Global exponential stability of high-order BAM neural networks with S-type distributed delays and reaction diffusion terms

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Abstract: In this paper, by constructing suitable Lyapunov functional, using differential mean value theorem and homeomorphism, we analyze the global exponential stability of high-order bi-directional associative memory (BAM) neural networks with reaction-diffusion terms and S-type distributed delays. Some sufficient theorems have been derived under different conditions to guarantee the global exponential stability of the networks. Moreover, two numerical examples are presented to illustrate the feasibility and effectiveness of the results.

Key–Words: High-order BAM Neural Networks; reaction- diffusion terms; S-type distributed delays; Lyapunov functional; exponential stability

1 Introduction

The dynamical behavior of bi-directional associative memory (BAM) neural networks introduced by Kosko [1] has played an important role in some applications such as image and signal processing, pattern recognition, optimization and automatic control. It is recognized that such applications of the BAM neural network depends heavily on the stability of the equilibrium point of BAM neural networks. The problem of stability analysis of first- order BAM neural networks has received much attention in recent years, and many results have been reported (see [2]-[7]). However, such neural networks are shown to have limitations such as limited capacity when used in pattern recognition problems (see [8]). Also, the dilemmas of optimization problems that can be solved using neural networks are limited. This led many investigators to use neural networks with high-order connections. The usage of high-order connections in neural networks improves dramatically their storage capacity (see [8]) and convergence rate, and increases the class of optimization problems (see [9]-[10]). Therefore, the stability of high-order BAM neural network is of great importance and applications, and has been widely investigated. For instance, by employing the linear matrix inequality (LMI) and the Lyapunov functional methods, Cao, Liang and Lam [11] obtained several sufficient conditions for ensuring the system to be globally exponentially stable. In paper [12], the existence and global exponential stability of periodic solution is studied for high-order bidirectional associative memory (BAM) neural networks with and without impulses based on coincidence degree theory as well as a priori estimates and Lyapunov functional. There are other results about the stability of high-order BAM neural network (see,[13]-[19]).

It is well known that time delays can't be avoided in interactions between neurons due to the finite transmission speed of signals among neurons, and will cause instability, divergence and oscillations in neural networks. So it is necessary to introduce time delays in the neural network models. In practice, the delays in artificial neural networks usually continuous distributed, because neural networks usually has a spatial extent due to the presence of an amount of parallel pathways with a variety of axon sizes and lengths. Besides, diffusion effects cannot be avoided in the neural network models when electrons are moving in asymmetric electromagnetic fields. To overcome this situation, we must consider that the activations vary in space as well as in time. In [19]–[31],[32]–[38] and [39]–[40] authors have considered the stability of reaction-diffusion neural networks with discrete delays, continuous distributed delays and S-type distributed delays (which is more general than continuous distributed delays), respectively.

Motivated by the discussions above, a class of high-order BAM neural networks with S-type distributed delays and reaction–diffusion terms is considered in this paper. To the best of our knowledge, there have been very few results on analysis for this type of BAM neural networks. In this paper, we will derive some sufficient conditions of existence, uniqueness and global exponential stability of equilibrium points for high-order BAM neutral networks with S-type distributed delays and reaction–diffusion terms by applying some analysis techniques, constructing suitable Lyapunov functional, using differential mean value theorem and homeomorphism. The remainder of the paper is organized as follows: In Sec. 2, the model formulation and some preliminaries are given. The main results are stated in Sec. 3. Finally, two illustrative example and simulation are given to show the effectiveness of the proposed theory.

2 Model formulation and preliminaries

Consider the following high-order BAM neutral networks with S-type distributed delays and reaction–diffusion terms

$$\begin{cases} \frac{\partial u_i(t,x)}{\partial t} = \sum\limits_{k=1}^{r} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial u_i(t,x)}{\partial x_k}\right) \\ -a_i u_i(t,x) + \sum\limits_{j=1}^{m} b_{ij} g_j\left(v_j(t,x)\right) \\ + \sum\limits_{j=1}^{m}\sum\limits_{l=1}^{m} e_{ijl} g_j\left(v_j(t,x)\right) g_l\left(v_l(t,x)\right) \\ + \sum\limits_{j=1}^{m} p_{ij}\int_{-\infty}^{0} w_j\left(v_j(t+\theta,x)\right) d\eta_{ij}(\theta) + I_i, \\ \frac{\partial v_j(t,x)}{\partial t} = \sum\limits_{k=1}^{r} \frac{\partial}{\partial x_k}\left(E_{jk}\frac{\partial v_j(t,x)}{\partial x_k}\right) \\ -c_j v_j(t,x) + \sum\limits_{i=1}^{n} d_{ji} f_i\left(u_j(t,x)\right) \\ + \sum\limits_{i=1}^{n}\sum\limits_{p=1}^{n} s_{jip} f_i\left(u_i(t,x)\right) f_p\left(u_p(t,x)\right) \\ + \sum\limits_{i=1}^{n} q_{ji}\int_{-\infty}^{0} h_i\left(u_i(t+\theta,x)\right) d\sigma_{ji}(\theta) + J_j, \end{cases} \tag{1}$$

where $i = 1,2,\cdots,n, j = 1,2,\cdots,m, x = (x_1,x_2,\cdots,x_r)^T \in \Omega_i \subset R^r$ and Ω_i is a bounded compact set with smooth boundary $\partial\Omega_i$ and mes$\Omega_i > 0$ in space R^r;

$$u(t,x) = (u_1(t,x), u_2(t,x), \cdots, u_n(t,x))^T \in R^n,$$

$$v(t,x) = (v_1(t,x), v_2(t,x), \cdots, v_m(t,x))^T \in R^m,$$

$u_i(t,x)$ and $v_j(t,x)$ are the state of the i-th neurons from the neural field $F_u$and the j-th neurons from the neural field F_v at time t and in space x, respectively; $D_{ik} > 0$ and $E_{jk} > 0$ correspond to the transmission reaction-diffusion operator along the i-th neurons and the j-th neurons, respectively; $a_i > 0$ and $c_j > 0$ denote the rate with witch the i-th neurons and the j-th neurons will reset its potential to the resting state in isolation when disconnected from the networks and external inputs, respectively; $b_{ij},\ d_{ji}, e_{ijl},\ s_{jip}$ are constants and denote the first-and second-and connection weights of the neural networks respectively;

$f_i,\ h_i$ denote the activation functions of the i-th neurons, and the $g_j,\ w_j$ the jth neurons at time t and in space x, respectively; $\int_{-\infty}^{0} w_j\left(v_j(t+\theta,x)\right) d\eta_{ij}(\theta)$ and $\int_{-\infty}^{0} h_i\left(u_i(t+\theta,x)\right) d\sigma_{ji}(\theta)$ are Lebesgue-Stieltjes integrable, $\eta_{ij}(\theta)$ and $\sigma_{ji}(\theta)$ are non-decreasing bounded variation functions which satisfy

$$\begin{array}{l} \int_{-\infty}^{0} d\eta_{ij}(\theta) = k_{ij} > 0, \\ i = 1,2,\cdots,n, j = 1,2,\cdots,m, \end{array} \tag{2}$$

$$\begin{array}{l} \int_{-\infty}^{0} d\sigma_{ji}(\theta) = r_{ji} > 0, \\ j = 1,2,\cdots,m, i = 1,2,\cdots,n, \end{array} \tag{3}$$

I_i and J_i are the i-th and j-th component of an external inputs source introduced from outside the network to the cell i and j, respectively.

The boundary conditions and initial conditions of system (1) are given by

$$\begin{cases} \frac{\partial u_i(t,x)}{\partial n} = \left(\frac{\partial u_i(t,x)}{\partial x_1}, \frac{\partial u_i(t,x)}{\partial x_2}, \cdots, \frac{\partial u_i(t,x)}{\partial x_r}\right)^T = 0, \\ t \geq 0, x \in \partial\Omega_i, i = 1,2,\cdots,n, \\ \frac{\partial v_j(t,x)}{\partial n} = \left(\frac{\partial v_j(t,x)}{\partial x_1}, \frac{\partial v_j(t,x)}{\partial x_2}, \cdots, \frac{\partial v_j(t,x)}{\partial x_r}\right)^T = 0, \\ t \geq 0, x \in \partial\Omega_i, i = 1,2,\cdots,m, \end{cases} \tag{4}$$

and

$$\begin{cases} u_i(s,x) = \phi_{u_i}(s,x), s \in (-\infty, 0], \\ v_j(s,x) = \phi_{v_j}(s,x), s \in (-\infty, 0], \end{cases} \tag{5}$$

for $x \in \Omega_i, i = 1,2,\cdots,n, j = 1,2,\cdots,m,$where $\phi_{u_i}(s,x)$ and $\phi_{v_j}(s,x)$ are bounder on $(-\infty, 0]$.

In order to establish the stability conditions for system (1), we first give some usual assumptions

$(\mathbf{H_1})$: The activation functions $f_i,\ h_i,\ g_j$ and $w_j\ (i = 1,2,\cdots,n, j = 1,2,\cdots,m)$ satisfy Lipschitz condition, that is, there exist constant $F_i > 0,\ H_i > 0,\ G_j > 0,\ W_j > 0$, such that

$$\left|f_i(\xi_1) - f_i(\xi_2)\right| \leq F_i \left|\xi_1 - \xi_2\right|,$$

$$\left|h_i(\xi_1) - h_i(\xi_2)\right| \leq H_i \left|\xi_1 - \xi_2\right|,$$

$$\left|g_j(\xi_1) - g_j(\xi_2)\right| \leq G_j \left|\xi_1 - \xi_2\right|,$$

$$\left|w_j(\xi_1) - w_j(\xi_2)\right| \leq W_j \left|\xi_1 - \xi_2\right|,$$

for any $\xi_1, \xi_2 \in R$.

$(\mathbf{H_2})$:There exist numbers $N_i > 0$ and $M_j > 0$,such that $|f_i(z)| \leq N_i,\ |g_j(z)| \leq M_j$,for all $z \in R$.

($\mathbf{H_3}$) :The activation functions $f_i(z)$,$h_i(z)$,$g_j(z)$ and $w_j(z)$,$(i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$ are continuously differentiable on $z \in R$.

($\mathbf{H_4}$) : The activation functions $f_i(z)$,$h_i(z)$,$g_j(z)$ and $w_j(z)$ are continuously differentiable on z, and there exist $A_i > 0$ and $B_i > 0$, such that

$$\left|\frac{df_i(z)}{dz}\right| \le A_i, \quad \left|\frac{dg_j(z)}{dz}\right| \le B_j.$$

Let

$$u^* = (u_1^*, u_2^*, \cdots, u_n^*)^T, v^* = (v_1^*, v_2^*, \cdots, v_m^*)^T.$$

Definition 2.1 The point $\left(u^{*T}, v^{*T}\right)$ is called an equilibrium point of system (1), if it satisfies the following equations

$$\begin{cases} -a_i u_i^* + \sum\limits_{j=1}^{n} b_{ij} g_j(v_j^*) + \sum\limits_{j=1}^{m}\sum\limits_{l=1}^{m} e_{ijl} g_j(v_j^*) g_l(v_l^*) \\ + \sum\limits_{j=1}^{m} p_{ij} k_{ij} w_j(v_j^*) + I_i = 0, \\ -c_j v_j^* + \sum\limits_{i=1}^{n} d_{ji} f_i(u_i^*) + \sum\limits_{i=1}^{n}\sum\limits_{p=1}^{n} s_{jip} f_i(u_i^*) f_p(u_p^*) \\ + \sum\limits_{i=1}^{n} q_{ji} r_{ji} h_i(u_i^*) + J_i = 0 \end{cases} \tag{6}$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$.

Definition 2.2 Let $\left(u^{*T}, v^{*T}\right)$ be the equilibrium point of system (1), we define the norm

$$\|u_i(t) - u_i^*\|_2^2 = \int_\Omega (u_i(t,x) - u_i^*)^2 \, dx,$$

$$\left\|v_j(t) - v_j^*\right\|_2^2 = \int_\Omega (v_j(t,x) - v_i^*)^2 \, dx,$$

$$\|\phi_u - u^*\|_2 = \sup_{-\infty \le t \le 0} \sum_{i=1}^{n} \|\phi_{u_i}(t) - u_i^*\|_2,$$

$$\|\phi_v - v^*\|_2 = \sup_{-\infty \le t \le 0} \sum_{j=1}^{m} \left\|\phi_{v_j}(t) - v_j^*\right\|_2,$$

$$\|u\|^2 = \sum_{i=1}^{n} |u_i(t,x)|^2, \|v\|^2 = \sum_{j=1}^{m} |v_j(t,x)|^2,$$

where $\phi_u = (\phi_{u_1}, \phi_{u_2}, \cdots, \phi_{u_n})^T$ and $\phi_v = (\phi_{v_1}, \phi_{v_2}, \cdots, \phi_{v_m})^T$ are initial values.

Definition 2.3 The equilibrium point $\left(u^{*T}, v^{*T}\right)$ of system (1) is said to be globally exponentially stable, if there exist constant $\alpha > 0$ and $M \ge 1$ such that

$$\begin{aligned} &\sum_{i=1}^{n} \|u_i(t) - u_i^*\|_2 + \sum_{j=1}^{m} \left\|v_j(t) - v_j^*\right\|_2 \\ &\le M e^{-\alpha t} \left[\|\phi_u - u^*\|_2 + \|\phi_v - v^*\|_2\right] \end{aligned}$$

for all $t \ge 0$, where

$$(u(t,x), v(t,x))^T = (u_1(t,x), u_2(t,x), \cdots, u_n(t,x),$$
$$v_1(t,x), v_2(t,x), \cdots, v_m(t,x))^T$$

is any solution of system (1) with boundary condition (4) and initial condition (5).

Lemma 2.1 [32] If $H(u) \in C^0$,and it satisfies the following conditions

1) $H(u)$is injective on R^n,

2) $\|H(u)\| \to +\infty$,as $\|u\| \to +\infty$,

Then $H(u)$ is a homeomorphism of R^n.

Lemma 2.2[19] If $f_i(x_i)$ are continuously differentiable on $x_i (i = 1, 2, \cdots, n)$,

$$x(t) = (x_1(t), x_2(t), \cdots x_n(t))^T \in R^n,$$

$$x^0(t) = \left(x_1^0(t), x_2^0(t), \cdots x_n^0(t)\right)^T \in R^n,$$

then we have

(A1)

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{p=1}^{n} s_{jip}\left[f_i(x_i) f_p(x_p) - f_i(x_i^0) f_p(x_p^0)\right] \\ &= \sum_{i=1}^{n}\sum_{p=1}^{n} (s_{jip} + s_{jpi}) \frac{\partial f_i(\xi_i)}{\partial x_i}\left(x_i - x_i^0\right) f_p(\xi_p) \end{aligned}$$

Or,

(A2)

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{p=1}^{n} s_{jip}\left[f_i(x_i) f_p(x_p) - f_i(x_i^0) f_p(x_p^0)\right] \\ &= \sum_{i=1}^{n}\sum_{p=1}^{n} (s_{jip} + s_{jpi})\left(f_i(x_i) - f_i(x_i^0)\right) \\ &\cdot f_p\left(x_p^0 + (x_p - x_p^0)\theta\right), \end{aligned}$$

where ξ_p lies between x_p and x_p^0, $p = 1, 2, \cdots, n, 0 < \theta < 1$.

Lemma 2.3 [19] For any

$$x(t) = (x_1(t), x_2(t), \cdots x_n(t))^T \in R^n,$$

$$x^0(t) = \left(x_1^0(t), x_2^0(t), \cdots x_n^0(t)\right)^T \in R^n,$$

we have

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{p=1}^{n} s_{jip}\left[f_i(x_i) f_p(x_p) - f_i\left(x_i^0\right) f_p\left(x_p^0\right)\right] \\ &= \sum_{i=1}^{n}\sum_{p=1}^{n}\left[s_{jip} f_p\left(x_p^0\right) + s_{jpi} f_p(x_p)\right] \\ &\cdot \left(f_i(x_i) - f_i\left(x_i^0\right)\right). \end{aligned}$$

Lemma 2.4 [41] If $f(t,\theta)$ is continuous on $[a,b;\,-\infty,0]$,and $\eta(\theta)$is a nondecreasing bounded variation function on $(-\infty,0]$ and $\int_{-\infty}^{0} d\eta(\theta)=k<\infty$, then

$$\frac{d}{dt}\int_{-\infty}^{0} f(t,\theta)\,d\eta(\theta)=\int_{-\infty}^{0}\frac{d}{dt}f(t,\theta)\,d\eta(\theta).$$

Lemma 2.5 Assume that

$$\begin{aligned}&-a_i+\sum_{j=1}^{m}|q_{ji}|\,r_{ji}H_i\\&+\sum_{j=1}^{m}\left[|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_p\right]F_i<0\\&i=1,2,\cdots,n,\end{aligned}$$

and

$$\begin{aligned}&-c_j+\sum_{i=1}^{n}|p_{ij}|\,k_{ij}W_j\\&+\sum_{i=1}^{n}\left[|b_{ij}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right]G_j<0,\\&j=1,2,\cdots,m,\end{aligned}$$

then there exists $\overline{\mu}>0$ such that

$$\begin{aligned}&(\overline{\mu}-a_i)+\sum_{j=1}^{m}\left[|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_p\right]F_i\\&+\sum_{j=1}^{m}|q_{ji}|\,H_i\int_{-\infty}^{0}e^{-\overline{\mu}\theta}d\sigma_{ji}(\theta)\le 0,\\&i=1,2,\cdots,n,\end{aligned}\tag{7}$$

$$\begin{aligned}&(\overline{\mu}-c_j)+\sum_{i=1}^{n}\left[|b_{ij}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right]G_j\\&+\sum_{i=1}^{n}|q_{ji}|\,W_j\int_{-\infty}^{0}e^{-\overline{\mu}\theta}d\eta_{ij}(\theta)\le 0,\\&j=1,2,\cdots,m.\end{aligned}\tag{8}$$

Proof: Let

$$\begin{aligned}&W(\mu_i)=\\&(\mu_i-a_i)+\sum_{j=1}^{m}\left[|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_p\right]F_i\\&+\sum_{j=1}^{m}|q_{ji}|\,H_i\int_{-\infty}^{0}e^{-\mu_i\theta}d\sigma_{ji}(\theta),\ i=1,2,\cdots,n,\end{aligned}$$

Then,

$$\begin{aligned}&W(0)=\\&-a_i+\sum_{j=1}^{m}\left[|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_p\right]F_i\\&+\sum_{j=1}^{m}|q_{ji}|\,H_ir_{ji}<0,\ i=1,2,\cdots,n,\end{aligned}$$

by Lemma 2.4, we have

$$\begin{aligned}&W'(\mu_i)=1-\sum_{j=1}^{m}|q_{ji}|\,H_i\int_{-\infty}^{0}\theta e^{-\mu_i\theta}d\sigma_{ji}(\theta)>0,\\&i=1,2,\cdots,n.\end{aligned}$$

Since $\lim\limits_{\mu_i\to+\infty}W(\mu_i)=+\infty$, there exists $\mu_i^*>0$ such that $W(\mu_i^*)=0, i=1,2,\ \cdots,n$. Let $\xi=\min\limits_{1\le i\le n}\{\mu_1^*,\mu_2^*,\cdots\mu_n^*\}$, then we have $W(\overline{\mu}_1)\le 0$ for $\overline{\mu}_1\in(0,\xi)$, i.e. $\overline{\mu}_1$ satisfies (7).

Similarly, let

$$\begin{aligned}&Z(\mu_j)=\\&(\mu_j-c_j)+\sum_{i=1}^{n}\left[|b_{ij}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right]G_j\\&+\sum_{i=1}^{n}|q_{ji}|\,W_j\int_{-\infty}^{0}e^{-\mu_j\theta}d\eta_{ij}(\theta),\ j=1,2,\cdots,m.\end{aligned}$$

Then

$$\begin{aligned}&Z(0)=\\&-c_j+\sum_{i=1}^{n}\left[|b_{ij}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right]G_j\\&+\sum_{i=1}^{n}|q_{ji}|\,W_jk_{ij}<0,\ j=1,2,\cdots,m.\end{aligned}$$

By Lemma 2.4, we have $Z'(\mu_j)>0$. Since $\lim\limits_{\mu_j\to+\infty}Z(\mu_j)=+\infty$, so there exists $\delta_j^*>0$ such that $Z(\delta_j^*)=0, j=1,2,\ \cdots,m$. Set $\delta=\min\limits_{1\le j\le m}\{\delta_1^*,\delta_2^*,\cdots\delta_m^*\}$, we have $Z(\overline{\mu}_2)\le 0$ for $\overline{\mu}_2\in(0,\delta)$, i.e. $\overline{\mu}_2$ satisfies (8).

Let $\overline{\mu}=\min\{\overline{\mu}_1,\overline{\mu}_2\}$. Then $W(\overline{\mu})\le 0$ and $Z(\overline{\mu})\le 0$, which means that

$$\begin{aligned}&(\overline{\mu}-a_i)+\sum_{j=1}^{m}\left[|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_p\right]F_i\\&+\sum_{j=1}^{m}|q_{ji}|\,H_i\int_{-\infty}^{0}e^{-\overline{\mu}\theta}d\sigma_{ji}(\theta)\le 0,\\&i=1,2,\cdots,n,\end{aligned}$$

$$\begin{aligned}&(\overline{\mu}-c_j)+\sum_{i=1}^{n}\left[|b_{ij}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right]G_j\\&+\sum_{i=1}^{n}|q_{ji}|\,W_j\int_{-\infty}^{0}e^{-\overline{\mu}\theta}d\eta_{ij}(\theta)\le 0,\\&j=1,2,\cdots,m.\end{aligned}$$

This completes the proof.

3 Main results

3.1 Existence and uniqueness of the equilibrium point

In this section, we will derive some sufficient conditions which ensure the existence, uniqueness and the exponential stability of the equilibrium point for system (1).

Theorem 3.1 under hypotheses $(H_1)-(H_3)$, then the system (1) has a unique equilibrium point if

$$-a_i+\frac{1}{2}\sum_{j=1}^{m}\left[|b_{ji}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right]G_j+\sum_{j=1}^{m}\left[|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_p\right]F_i+\frac{1}{2}\sum_{j=1}^{m}|p_{ij}|\,k_{ij}W_j+\sum_{j=1}^{m}|q_{ji}|\,r_{ji}H_i<0 \quad (9)$$

$$-c_j+\sum_{i=1}^{n}\left[|b_{ji}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right]G_j+\frac{1}{2}\sum_{i=1}^{n}\left[|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_p\right]F_i+\sum_{i=1}^{n}|p_{ij}|\,k_{ij}W_j+\frac{1}{2}\sum_{i=1}^{n}|q_{ji}|\,r_{ji}H_i<0 \quad (10)$$

for $i=1,2,\cdots,n,\ j=1,2,\cdots,m.$

Proof: Let

$$H(u,v)=(H_1(u,v),H_2(u,v),\cdots,H_n(u,v),$$

$$H_{n+1}(u,v),\cdots,H_{n+m}(u,v))^T$$

where

$$H_i(u,v)=-a_iu_i+\sum_{j=1}^{n}b_{ij}g_j(v_j)+\sum_{j=1}^{m}\sum_{l=1}^{m}e_{ijl}g_j(v_j)g_l(v_l)+\sum_{j=1}^{m}p_{ij}k_{ij}\,w_j(v_j)+I_i,\quad i=1,2,\cdots,n$$

$$H_{n+j}(u,v)=-c_jv_j+\sum_{i=1}^{n}d_{ji}f_i(u_i)+\sum_{i=1}^{n}\sum_{p=1}^{n}s_{jip}f_i(u_i)f_p(u_p)+\sum_{i=1}^{n}q_{ji}\,r_{ji}\,h_i(u_i)+J_i,\quad j=1,2,\cdots,m.$$

It is known that the solutions of $H(u,v)=0$ are equilibriums of system (1). If the mapping $H(u,v)$ is a homeomorphism on R^{n+m}, then there exists a unique point (u^*,v^*), such that $H(u^*,v^*)=0$, $i.e.$, system (1) has a unique equilibrium point (u^*,v^*). In the following, we shall prove that $H(u,v)$ is an injective mapping on R^{n+m}.

In fact, if there exist

$$(u,v)=(u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_m)^T,$$

$$(\bar{u},\bar{v})=(\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_n,\bar{v}_1,\bar{v}_2,\cdots,\bar{v}_m)^T\in R^{n+m},$$

such that $H(u,v)=H(\bar{u},\bar{v})$ for $(u,v)\neq(\bar{u},\bar{v})$, then by using (A2) of Lemma 2.2, we have

$$-a_i(u_i-\bar{u}_i)+\sum_{j=1}^{m}b_{ij}\left(g_j(v_j)-g_j(\bar{v}_j)\right)+\sum_{j=1}^{m}\sum_{l=1}^{m}(e_{ijl}+e_{ilj})\left(g_j(v_j)-g_j(\bar{v}_j)\right)\cdot g_l\left(\bar{v}_l+(v_l-\bar{v}_l)\theta\right)+\sum_{j=1}^{m}p_{ij}\,k_{ij}\left[w_j(v_j)-w_j(\bar{v}_j)\right]=0,\ i=1,2,\cdots,n,\ 0<\theta<1. \quad (11)$$

$$-c_j(v_j-\bar{v}_j)+\sum_{i=1}^{n}d_{ji}\left(f_i(u_i)-f_i(\bar{u}_i)\right)+\sum_{i=1}^{n}\sum_{p=1}^{n}(s_{jip}+s_{jpi})\left(f_i(u_i)-f_i(\bar{u}_i)\right)\cdot f_p\left(\bar{u}_p+(u_p-\bar{u}_p)\theta\right)+\sum_{i=1}^{n}q_{ji}\,r_{ji}\left[h_i(u_i)-h_i(\bar{u}_i)\right]=0,\ j=1,2,\cdots,m,\ 0<\theta<1. \quad (12)$$

Multiplying both sides of (11) by $(u_i-\bar{u}_i)$ we have

$$-a_i(u_i-\bar{u}_i)^2+(u_i-\bar{u}_i)\sum_{j=1}^{m}b_{ij}\left(g_j(v_j)-g_j(\bar{v}_j)\right)+(u_i-\bar{u}_i)\sum_{j=1}^{m}\sum_{l=1}^{m}(e_{ijl}+e_{ilj})\left(g_j(v_j)-g_j(\bar{v}_j)\right)\cdot g_l\left(\bar{v}_l+(v_l-\bar{v}_l)\theta\right)+(u_i-\bar{u}_i)\sum_{j=1}^{m}p_{ij}\,k_{ij}\left[w_j(v_j)-w_j(\bar{v}_j)\right]=0. \quad (13)$$

for $i=1,2,\cdots,n,0<\theta<1$, which means

$$-a_i(u_i-\bar{u}_i)^2+|u_i-\bar{u}_i|\sum_{j=1}^{m}|b_{ij}|\,|g_j(v_j)-g_j(\bar{v}_j)|+|u_i-\bar{u}_i|\sum_{j=1}^{m}\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,|g_j(v_j)-g_j(\bar{v}_j)|\,M_l+|u_i-\bar{u}_i|\sum_{j=1}^{m}|p_{ij}|\,k_{ij}\,|w_j(v_j)-w_j(\bar{v}_j)|\geq 0,\quad i=1,2,\cdots,n. \quad (14)$$

Since $|g_j(v_j)-g_j(\bar{v}_j)|\leq G_j\,|v_j-\bar{v}_j|,$

$$|w_j(v_j)-w_j(\bar{v}_j)|\leq W_j\,|v_j-\bar{v}_j|,$$

from (14) we have

$$-a_i(u_i-\bar{u}_i)^2+\sum_{j=1}^{m}|b_{ij}|\,G_j\,|u_i-\bar{u}_i|\,|v_j-\bar{v}_j|+\sum_{j=1}^{m}\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,G_jM_l\,|u_i-\bar{u}_i|\,|v_j-\bar{v}_j|+\sum_{j=1}^{m}|p_{ij}|\,k_{ij}\,W_j\,|u_i-\bar{u}_i|\,|v_j-\bar{v}_j|\geq 0,\quad i=1,2,\cdots,n. \quad (15)$$

Applying the inequality: $a^2+b^2 \geq 2\,|a|\;|b|$ to (15), it follows that

$$-a_i(u_i-\bar{u}_i)^2+\frac{1}{2}\sum_{j=1}^{m}\left(|b_{ij}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right)\cdot G_j\left((u_i-\bar{u}_i)^2+(v_j-\bar{v}_j)^2\right)+\frac{1}{2}\sum_{j=1}^{m}|p_{ij}|\;k_{ij}\,W_j\left((u_i-\bar{u}_i)^2+(v_j-\bar{v}_j)^2\right)\geq 0. \tag{16}$$

Similarly, from (10) we derive

$$-c_j(v_j-\bar{v}_j)^2+\frac{1}{2}\sum_{i=1}^{n}\left(|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|N_p\right)\cdot F_i\left((u_i-\bar{u}_i)^2+(v_j-\bar{v}_j)^2\right)+\frac{1}{2}\sum_{i=1}^{n}|q_{ji}|\;r_{ji}H_i\left((u_i-\bar{u}_i)^2+(v_j-\bar{v}_j)^2\right)\geq 0,\quad j=1,2,\cdots,m. \tag{17}$$

Plus the left hand sides of (16)-(17), and merge the similar items, we can obtain

$$\begin{aligned}&\sum_{i=1}^{n}\left\{-a_i+\frac{1}{2}\sum_{j=1}^{m}\left(|b_{ij}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right)G_j\right.\\&+\frac{1}{2}\sum_{j=1}^{m}\left(|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_p\right)F_i\\&\left.+\frac{1}{2}\sum_{i=1}^{n}|p_{ij}|\;k_{ij}W_j+\frac{1}{2}\sum_{i=1}^{n}|q_{ji}|\;r_{ji}H_i\right\}(u_i-\bar{u}_i)^2\\&+\sum_{j=1}^{m}\left\{-c_j+\frac{1}{2}\sum_{i=1}^{n}\left(|b_{ji}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right)G_j\right.\\&+\frac{1}{2}\sum_{i=1}^{n}\left(|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_p\right)F_i\\&\left.+\frac{1}{2}\sum_{i=1}^{n}|p_{ij}|\;k_{ij}W_j+\frac{1}{2}\sum_{i=1}^{n}|q_{ji}|\;r_{ji}H_i\right\}(v_j-\bar{v}_j)^2\\&\geq 0.\end{aligned} \tag{18}$$

According to (9) and (10), from (18) it is easy to see that $u_i=\bar{u}_i,\ v_j=\bar{v}_j$, for $i=1,2,\cdots,n,\ j=1,2,\cdots,m.$ which contradict $(u,v)\neq(\bar{u},\bar{v})$. So $H(u,v)$ is an inject mapping on R^{n+m}.

Secondly, we prove that $\|H(u,v)\|\to+\infty$ as $\|(u,v)\|\to+\infty$.

Let $\tilde{H}(u,v)=H(u,v)-H(0,0)$

$$=\left(\tilde{H}_1(u,v),\tilde{H}_2(u,v),\tilde{H}_n(u,v),\tilde{H}_{n+1}(u,v),\tilde{H}_{n+2}(u,v),\cdots,\tilde{H}_{n+m}(u,v)\right)^T$$

where

$$\tilde{H}_i(u,v)=-a_iu_i+\sum_{j=1}^{m}b_{ij}\left(g_j(v_j)-g_j(0)\right)+\sum_{j=1}^{m}\sum_{l=1}^{m}(e_{ijl}+e_{ilj})\left(g_j(v_j)-g_j(0)\right)g_l(\theta v_l)+\sum_{j=1}^{m}p_{ij}k_{ij}\left(w_j(v_j)-w_j(0)\right) \tag{19}$$

and

$$\tilde{H}_{n+j}(u,v)=-c_jv_j+\sum_{i=1}^{n}d_{ji}\left(f_i(u_i)-f_i(0)\right)+\sum_{i=1}^{n}\sum_{p=1}^{n}(s_{jip}+s_{jpi})\left(f_i(u_i)-f_i(0)\right)f_p(\theta u_p)+\sum_{i=1}^{n}q_{ji}\,r_{ji}\left(h_i(u_i)-h_i(0)\right) \tag{20}$$

for $i=1,2,\cdots,n,\ j=1,2,\cdots,m,\ 0<\theta<1.$

By (19) and (20), we can find

$$\begin{aligned}&(u,v)^T\tilde{H}(u,v)=\sum_{i=1}^{n}u_i\tilde{H}_i(u,v)+\sum_{j=1}^{m}v_j\tilde{H}_{n+j}(u,v)\\&=\sum_{i=1}^{n}\left\{-a_iu_i^2+\sum_{j=1}^{m}b_{ij}u_i\left(g_j(v_j)-g_j(0)\right)\right.\\&+\sum_{j=1}^{m}\sum_{l=1}^{m}(e_{ijl}+e_{ilj})\left(g_j(v_j)-g_j(0)\right)g_l(\theta\,v_l)\\&\left.+\sum_{j=1}^{m}p_{ij}u_ik_{ij}\left(w_j(v_j)-w_j(0)\right)\right\}\\&+\sum_{j=1}^{m}\left\{-c_jv_j^2+\sum_{i=1}^{n}d_{ji}v_j\left(f_i(u_i)-f_i(0)\right)\right.\\&+\sum_{i=1}^{n}\sum_{p=1}^{n}(s_{jip}+s_{jpi})\left(f_i(u_i)-f_i(0)\right)f_p(\theta\,u_p)\\&\left.+\sum_{i=1}^{n}q_{ji}v_jr_{ji}\left(h_i(u_i)-h_i(0)\right)\right\}\\&\leq\sum_{i=1}^{n}\left\{-a_iu_i^2+\sum_{j=1}^{m}|b_{ij}|\,G_j\,|u_i|\,|v_j|\right.\\&+\sum_{j=1}^{m}\sum_{l}^{m}(e_{ijl}+e_{ilj})G_jM_l\,|u_i|\,|v_j|\\&\left.+\sum_{j=1}^{m}|p_{ij}|\,k_{ij}W_j\,|u_i|\,|v_j|\right\}\\&+\sum_{j=1}^{m}\left\{-c_jv_j^2+\sum_{i=1}^{n}|d_{ji}|\,F_i\,|u_i|\,|v_j|\right.\\&\cdot\sum_{i=1}^{n}\sum_{p=1}^{n}(s_{jip}+s_{jpt})F_iN_p\,|u_i|\,|v_j|\\&\left.+\sum_{i=1}^{n}|q_{ji}|\,r_{ji}H_i\,|u_i|\,|v_j|\right\}\\&\leq\sum_{i=1}^{n}\left\{-a_iu_i^2+\left(\frac{1}{2}\sum_{j=1}^{m}\left(|b_{ij}|+\sum_{j=1}^{m}\sum_{l=1}^{m}(e_{ijl}+e_{ilj})M_l\right)\right.\right.\\&\left.\left.\cdot G_j+\frac{1}{2}\sum_{j=1}^{m}|p_{ij}|\,k_{ij}W_j\right)(u_i^2+v_j^2)\right\}\\&+\sum_{j=1}^{m}\left\{-c_jv_j^2+\left(\frac{1}{2}\sum_{i=1}^{n}\left(|d_{ji}|+\sum_{i=1}^{n}\sum_{p=1}^{n}(s_{jip}+s_{jpi})N_p\right)\right.\right.\\&\left.\left.\cdot F_i+\frac{1}{2}\sum_{i=1}^{n}|q_{ji}|\,r_{ji}H_i\right)(u_i^2+v_j^2)\right\}\\&=\sum_{i=1}^{n}\left\{-a_i+\frac{1}{2}\sum_{j=1}^{m}\left(|b_{ij}|+\sum_{j=1}^{m}\sum_{l=1}^{m}(e_{ijl}+e_{ilj})M_l\right)\right.\\&\left.\cdot G_j+\frac{1}{2}\sum_{j=1}^{m}|p_{ij}|\,k_{ij}W_j+\frac{1}{2}\sum_{i=1}^{n}|q_{ji}|\,r_{ji}H_i\right\}u_i^2\\&+\sum_{j=1}^{m}\left\{-c_j+\frac{1}{2}\sum_{i=1}^{n}\left(|d_{ji}|+\sum_{i=1}^{n}\sum_{p=1}^{n}(s_{jip}+s_{jpi})N_p\right)\right.\end{aligned}$$

$$\cdot F_i + \tfrac{1}{2}\sum_{i=1}^{n} |p_{ij}|\, k_{ij} W_j + \tfrac{1}{2}\sum_{i=1}^{n} |q_{ji}|\, r_{ji} H_i \Big\} v_j^2$$
$$\le -\min_{1\le i\le n}\Big\{ a_i - \tfrac{1}{2}\sum_{j=1}^{m}\Big(|b_{ij}| + \sum_{j=1}^{m}\sum_{l}^{m} |e_{ijl}+e_{ilj}|\, M_l \Big) G_j$$
$$-\tfrac{1}{2}\sum_{j=1}^{m} |p_{ij}|\, k_{ij} W_j - \tfrac{1}{2}\sum_{i=1}^{n} |q_{ji}|\, r_{ji} H_i \Big\} \|u\|^2$$
$$-\min_{1\le j\le m}\Big\{ c_j - \tfrac{1}{2}\sum_{i=1}^{n}\Big(|d_{ji}| + \sum_{i=1}^{n}\sum_{p=1}^{n} |s_{jip}+s_{jpi}|\, N_p \Big) F_i$$
$$-\tfrac{1}{2}\sum_{i=1}^{n} |p_{ij}|\, k_{ij} W_j - \tfrac{1}{2}\sum_{i=1}^{n} |q_{ji}|\, r_{ji} H_i \Big\} \|v\|^2 . \quad (21)$$

Using the Schwartz inequality

$$-X^T Y \le \left|X^T Y\right| \le \left\|X^T\right\| \cdot \|Y\| , \quad (22)$$

where $\|X\|$, $\|Y\|$are the norm of vectors X and Y, respectively, from (18) and (19) we get

$$\|(u,v)\| \cdot \left\|\tilde{H}(u,v)\right\|$$
$$\ge \min_{1\le i\le n}\Big\{ a_i - \tfrac{1}{2}\sum_{j=1}^{m}\Big(|b_{ij}| + \sum_{j=1}^{m}\sum_{l=1}^{m} |e_{ijl}+e_{ilj}|\, M_l \Big)$$
$$\cdot G_j - \tfrac{1}{2}\sum_{j=1}^{m} |p_{ij}|\ k_{ij} W_j - \tfrac{1}{2}\sum_{i=1}^{n} |q_{ji}|\ r_{ji} H_i \Big\} \|u\|^2$$
$$+\min_{1\le j\le m}\Big\{ c_j - \tfrac{1}{2}\sum_{i=1}^{n}\Big(|d_{ji}| + \sum_{i=1}^{n}\sum_{p=1}^{n} |s_{jip}+s_{jpi}|\, N_p \Big)$$
$$\cdot F_i - \tfrac{1}{2}\sum_{j=1}^{m} |p_{ij}|\ k_{ij} W_j - \tfrac{1}{2}\sum_{i=1}^{n} |q_{ji}|\ r_{ji} H_i \Big\} \|v\|^2$$
$$\ge M(\|u\|^2 + \|v\|^2) = M\ \|(u,v)\|^2 ,$$

where

$$M = \min\Big\{$$
$$\min_{1\le i\le n}\Big\{ a_i - \tfrac{1}{2}\sum_{j=1}^{m}\Big(b_{ij} + \sum_{j=1}^{m}\sum_{l=1}^{m} |e_{ijl}+e_{ilj}|\, M_l \Big) G_j$$
$$-\tfrac{1}{2}\sum_{j=1}^{m} |p_{ij}|\ k_{ij} W_j - \tfrac{1}{2}\sum_{i=1}^{n} |q_{ji}|\, r_{ji} H_i \Big\}\ \|u\|^2 ,$$
$$+\min_{1\le j\le m} c_j \Big\{ -\tfrac{1}{2}\sum_{i=1}^{n}\Big(|d_{ji}| + \sum_{i=1}^{n}\sum_{p=1}^{n} |s_{jip}+s_{jpi}|\, N_p \Big)$$
$$\cdot F_i\ -\tfrac{1}{2}\sum_{j=1}^{m} |p_{ij}|\ k_{ij} W_j - \tfrac{1}{2}\sum_{i=1}^{n} |q_{ji}|\, r_{ji} H_i \Big\} \|v\|^2 \Big\} .$$

When $\|(u,v)\| \ne 0$, we have $\left\|\tilde{H}(u,v)\right\| \ge M(u,v)$. Therefore, $\left\|\tilde{H}(u,v)\right\| \to +\infty$ as $\|(u,v)\| \to +\infty$, which implies that $\left\|\tilde{H}(u,v)\right\| \to +\infty$ as $\|(u,v)\| \to +\infty$. From Lemma 2.1 we know that $\|H(u,v)\|$ is a homeomorphism on R^{n+m}. Thus, system (1) has a unique equilibrium point. This completes the proof.

3.2 Global exponential stability of the equilibrium point

Theorem 3.2 Under hypotheses $(H1) - (H3)$, the unique equilibrium point of system (1) is global exponentially stable if (9) and (10) in Theorem 3.1 hold.

Proof: By using Theorem 3.1, system (1) has a unique equilibrium point. In the following we will prove the unique equilibrium point $u^* = (u_1^*, u_2^*, \cdots, u_n^*, v_1^*, v_2^*, \cdots, v_m^*)^T$ is globally exponentially stable.

Let

$y_i(t,x) = u_i(t,x) - u_i^*,\ z_j(t,x) = v_j(t,x) - v_j^*$
$\bar{f}_i\left(y_i(t,x)\right) = f_i\left(y_i(t,x) + u_i^*\right) - f_i\left(u_i^*\right),$
$\bar{g}_i\left(z_j(t,x)\right) = g_j\left(z_j(t,x) + v_j^*\right) - g_j\left(v_j^*\right),$
$\bar{h}_i\left(y_i(t+\theta,x)\right) = h_i\left((y_i(t+\theta,x) + u_i^*)\right) - h_i\left(u_i^*\right),$
$\bar{w}_j(z_j(t+\theta,x)) = w_j(z_j(t+\theta,x)+v_j^*) - w_j(v_j^*),$
for $i = 1,2,\cdots,n,\ j = 1,2,\cdots,m.$

From (1), (4) and $(A2)$ of Lemma 2.2 we derive

$$\frac{\partial y_i(t,x)}{\partial t} = \sum_{k=1}^{r}\frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\Big) - a_i y_i(t,x)$$
$$+ \sum_{j=1}^{m} b_{ij}\bar{g}_j\left(z_j(t,x)\right) + \sum_{j=1}^{m}\sum_{l=1}^{m}\left(e_{ijl}+e_{ilj}\right)\bar{g}_j\left(z_j(t,x)\right)$$
$$\cdot g_l\left(v_l^* + (v_l - v_l^*)\theta\right)$$
$$+ \sum_{j=1}^{m} p_{ij}\Big(\int_{-\infty}^{0} \overline{w}_j\left(z_j(t+\theta,x)\right) d\eta_{ij}(\theta)\Big)$$
$$x \in \Omega_i, \quad (23)$$

$$\frac{\partial z_j(t,x)}{\partial t} = \sum_{k=1}^{r}\frac{\partial}{\partial x_k}\Big(E_{jk}\frac{\partial z_j(t,x)}{\partial x_k}\Big) - c_j z_j(t,x)$$
$$+ \sum_{i=1}^{n} d_{ji}\bar{f}_i\left(y_i(t,x)\right) + \sum_{i=1}^{n}\sum_{p=1}^{n}(s_{jip}+s_{jpi})\bar{f}_i\left(y_i(t,x)\right)$$
$$\cdot f_p\left(u_p^* + (u_p - u_p^*)\theta\right)$$
$$+ \sum_{i=1}^{n} q_{ji}\Big(\int_{-\infty}^{0} \bar{h}_i\left(y_i(t+\theta,x)\right) d\sigma_{ij}(\theta)\Big),$$
$$x \in \Omega_i, \quad (24)$$

for $i = 1,2,\cdots,n,\ j = 1,2,\cdots,m,\ 0 < \theta < 1.$

Multiply both side of (23) by $y_i(t,x)$ and integrate with respect x, we get

$$\tfrac{1}{2}\tfrac{d}{dt}\int_{\Omega_i} y_i(t,x)^2 dx = \sum_{k=1}^{l}\int_{\Omega_i} y_i\frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\Big) dx$$
$$-a_i\int_{\Omega_i} y_i^2(t,x)dx + \sum_{j=1}^{m} b_{ij}\int_{\Omega_i} y_i\bar{g}_j\left(z_j(t,x)\right) dx$$
$$+ \sum_{j=1}^{m}\sum_{l=1}^{m}\left(e_{ijl}+e_{ilj}\right)$$
$$\cdot\int_{\Omega_i} y_i\bar{g}_j\left(z_j(t,x)\right) g_l\left(v_l^* + (v_l - v_l^*)\theta\right) dx$$
$$+ \sum_{j=1}^{m} p_{ij}\int_{\Omega_i} y_i\Big(\int_{-\infty}^{0}\bar{w}_j\left(z_j(t+\theta,x)\right) d\eta_{ij}(\theta)\Big) dx \quad (25)$$

for $i = 1,2,\cdots,n.$

It follows from the boundary condition that

$$\begin{aligned}
&\sum_{k=1}^{r}\int_{\Omega_i} y_i\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\right)dx\\
&=\int_{\Omega_i} y_i\nabla\left(D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\right)_{k=1}^{r}dx\\
&=\int_{\Omega_i}\nabla\cdot y_i\left(D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\right)_{k=1}^{r}dx\\
&-\int_{\Omega_i}\left(D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\right)_{k=1}^{r}\cdot\nabla y_i dx\\
&=\int_{\Omega_i}\left(y_i D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\right)_{k=1}^{r}ds\\
&-\sum_{k=1}^{r}\int_{\Omega_i} D_{ik}\left(\frac{\partial y_i(t,x)}{\partial x_k}\right)^2dx\\
&=-\sum_{k=1}^{r}\int_{\Omega_i} D_{ik}\left(\frac{\partial y_i(t,x)}{\partial x_k}\right)^2dx,
\end{aligned}$$

where $\left(D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\right)_{k=1}^{r}$

$$=\left(D_{i1}\frac{\partial y_i(t,x)}{\partial x_1},D_{i2}\frac{\partial y_i(t,x)}{\partial x_2},\cdots,D_{il}\frac{\partial y_i(t,x)}{\partial x_r}\right)^T.$$

By (9),(10), assumptions $(H1),(H2)$ and Hölder inequality, from (25) we obtain

$$\begin{aligned}
&\frac{d\|y_i\|_2^2}{dt}\leq -2\sum_{k=1}^{r}\int_{\Omega_i}D_{ik}\left(\frac{\partial y_i(t,x)}{\partial x_k}\right)^2dx-2a_i\|y_i\|_2^2\\
&+2\sum_{j=1}^{m}|b_{ij}|\int_{\Omega_i}|y_i|\cdot|\bar{g}_j(z_j(t,x))|dx\\
&+2\sum_{j=1}^{m}\sum_{l=1}^{m}|(e_{ijl}+e_{ilj})|\cdot M_l\int_{\Omega_i}|y_i|\cdot|\bar{g}_j(z_j(t,x))|dx\\
&+2\sum_{j=1}^{m}|p_{ij}|\int_{\Omega_i}|y_i|\left|\int_{-\infty}^{0}\bar{w}_j(z_j(t+\theta,x))\,d\eta_{ij}(\theta)\right|dx,\\
&\leq -2a_i\|y_i\|_2^2+2\sum_{j=1}^{m}|b_{ij}|G_j\|y_i\|_2\cdot\|z_j\|_2\\
&+2\sum_{j=1}^{m}\sum_{l=1}^{m}|(e_{ijl}+e_{ilj})|\cdot M_lG_j\|y_i\|_2\cdot\|z_j\|_2\\
&+2\sum_{j=1}^{m}|p_{ij}|W_j\cdot\int_{\Omega_i}|y_i|\cdot\left|\int_{-\infty}^{0}z_j(t+\theta,x)\,d\eta_{ij}(\theta)\right|dx.
\end{aligned}\tag{26}$$

By Hölder inequality, we have

$$\begin{aligned}
&2\sum_{j=1}^{m}|p_{ij}|W_j\cdot\int_{\Omega_i}|y_i|\cdot\left|\int_{-\infty}^{0}z_j(t+\theta,x)d\eta_{ij}(\theta)\right|dx\\
&\leq 2\sum_{j=1}^{m}|p_{ij}|W_j\int_{-\infty}^{0}\left[\int_{\Omega_i}|z_j(t+\theta,x)|\,|y_i(t,x)|\,dx\right]\\
&\cdot d\eta_{ij}(\theta),\\
&\leq 2\sum_{j=1}^{m}|p_{ij}|W_j\int_{-\infty}^{0}\|z_j(t+\theta)\|_2\cdot\|y_i\|_2\cdot d\eta_{ij}(\theta).
\end{aligned}\tag{27}$$

By (27), from (26) we obtain

$$\begin{aligned}
&\frac{d\|y_i\|_2}{dt}\leq -a_i\|y_i\|_2+\sum_{j=1}^{m}\left(|b_{ij}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|M_l\right)\\
&\cdot G_j\|z_i\|_2+\sum_{j=1}^{m}|p_{ij}|W_j\int_{-\infty}^{0}\|z_j(t+\theta)\|_2d\eta_{ij}(\theta)
\end{aligned}\tag{28}$$

for $i=1,2,\cdots,n$.

Multiply both sides of (24) by $z_j(t,x)$, similarly, we also get

$$\begin{aligned}
&\frac{d\|z_j\|_2}{dt}\leq -c_j\|z_j\|_2\\
&+\sum_{i=1}^{n}\left(|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|N_p\right)F_i\|y_i\|_2\\
&+\sum_{i=1}^{n}|q_{ji}|H_i\int_{-\infty}^{0}\|y_i(t+\theta)\|_2d\sigma_{ji}(\theta).
\end{aligned}\tag{29}$$

Consider the following Lyapunov functional

$$\begin{aligned}
&V(t)=\sum_{i=1}^{n}e^{\overline{\mu}\,t}\|y_i\|_2+\sum_{i=1}^{n}\sum_{j=1}^{m}|p_{ij}|W_j\\
&\cdot\left(\int_{-\infty}^{0}\int_{t+\theta}^{t}e^{\overline{\mu}\,(s-\theta)}\|z_j(s)\|_2\,dsd\eta_{ij}(\theta)\right)\\
&+\sum_{j=1}^{m}e^{\overline{\mu}\,t}\|z_j\|_2+\sum_{j=1}^{m}\sum_{i=1}^{n}|q_{ji}|H_i\\
&\cdot\left(\int_{-\infty}^{0}\int_{t+\theta}^{t}e^{\bar{\mu}(s-\theta)}\|y_i(s)\|_2\,dsd\sigma_{ij}(\theta)\right),
\end{aligned}\tag{30}$$

where $\bar{\mu}>0$ is given by Lemma 2.5.

$$\begin{aligned}
&D^+V(t)=\sum_{i=1}^{n}D^+\|y_i\|_2\,e^{\overline{\mu}\,t}+\sum_{i=1}^{n}\|y_i\|_2\,\overline{\mu}\,e^{\overline{\mu}\,t}\\
&+\sum_{i=1}^{n}\sum_{j=1}^{m}|p_{ij}|W_j\left[\int_{-\infty}^{0}e^{\overline{\mu}(t-\theta)}\|z_j\|_2d\eta_{ij}(\theta)\right.\\
&\left.-\int_{-\infty}^{0}e^{\overline{\mu}\,t}\|z_j(t+\theta)\|_2d\eta_{ij}(\theta)\right]\\
&+\sum_{j=1}^{m}D^+\|z_j\|_2\,e^{\overline{\mu}\,t}+\sum_{j=1}^{m}\|z_j\|_2\,\overline{\mu}\,e^{\overline{\mu}\,t}\\
&+\sum_{j=1}^{m}\sum_{i=1}^{n}|q_{ji}|H_i\left[\int_{-\infty}^{0}e^{\overline{\mu}(t-\theta)}\|y_i\|_2d\sigma_{ji}(\theta)\right.\\
&\left.-\int_{-\infty}^{0}e^{\overline{\mu}\,t}\|y_i(t+\theta)\|_2d\sigma_{ji}(\theta)\right],\\
&\leq e^{\overline{\mu}\,t}\sum_{i=1}^{n}\left\{-a_i\|y_i\|_2\right.\\
&+\sum_{j=1}^{m}\left(|b_{ij}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|M_l\right)G_j\|z_j\|_2\\
&+\sum_{j=1}^{m}|p_{ij}|W_j\int_{-\infty}^{0}\|z_j(t+\theta)\|_2d\eta_{ij}(\theta)+\overline{\mu}\,\|y_i\|_2\\
&+\sum_{j=1}^{m}|p_{ij}|W_j\int_{-\infty}^{0}e^{-\overline{\mu}\theta}\|z_j\|_2\,d\eta_{ij}(\theta)\\
&\left.-\sum_{j=1}^{m}|p_{ij}|W_j\int_{-\infty}^{0}\|z_j(t+\theta)\|_2d\eta_{ij}(\theta)\right\}\\
&+e^{\overline{\mu}\,t}\sum_{j=1}^{m}\left\{-c_j\|z_j\|_2\right.\\
&+\sum_{i=1}^{n}\left(|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|N_p\right)F_i\|y_i\|_2\\
&+\sum_{i=1}^{n}|q_{ji}|H_i\int_{-\infty}^{0}\|y_i(t+\theta)\|_2d\sigma_{ji}(\theta)+\overline{\mu}\,\|z_j\|_2\\
&+\sum_{i=1}^{n}|q_{ji}|H_i\int_{-\infty}^{0}e^{-\overline{\mu}\theta}\|y_i\|_2\,d\sigma_{ji}(\theta)\\
&\left.-\sum_{i=1}^{n}|q_{ji}|H_i\int_{-\infty}^{0}\|y_i(t+\theta)\|_2d\sigma_{ji}(\theta)\right\}
\end{aligned}$$

$$\begin{aligned}&\le e^{\overline{\mu}\,t}\sum_{i=1}^{n}\left\{(\overline{\mu}-a_i)+\sum_{j=1}^{m}\left[|d_{ji}|\right.\right.\\&\left.+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_t\right]F_i\\&\left.+\sum_{j=1}^{m}|q_{ji}|\,H_i\int_{-\infty}^{0}e^{-\overline{\mu}\theta}d\sigma_{ji}(\theta)\right\}\cdot\|y_i\|_2\\&+e^{\overline{\mu}\,t}\sum_{j=1}^{m}\left\{(\overline{\mu}-c_j)+\sum_{i=1}^{n}\left[|b_{ij}|\right.\right.\\&\left.+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right]G_j\\&\left.+\sum_{i=1}^{n}|q_{ji}|\,W_j\int_{-\infty}^{0}e^{-\overline{\mu}\theta}d\eta_{ij}(\theta)\right\}\cdot\|z_j\|_2\,.\end{aligned}\tag{31}$$

By Lemma 2.5, from (31) we can find $D^+V(t)\le 0$, and so $V(t)\le V(0)$, for all $t\ge 0$.

From (30) it is easy to find that

$$V(t)\ge\sum_{i=1}^{n}e^{\overline{\mu}t}\|y_i\|_2+\sum_{j=1}^{m}e^{\overline{\mu}t}\|z_j\|_2,\ t\ge 0\tag{32}$$

and

$$\begin{aligned}V(0)&=\sum_{i=1}^{n}\left[\|y_i(0)\|_2+\sum_{j=1}^{m}|p_{ij}|\,W_j\right.\\&\left.\cdot\int_{-\infty}^{0}\left(\int_{\theta}^{0}e^{\overline{\mu}(s-\theta)}\|z_j(s)\|_2\,ds\right)d\eta_{ij}(\theta)\right.\\&+\sum_{j=1}^{m}\left[\|z_j(0)\|_2+\sum_{i=1}^{n}|q_{ji}|\,H_j\right.\\&\left.\cdot\int_{-\infty}^{0}\left(\int_{\theta}^{0}e^{\overline{\mu}(s-\theta)}\|y_i(s)\|_2\,ds\right)\right]d\sigma_{ij}(\theta),\\&=\sum_{i=1}^{n}\|\varphi_{u_i}(0,x)-u_i^*\|_2+\sum_{j=1}^{m}|p_{ij}|\,W_j\\&\cdot\int_{-\infty}^{0}\left(\int_{\theta}^{0}e^{\overline{\mu}(s-\theta)}\left\|\varphi_{v_j}(s)-v_j^*\right\|_2ds\right)d\eta_{ij}(\theta)\\&+\sum_{j=1}^{m}\left\|\varphi_{v_j}(0)-v_j^*\right\|_2+\sum_{i=1}^{n}|q_{ji}|\,H_j\\&\cdot\int_{-\infty}^{0}\left(\int_{\theta}^{0}e^{\overline{\mu}(s-\theta)}\|\varphi_{u_i}(s)-u_i^*\|_2\,ds\right)d\sigma_{ij}(\theta),\\&\le\sum_{i=1}^{n}\left[1+\sum_{j=1}^{m}\frac{|q_{ji}|H_i}{\overline{\mu}}\int_{-\infty}^{0}\left(e^{-\overline{\mu}\,\theta}-1\right)d\sigma_{ji}(\theta)\right]\\&\cdot\|\varphi_u-u^*\|\\&+\sum_{j=1}^{m}\left[1+\sum_{i=1}^{n}\frac{|p_{ij}|W_j}{\overline{\mu}}\int_{-\infty}^{0}\left(e^{-\overline{\mu}\,\theta}-1\right)d\eta_{ij}(\theta)\right]\\&\cdot\|\varphi_v-v^*\|\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left[\sum_{i=1}^{n}\|y_i\|_2+\sum_{j=1}^{m}\|z_j\|_2\right]e^{\overline{\mu}t}\le V(t)\le V(0),\\&\le\sum_{i=1}^{n}\left[1+\sum_{j=1}^{m}\frac{|q_{ji}|H_i}{\overline{\mu}}\int_{-\infty}^{0}\left(e^{-\overline{\mu}\,\theta}-1\right)d\sigma_{ji}(\theta)\right]\\&\cdot\|\varphi_u-u^*\|\\&+\sum_{j=1}^{m}\left[1+\sum_{i=1}^{n}\frac{|p_{ij}|W_j}{\overline{\mu}}\int_{-\infty}^{0}\left(e^{-\overline{\mu}\,\theta}-1\right)d\eta_{ij}(\theta)\right]\\&\cdot\|\varphi_v-v^*\|\,,\\&\le\overline{M}\left[\|\varphi_u-u^*\|_2+\|\varphi_v-v^*\|_2\right],\ t\ge 0\end{aligned}$$

where

$$\begin{aligned}\overline{M}=\max&\left\{\right.\\&\sum_{i=1}^{n}\left[1+\sum_{j=1}^{m}\frac{|q_{ji}|H_i}{\overline{\mu}}\int_{-\infty}^{0}\left(e^{-\overline{\mu}\,\theta}-1\right)d\sigma_{ji}(\theta)\right],\\&\sum_{j=1}^{m}\left[1+\sum_{i=1}^{n}\frac{|p_{ij}|W_j}{\overline{\mu}}\int_{-\infty}^{0}\left(e^{-\overline{\mu}\,\theta}-1\right)d\eta_{ij}(\theta)\right]\\&\left.\right\}\ge 1.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{i=1}^{n}\|u_i-u_i^*\|_2+\sum_{j=1}^{m}\left\|v_j-v_j^*\right\|_2\\&\le\bar{M}e^{-\bar{\mu}t}\left[\|\phi_u-u^*\|_2+\|\phi_v-v^*\|_2\right],\ t\ge 0\end{aligned}$$

By Definition 2.3, the equilibrium point (u^*,v^*) of system (1) is globally exponentially stable.

On the other hand, by using $(A1)$ of Lemma 2.2 and Lemma 2.3 respectively, the proof is similar to that proving Theorem 3.1 and Theorem 3.2, we may obtain the following two results (Theorem 3.3 and Theorem 3.4).

Theorem 3.3 Under hypothesis $(H1)-(H2)$ and $(H4)$, the equilibrium point of system (1) is unique and globally exponentially stable if

$$\begin{aligned}&-a_i+\frac{1}{2}\sum_{j=1}^{m}\left(|b_{ji}|\,G_j+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_lB_j\right)\\&+\sum_{j=1}^{m}\left(|d_{ji}|\,F_i+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_pA_i\right)\\&+\frac{1}{2}\sum_{j=1}^{m}|p_{ij}|\,k_{ij}W_j+\sum_{j=1}^{m}|q_{ji}|\,r_{ji}H_i<0,\end{aligned}\tag{33}$$

$$\begin{aligned}&-c_j+\sum_{i=1}^{n}\left(|b_{ji}|\,G_j+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_lB_j\right)\\&+\frac{1}{2}\sum_{i=1}^{n}\left(|d_{ji}|\,F_i+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_pA_i\right)\\&+\sum_{i=1}^{n}|p_{ij}|\,k_{ij}W_j+\frac{1}{2}\sum_{i=1}^{n}|q_{ji}|\,r_{ji}H_i<0\end{aligned}\tag{34}$$

for $i=1,2,\cdots,n,\ j=1,2,\cdots,m.$

Theorem 3.4 Under hypotheses $(H1)-(H2)$, the equilibrium point of system (1) is unique and globally exponential stable if

$$\begin{aligned}&-a_i+\frac{1}{2}\sum_{j=1}^{m}\left(|b_{ji}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right)G_j\\&+\sum_{j=1}^{m}\left(|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_p\right)F_i\\&+\frac{1}{2}\sum_{j=1}^{m}|p_{ij}|\,k_{ij}W_j+\sum_{j=1}^{m}|q_{ji}|\,r_{ji}H_i<0,\end{aligned}\tag{35}$$

$$-c_j+\sum_{i=1}^{n}\left(|b_{ji}|+\sum_{l=1}^{m}|e_{ijl}+e_{ilj}|\,M_l\right)G_j$$
$$+\frac{1}{2}\sum_{i=1}^{n}\left(|d_{ji}|+\sum_{p=1}^{n}|s_{jip}+s_{jpi}|\,N_p\right)F_i \qquad (36)$$
$$+\sum_{i=1}^{n}|p_{ij}|\,k_{ij}W_j+\frac{1}{2}\sum_{i=1}^{n}|q_{ji}|\,r_{ji}H_i<0$$

for $i=1,2,\cdots,n,\ j=1,2,\cdots,m.$

Remark 1. When $D_{ik}=E_{ik}\equiv 0$, $e_{ijl}=s_{ijp}\equiv 0$, $v_j(t,x)=u_j(t,x)$, $m=n$ and $c_j=d_{ji}=q_{ij}=J_j\equiv 0$ in (1), then system ((1)) is greatly similar to the system (1) in [41]. On the other hand, when $e_{ijl}=s_{ijp}\equiv 0$, $v_j(x,t)=u_j(x,t)$, $m=n$ and $c_j=d_{ji}=q_{ij}=J_j\equiv 0$ in (1), then system (1) is greatly similar to the following system

$$\frac{\partial u_i(t,x)}{\partial t}=\sum_{k=1}^{r}\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial u_i(t,x)}{\partial x_k}\right)-a_iu_i(t,x)$$
$$+\sum_{j=1}^{m}b_{ij}g_j\left(u_j(t,x)\right)$$
$$+\sum_{j=1}^{m}p_{ij}W_j\left(\int_{-\infty}^{0}d\eta_{ij}(\theta)u_j(t+\theta,x)\right)+I_i \qquad (37)$$

which has been considered in [39]-[40]. From this point, our model and results are more general.

Remark 2. Introduce high order term into system (1), the study of global exponential stability becomes difficult for reaction-diffusion cellular networks with S-type distributed time delays. Lemma 2.2 and Lemma 2.3 are greatly useful to resolve the difficulty.

Remark 3. Because of the infiniteness of S-type distributed delays, it is difficult to consider the global exponential stability for system (1). In this paper , Lemma 2.5 is very useful .

Remark 4. Theorem 3.1-3.2, Theorem 3.3 and Theorem 3.4 are developed under different assumptions and use of various lemmas. They provide different sufficient conditions ensuring the equilibrium point of system (1) to be unique and globally exponentially stable. Therefore, we can select suitable theorems for a high-order BAM neural networks with reaction diffusion terms and S-type distributed delays to determine its globally exponential stability.

4 Examples

In this section, we give two examples for showing our results.

Example 4.1 Consider the following second-order BAM neural networks with reaction-diffusion terms and S-type distributed delays

$$\begin{cases}\frac{\partial u_i(t,x)}{\partial t}=\frac{\partial}{\partial x}\left(D_i\frac{\partial u_i(t,x)}{\partial x}\right)-a_iu_i(t,x)\\ +\sum_{j=1}^{2}b_{ij}g_j\left(v_j(t,x)\right)\\ +\sum_{j=1}^{2}\sum_{l=1}^{2}e_{ijl}g_j\left(v_j(t,x)\right)g_l\left(v_l(t,x)\right)\\ +\sum_{j=1}^{2}p_{ij}\ \int_{-\infty}^{0}w_j\left(v_j(t+\theta,x)\right)d\eta_{ij}(\theta)+I_i,\\ \frac{\partial v_j(t,x)}{\partial t}=\frac{\partial}{\partial x}\left(E_j\frac{\partial v_j(t,x)}{\partial x}\right)-c_jv_j(t,x)\\ +\sum_{i=1}^{2}d_{ji}f_i\left(u_i(t,x)\right)\\ +\sum_{i=1}^{2}\sum_{p=1}^{2}s_{jip}f_i\left(u_i(t,x)\right)f_p\left(u_p(t,x)\right)\\ +\sum_{i=1}^{2}q_{ji}\ \int_{-\infty}^{0}h_i\left(u_i(t+\theta,x)\right)d\sigma_{ji}(\theta)+J_i\end{cases} \qquad (38)$$

For $i=1,2,\ j=1,2,$ let

$$f_1(r)=f_2(r)=g_1(r)=g_2(r)=\sin 2r,$$

$$h_1(r)=h_2(r)=w_1(r)=w_2(r)=r.$$

Since

$$\begin{array}{l}|f_1(r_1)-f_2(r_2)|=|g_1(r_1)-g_2(r_2)|\\ \le 4\,|r_1-r_2|\,,|f_i(r)|=|g_i(r)|\le 1\end{array},$$

$$\begin{array}{l}|h_1(r_1)-h_2(r_2)|=|w_1(r_1)-w_2(r_2)|\\ =|r_1-r_2|\,,\\ \left|\frac{df_i(r)}{dr}\right|=\left|\frac{dg_i(r)}{dr}\right|=2\,|\cos 2r|\le 2\end{array},$$

we select $F_i=G_i=4,N_i=M_i=1,A_i=B_i=2,H_i=W_i=1,i=1,2.$ Take

$a_1=23$	$a_2=17$	$b_{11}=2$	$b_{12}=\frac{3}{2}$
$b_{21}=-2$	$b_{22}=-1$	$c_1=26$	$c_2=24$
$d_{11}=1$	$d_{12}=-\frac{1}{3}$	$d_{21}=\frac{1}{2}$	$d_{22}=1$
$e_{111}=1$	$e_{112}=\frac{1}{2}$	$e_{121}=-\frac{1}{2}$	$e_{122}=1$
$e_{211}=\frac{1}{4}$	$e_{212}=1$	$e_{221}=-1$	$e_{222}=\frac{1}{2}$
$s_{111}=-\frac{1}{4}$	$s_{112}=\frac{1}{3}$	$s_{121}=-\frac{1}{3}$	$s_{122}=\frac{1}{3}$
$s_{211}=\frac{1}{2}$	$s_{212}=\frac{1}{4}$	$s_{221}=-\frac{1}{4}$	$s_{222}=-\frac{1}{2}$
$p_{11}=2$	$p_{12}=1$	$p_{21}=\frac{3}{4}$	$p_{22}=\frac{5}{4}$
$q_{11}=1$	$q_{12}=-\frac{1}{2}$	$q_{21}=-\frac{3}{4}$	$q_{22}=\frac{5}{4}$
$I_1=0$	$I_2=0$	$J_1=0$	$J_2=0$

$\eta_{ij}(\theta)$,$\sigma_{ji}(\theta)$ $(i,j=1,2)$ are non-decreasing functions with bounded that satisfy
$k_{11}=\int_{-\infty}^{0}d\eta_{11}(\theta)=\frac{1}{4},k_{12}=\int_{-\infty}^{0}d\eta_{12}(\theta)=\frac{1}{2},$
$k_{21}=\int_{-\infty}^{0}d\eta_{21}(\theta)=\frac{1}{3},k_{22}=\int_{-\infty}^{0}d\eta_{22}(\theta)=\frac{1}{5},$
$r_{11}=\int_{-\infty}^{0}d\sigma_{11}(\theta)=\frac{1}{4},r_{12}=\int_{-\infty}^{0}d\sigma_{12}(\theta)=\frac{1}{4},$

$r_{21} = \int_{-\infty}^{0} d\sigma_{21}(\theta) = \frac{1}{3}, r_{22} = \int_{-\infty}^{0} d\sigma_{22}(\theta) = \frac{1}{5}$
By calculation, we have the following results

$$\begin{aligned}
&-a_i + \tfrac{1}{2}\sum_{j=1}^{2}\left(|b_{ji}|\,G_j + \sum_{l=1}^{2}|e_{ijl}+e_{ilj}|\,M_l B_j\right)\\
&+\sum_{j=1}^{2}\left(|d_{ji}|\,F_i + \sum_{p=1}^{2}|s_{jip}+s_{jpi}|\,N_p A_i\right)\\
&+\tfrac{1}{2}\sum_{j=1}^{2}|p_{ij}|\,k_{ij}W_j + \sum_{j=1}^{2}|q_{ji}|\,r_{ji}H_i < 0,\\
&-c_j + \sum_{i=1}^{2}\left(|b_{ji}|\,G_j + \sum_{l=1}^{2}|e_{ijl}+e_{ilj}|\,M_l B_j\right)\\
&+\tfrac{1}{2}\sum_{i=1}^{2}\left(|d_{ji}|\,F_i + \sum_{p=1}^{2}|s_{jip}+s_{jpi}|\,N_p A_i\right)\\
&+\sum_{i=1}^{2}|p_{ij}|\,k_{ij}W_j + \tfrac{1}{2}\sum_{i=1}^{2}|q_{ji}|\,r_{ji}H_i < 0
\end{aligned}$$

for $i = 1, 2,\ j = 1, 2$.

It follows from Theorem 3.3 that this system (37) in Example 4.1 has one equilibrium point, which is globally exponentially stable.

Example 4.2. For the network described by system (37), let

$$f_1(r) = f_2(r) = g_1(r) = g_2(r) = \sin\frac{r}{2},$$

$$h_1(r) = h_2(r) = w_1(r) = w_2(r) = r,$$

$$\begin{aligned}&\text{since } |f_1(r_1) - f_2(r_2)| = |g_1(r_1) - g_2(r_2)|\\ &\le 4\,|r_1 - r_2|\,, |f_i(r)| = |g_i(r)| \le 1?\end{aligned}$$

$$|h_i(r_1) - h_i(r_2)| = |w_i(r_1) - w_i(r_2)| = |r_1 - r_2|\,,$$

$$\left|\frac{df_i(r)}{dr}\right| = \left|\frac{dg_i(r)}{dr}\right| = \frac{1}{2}\left|\cos\frac{r}{2}\right| \le \frac{1}{2}.$$

We select $F_i = G_i = \frac{1}{4},\ N_i = M_i = 1,\ A_i = B_i = \frac{1}{2},\ H_i = W_i = 1,\ i = 1, 2$. Take $a_1 = 3,\ a_2 = 2,\ c_1 = 3, c_2 = 4,$ the other parameters are the same as that in Example 4.1,

By calculation, it is easy to find that the parameters in Example 4.2 satisfy (9) and (10), therefore, by Theorem 3.1-3.2, the system of Example 4.2 has one equilibrium point, which is globally exponentially stable. Simulation results with 100 random initial points are depicted in Fig.4.1- 4.2.

Remarks 5 By simple calculation, it follows that the conditions (9) and(10) of Theorem 3.1-3.2 don't satisfy for the system in Example 4.1, while for the system in Example 4.2, conditions (33) and (34) of Theorem 3.3 don't satisfy. Therefore, Theorem 3.3 is suitable for the globally exponential stability of system in Example 4.1, but Theorem 3.1-3.2 aren't. Theorem 3.1-3.2 are suitable for the globally exponential stability of system in Example 4.2, but Theorem 3.3 isn't. The above two examples show that all the Theorem 3.1-3.4 in this paper have advantages in different problems and applications.

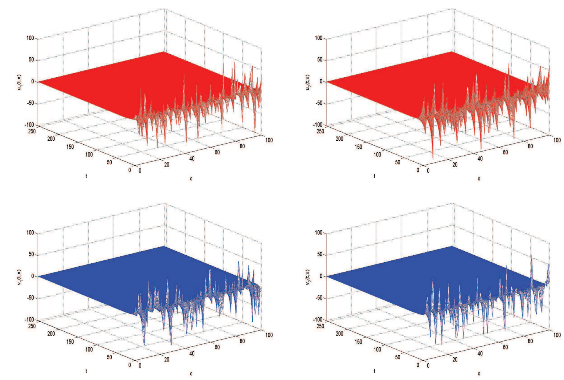

Figure 1: Three-dimensional view of (t, x, u_i),(t, x, v_i) in Example 4.2 when$I_i = J_i = 0 \quad (i = 1, 2)$

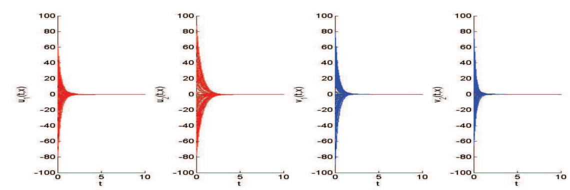

Figure 2: Transient response of state variable $u_i(t, x), v_i(t, x)$ in Example 4.2 when$I_i = J_i = 0 \quad (i = 1, 2)$

5 Conclusion

Under different assumption conditions, four theorems are given to ensure the existence, uniqueness and the global exponential stability of the high-order BAM neural networks with reaction–diffusion terms and S-type distributed delays by constructing a suitable Lyapunov functional, utilizing differential mean value theorem and some analytical techniques. The given algebra conditions are easily verifiable and useful in theories and applications. Finally, two examples and simulation are given to show the effectiveness of the results.

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