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ \leq\ & K|\alpha_n - \alpha_{n-1}| + (1 - \alpha_n(\tau - \gamma\rho)) (\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1 + |\beta_n - \beta_{n-1}|M_1) \\ \leq\ & (1 - \alpha_n(\tau - \gamma\rho))\|x_n - x_{n-1}\| + M(|\alpha_n - \alpha_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}|),\end{aligned}$$

where M=max$[K, M_1]$. By Lemma 5 we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \quad (25)$$

Using (22) and (24) together with $|\lambda_n - \lambda_{n-1}| \to 0$ and $|\beta_n - \beta_{n-1}| \to 0$, we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$
$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0.$$

From the equality

$$x_{n+1} = [I - \alpha_n(\mu F - \gamma f)]y_n$$

it follows that

$$\begin{aligned}\|x_n - y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n\| - (\gamma f - \mu F)y_n\|.\end{aligned}$$

From $\alpha_n \to 0$ and (25) we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \quad (26)$$

For $p \in F(S) \cap EP(\phi)$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{\lambda_n}x_n - T_{\lambda_n}p\|^2 \\ &\leq \langle x_n - p, u_n - p\rangle \\ &= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|u_n - x_n\|^2),\end{aligned}$$

which implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2. \quad (27)$$

Thus from (10) and (27) we derive that

$$\begin{aligned} & \|x_{n+1}-p\|^2 = \|[I-\alpha_n(\mu F-\gamma f)]y_n - p\|^2 \\ = & \|[I-\alpha_n(\mu F-\gamma f)]y_n \\ & -[I-\alpha_n(\mu F-\gamma f)]p - \alpha_n(\mu F-\gamma f)p\|^2 \\ \leq & [1-\alpha_n(\tau-\gamma\rho)]^2\|y_n-p\|^2 \\ & +2\alpha_n\|-(\mu F-\gamma f)p\|\|y_n-p\| \\ & +\alpha_n^2\|-(\mu F-\gamma f)p\|^2 \\ \leq & \|u_n-p\|^2 \\ & +2\alpha_n\|-(\mu F-\gamma f)p\|\|x_n-p\| \\ & +\alpha_n^2\|-(\mu F-\gamma f)p\|^2 \\ \leq & \|x_n-p\|^2 - \|u_n-x_n\|^2 \\ & +2\alpha_n\|-(\mu F-\gamma f)p\|\|x_n-p\| \\ & +\alpha_n^2\|-(\mu F-\gamma f)p\|^2. \end{aligned}$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$, we have

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{28}$$

From (26) and (28), we obtain

$$\|u_n-y_n\| \leq \|u_n-x_n\| + \|x_n-y_n\| \to 0, \quad as\ n\to\infty. \tag{29}$$

Define a mapping $T : C \to H$ by $Tx = \lambda x + (1-\lambda)Sx$. Then T is nonexpansive with $F(T) = F(S)$ by Lemma 3. Since

$$\begin{aligned} \|Tu_n-u_n\| & \leq \|Tu_n-y_n\| + \|y_n-u_n\| \\ & \leq |\lambda-\beta_n|\|u_n-Su_n\| + \|y_n-u_n\|, \end{aligned}$$

from (29) and $\beta_n \to \lambda$ we obtain

$$\lim_{n\to\infty}\|Tu_n-u_n\| = 0. \tag{30}$$

Finally we show that

$$\limsup_{n\to\infty}\langle(\mu F-\gamma f)q, q-x_n\rangle \leq 0$$

where $q = P_{F(S)\cap EP(\phi)}[I-(\mu F-\gamma f)]q$ is a unique solution of the variational inequality (8). Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned} & \lim_{i\to\infty}\langle(\mu F-\gamma f)q, q-x_{n_i}\rangle \\ = & \limsup_{n\to\infty}\langle(\mu F-\gamma f)q, q-x_n\rangle. \end{aligned}$$

Due to $\{u_{n_i}\}$ is bounded, there exist a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to w. Without lose of generality, we can assume that $u_{n_i} \rightharpoonup w$. From (28) and (30), we obtain $x_{n_i} \rightharpoonup w$ and $Tu_{n_i} \rightharpoonup w$. By the same argument as in the proof of Theorem 7, we have $w \in EP(\phi)\cap F(S)$. Since $q = P_{F(S)\cap EP(\phi)}[I-(\mu F-\gamma f)]q$, it follows that

$$\begin{aligned} & \limsup_{n\to\infty}\langle(\mu F-\gamma f)q, q-x_n\rangle \\ = & \langle(\mu F-\gamma f)q, q-w\rangle \leq 0. \end{aligned} \tag{31}$$

From

$$\begin{aligned} & x_{n+1}-q \\ & = -\alpha_n(\mu F-\gamma f)q + [I-\alpha_n(\mu F-\gamma f)]y_n \\ & \quad -[I-\alpha_n(\mu F-\gamma f)]q \end{aligned}$$

we get

$$\begin{aligned} & \|x_{n+1}-q\|^2 \\ \leq & \|[I-\alpha_n(\mu F-\gamma f)]y_n - [I-\alpha_n(\mu F-\gamma f)]q\|^2 \\ & +2\alpha_n\langle-(\mu F-\gamma f)q, x_{n+1}-q\rangle \\ \leq & [1-\alpha_n(\tau-\gamma\rho)]^2\|x_n-q\|^2 \\ & +2\alpha_n\langle-(\mu F-\gamma f)q, x_{n+1}-q\rangle. \end{aligned}$$

This implies that

$$\begin{aligned} & \|x_{n+1}-q\|^2 \\ \leq & [1-2\alpha_n(\tau-\gamma\rho) + (\alpha_n(\tau-\gamma\rho))^2] \\ & \|x_n-q\|^2 + 2\alpha_n\langle-(\mu F-\gamma f)q, x_{n+1}-q\rangle \\ = & [1-2\alpha_n(\tau-\gamma\rho)]\|x_n-q\|^2 \\ & +(\alpha_n(\tau-\gamma\rho))^2\|x_n-q\|^2 \\ & +2\alpha_n\langle-(\mu F-\gamma f)q, x_{n+1}-q\rangle \\ = & [1-2\alpha_n(\tau-\gamma\rho)]\|x_n-q\|^2 \\ & +2\alpha_n(\tau-\gamma\rho)\Big[\frac{\alpha_n(\tau-\gamma\rho)}{2}M^* \\ & +\frac{1}{\tau-\gamma\rho}\langle-(\mu F-\gamma f)q, x_{n+1}-q\rangle\Big] \\ = & (1-\gamma_n)\|x_n-q\|^2 + \gamma_n\delta_n \end{aligned}$$

where $M^* = \sup\{\|x_n-q\|^2 : n \in N\}$, $\gamma_n = 2\alpha_n(\tau-\gamma\rho)$ and $\delta_n = \frac{\alpha_n(\tau-\gamma\rho)}{2}M^* + \frac{1}{\tau-\gamma\rho}\langle-(\mu F-\gamma f)q, x_{n+1}-q\rangle$.

It is easy to see that $\lim_{n\to\infty}\gamma_n = 0$, $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$ by (31). By Lemma 5, the sequence $\{x_n\}$ converges strongly to q. □

Remark 9 *If $C = H$, S is a nonexpansive mapping, $\{\beta_n\} = 0$, $\phi(x,y) = 0$ for all $x, y \in C$, $\lambda_n = 1$, $\mu = 1$ and $F = A$, then Theorem 8 reduced to Theorem 3.4 of G. Marino and H. K. Xu [6].*

Remark 10 *If $\mu = 1$ and $F = B$ in Theorem 7 and Theorem 8, we can obtain the corresponding results in Y. Liu [16].*

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