Robust portfolio selection problem for an insurer with exponential utility preference

Hui Zhao
Tianjin University
School of Science
Tianjin, 300072
P.R. China
zhaohui_tju@hotmail.com

Ximin Rong
Tianjin University
School of Science
Tianjin, 300072
P.R. China
rongximin@tju.edu.cn

Jiling Cao
Auckland University of Technology
School of Computing and Mathematical Sciences
Private Bag 92006, 1142 Auckland
New Zealand
jiling.cao@aut.ac.nz

Abstract: In this paper, we consider the robust portfolio selection problem for an insurer in the sense of maximizing the exponential utility of his wealth. This special robust investment problem, where underwriting results and a risk-free asset are considered, differs from ordinary robust portfolio selection problems. The insurer has the option of investing in a risk-free asset and multiple risky assets whose returns are described by the factor model. The rate of underwriting return is also assumed to be correlated with returns of risky assets. When the parameters are perturbed in a joint uncertainty set, the robust investment problem for an insurer is established and this problem is reformulated and solved as a cone programming problem. Finally, some computational results are given for raw market data.

Key–Words: Robust optimization, Investment for insurers, Joint uncertainty set, Underwriting result, Cone programming, Factor model

1 Introduction

Since insurers are permitted to invest in financial markets in practice, there has been much attention to the optimal investment problem for insurers recently. It is well-known that the overall result of an insurer taking investment into consideration is analyzed into two parts: the underwriting result and the investment income. Underwriting results cause a large amount of claim and then bring lots of risks, thus an insurer needs to take the underwriting risk into account. As considering the underwriting risk, the investment problem for an insurer is similar to, but not the same as, the ordinary portfolio selection problem. The portfolio selection problem for insurers has been studied for a long time. Kahane and Nye [1], Krous [2], Lambert and Hofflander [3] tackled this problem via using the portfolio theory developed by Markowitz [4]. Briys [5] studied investment behavior of insurers by analyzing the expected utility.

In the Markowitz portfolio selection model, the optimal portfolio can be identified by solving a convex quadratic program. However, this mean-variance model is not successful in practice in spite of its theoretical elegance. One of the main reasons is that the optimal portfolios are often very sensitive to perturbations in the parameters of the problem. Since there are some statistical errors when estimating the market parameters, the result of the subsequent optimization is not very reliable.

In order to alleviate the sensitivity of optimal portfolios to the inputs, many researches apply the robust optimization to model portfolio selection problems. In particular, this approach has been successful in optimization with uncertain parameters. Generally speaking, the parameters of robust optimization problems are uncertain in bounded sets, and optimization problems are solved under the worst case behavior of these uncertain parameters. This robust optimization framework was introduced and studied in [6], [7], [8], [9], [10], and has been used in a wide spectrum of domains since late 1990s, see [11] for a survey. Regarding portfolio selections, the major contributions have come in 2000s, see [12] for details. Goldfarb and Iyengar [13] introduced a robust factor model for asset returns and parameterize the uncertainty sets from the market data through statistical procedures. They showed that the robust optimization problems corresponding to their uncertain sets can be reformulated as second-order cone programs. El Ghaoui et al. [14] defined the worst-case Value-at-Risk (VaR) for given partial information on the returns' distribution, and they showed that the robust problems of optimizing the worst-case VaR can be cast as semidefinite programs. Zhu and Fukushima [15] investigated the minimization of worst-case conditional Value-at-Risk (CVaR) and present its application to robust portfolio optimization.

Determining the structure of uncertainty sets is essential in formulating and solving robust portfolio selection problems. Goldfarb and Iyengar [13] proposed a box uncertainty set and an ellipsoidal set for different parameters, respectively. In [16] (see also [17]), two box-type uncertainty sets were proposed for the mean vector and the covariance matrix of the asset returns, respectively. But in most of the mentioned works, the uncertainty sets of the parameters are separable. Lu [18] pointed out the common drawbacks of these separable uncertainty structures and proposed a joint ellipsoidal uncertainty set for his model parameters to overcome the disadvantages of the afore mentioned models. He also showed that the joint uncertainty set could be constructed as a confidence region for any desired confidence level through a statistical procedure. With this uncertainty set, the robust maximum risk-adjusted return portfolio selection problem presented in [18] is reformulated and solved as a cone programming problem.

In this paper, we apply the robust optimization to study the portfolio selection problem for a general insurer and choose the joint uncertainty set for parameters. In [13] and [18], there is no risk-free asset. However, a risk-free asset is an important component for the safety of the investment of an insurer. Therefore, in this paper we consider a model with a risk-free asset. Moreover, the underwriting rate of return is regarded as a random variable and assumed to be correlated with returns of the risky assets. These factors make our model more complicated than the ordinary robust portfolio selection problem. Consequently, numerical results show that the robust optimal portfolio for an insurer is different from that for a general investor and an insurer is much more conservative than a general investor.

This paper is organized as follows. Section 2 describes the robust portfolio selection problem for an insurer. In Section 3, our robust portfolio selection problem with joint uncertainty structure is reformulated as a cone programming problem. Some computational results for real market data are presented in Section 4. Section 5 contains some conclusions.

2 Robust investment model for an insurer

Suppose that an insurer can invest in a discrete-time market with n risky assets (hereinafter called 'stocks') and a risk-free asset (hereinafter called 'bond'). The rates of returns on stocks are described by the factor model which is introduced by Goldfarb and Iyengar [13]. The vector of the rates of returns on stocks over a single market period is denoted by $\mathbf{r}_0 \in \mathbb{R}^n$. The rates of returns on the stocks in different market periods are assumed to be independent. The single period rate of return $\mathbf{r}_0$ is assumed to be a random vector given by

$$\mathbf{r}_0 = \boldsymbol{\mu} + \mathbf{V}^{\mathrm{T}}\mathbf{f} + \boldsymbol{\epsilon}, \tag{1}$$

where $\boldsymbol{\mu} \in \mathbb{R}^n$ is the vector of the mean rates of returns, $\mathbf{f} \sim N(\mathbf{0}, \mathbf{F}) \in \mathbb{R}^m$ denotes the rates of returns of the m factors that drive the market, $\mathbf{V} \in \mathbb{R}^{m\times n}$ represents the factor loading matrix of the n stocks, and $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \mathbf{D}) \in \mathbb{R}^n$ is the vector of the residual rates of return. Here $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes that $\mathbf{x}$ is a multivariate normal random variable with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. In addition, we assume that the covariance matrix $\mathbf{D} = \mathrm{diag}(\mathbf{d}) \succeq \mathbf{0}$, where $\mathrm{diag}(\mathbf{d})$ denotes a diagonal matrix with the vector $\mathbf{d}$ along the diagonal, and the vector $\boldsymbol{\epsilon}$ which is related to the residual returns is independent of the vector $\mathbf{f}$. Thus, $\mathbf{r}_0 \sim N(\boldsymbol{\mu}, \mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V} + \mathbf{D})$.

Goldfarb and Iyengar [13] have developed a robust counterpart for the afore mentioned factor model. In their work, even though $\mathbf{F}$ and $\mathbf{D}$ are perturbed in some uncertainty structures, they are usually assumed to be known and fixed in computation. Thus, for the convenience of presentation, we only assume the uncertainty structures for $\boldsymbol{\mu}$ and $\mathbf{V}$ in this paper.

Suppose that the collected premium is P and r is the underwriting rate of return. Then the underwriting return of the insurer is rP. Let g be the investable proportion of the premium, which is fixed and decided by the government. Then the insurer's amount of wealth for investment is gP[1]. The insurer is allowed to invest in those stocks as well as in the bond. An investment plan can be expressed in terms of the proportion invested in each asset. Let ϕ_i be the percentage invested in stock i for $i = 1, \ldots, n$ and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)^{\mathrm{T}} \in \mathbb{R}^n$. Then $1 - \mathbf{1}_n^{\mathrm{T}}\boldsymbol{\phi}$ represents the proportion invested in the bond, where $\mathbf{1_n} \in \mathbb{R}^n$ is the all-one vector. In terms of the security of an insurer's investment, it is required that certain amount of wealth should be invested in the bond and thus $\mathbf{1_n^{\mathrm{T}}}\boldsymbol{\phi} < 1$. In addition, short sales are not allowed throughout this paper. Therefore, the set of all admissible trading strategies is denoted by

$$\Phi = \{\boldsymbol{\phi} : 0 < \mathbf{1_n^{\mathrm{T}}}\boldsymbol{\phi} < 1, \boldsymbol{\phi} \geq \mathbf{0}\}. \tag{2}$$

Corresponding to an admissible trading strategy $\boldsymbol{\phi}$, the total profit of the insurer (underwriting result and

[1] In this paper we assume that initial endowment equals zero as in [5]. It does not change the spirit of the model and allows easier derivations.

the investment income) at the end of one single period is:

$$\pi = rP + gP[\mathbf{r}_0^{\mathrm{T}}\boldsymbol{\phi} + (1 - \mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\phi})r_f], \tag{3}$$

where $\mathbf{r}_0$ is given in (1) and the constant r_f denotes the risk-free rate. Suppose that the insurer has a utility function $U(x)$ of the total profit. The aim of the insurer is to maximize the expected utility $\mathrm{E}[U(\pi)]$, i.e.,

$$\max_{\boldsymbol{\phi}\in\Phi} \mathrm{E}[U(\pi)]. \tag{4}$$

In this paper, we consider the common used CARA utility function $U(x) = -\frac{1}{\beta}\exp(-\beta x)$ with $\beta > 0$ and the investment problem of this paper is

$$\max_{\boldsymbol{\phi}\in\Phi} \mathrm{E}\left[-\frac{1}{\beta}\exp\left(-\beta\pi\right)\right]. \tag{5}$$

By (1), on the portfolio $\boldsymbol{\phi}$, we have

$$\begin{aligned} \mathbf{r}_0^{\mathrm{T}}\boldsymbol{\phi} &= \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi} + \mathbf{f}^{\mathrm{T}}\mathbf{V}\boldsymbol{\phi} + \boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\phi} \\ &\sim N(\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi}, \boldsymbol{\phi}^{\mathrm{T}}(\mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V} + \mathbf{D})\boldsymbol{\phi}). \end{aligned} \tag{6}$$

Denote by $\mathrm{Cov}(r, r_i)$ the covariance of r and r_i (the ith component of $\mathbf{r}_0$) for $i = 1, ..., n$, we derive

$$\begin{aligned} \mathrm{Cov}(r, r_i) &= \mathrm{Cov}(r, \mu_i + (\mathbf{V}^{\mathrm{T}}\mathbf{f})_i + \epsilon_i) \\ &= \sum_{j=1}^{m} V_{ji}\mathrm{Cov}(r, f_j) + \mathrm{Cov}(r, \epsilon_i) \end{aligned} \tag{7}$$

from (1), where μ_i, $(\mathbf{V}^{\mathrm{T}}\mathbf{f})_i$, ϵ_i represent the ith components of $\boldsymbol{\mu}$, $\mathbf{V}^{\mathrm{T}}\mathbf{f}$ and $\boldsymbol{\epsilon}$ respectively, V_{ji} denotes the element of $\mathbf{V}$ in the jth row and ith column, and $\mathrm{Cov}(X, Y)$ denotes the covariance of random variables X and Y. Define $\boldsymbol{\rho} = (\mathrm{Cov}(r, r_1), \ldots, \mathrm{Cov}(r, r_n))^{\mathrm{T}}$, $\mathbf{Cov}_{(r,\mathbf{f})} = (\mathrm{Cov}(r, f_1), \ldots, \mathrm{Cov}(r, f_m))^{\mathrm{T}}$ and $\mathbf{Cov}_{(r,\boldsymbol{\epsilon})} = (\mathrm{Cov}(r, \epsilon_1), \ldots, \mathrm{Cov}(r, \epsilon_n))^{\mathrm{T}}$. With the help of (7), we have

$$\boldsymbol{\rho} = \mathbf{V}^{\mathrm{T}}\mathbf{Cov}_{(r,\mathbf{f})} + \mathbf{Cov}_{(r,\boldsymbol{\epsilon})}. \tag{8}$$

Thus the covariance of r and the rate of return on the stock portfolio $\mathbf{r}_0^{\mathrm{T}}\boldsymbol{\phi}$ is $\boldsymbol{\rho}^{\mathrm{T}}\boldsymbol{\phi}$.

According to the above results, we rewrite the objective function of problem (5). Substituting (3) into (5) yields

$$\begin{aligned} &\mathrm{E}\left[-\frac{1}{\beta}\exp(-\beta\pi)\right] \\ &= -\frac{1}{\beta}\mathrm{E}\left\{\exp\left[-\beta P(r + g\mathbf{r}_0^{\mathrm{T}}\boldsymbol{\phi})\right]\right. \\ &\left.\cdot\exp\left(-\beta r_f gP + \beta r_f gP\mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\phi}\right)\right\} \\ &= -\frac{1}{\beta}\exp\left(-\beta r_f gP + \beta r_f gP\mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\phi}\right) \\ &\cdot\mathrm{E}\left\{\exp\left[-\beta P(r + g\mathbf{r}_0^{\mathrm{T}}\boldsymbol{\phi})\right]\right\}. \end{aligned} \tag{9}$$

It is reasonable to assume that $(r, \mathbf{r}_0^{\mathrm{T}}\boldsymbol{\phi})$ has the bivariate normal distribution. Thus, $r + g\mathbf{r}_0^{\mathrm{T}}\boldsymbol{\phi}$ is a normal random variable. Following from (6), we have

$$\begin{aligned} &\mathrm{E}(r + g\mathbf{r}_0^{\mathrm{T}}\boldsymbol{\phi}) = \mathrm{E}(r) + g\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi}, \\ &\mathrm{Var}(r + g\mathbf{r}_0^{\mathrm{T}}\boldsymbol{\phi}) \\ &= \mathrm{Var}(r) + g^2[\boldsymbol{\phi}^{\mathrm{T}}(\mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V} + \mathbf{D})\boldsymbol{\phi}] + 2g\boldsymbol{\rho}^{\mathrm{T}}\boldsymbol{\phi}. \end{aligned} \tag{10}$$

The result of the moment generating function of normal random variables together with (10) implies

$$\begin{aligned} &\mathrm{E}\left\{\exp\left[-\beta P(r + g\mathbf{r}_0^{\mathrm{T}}\boldsymbol{\phi})\right]\right\} \\ = \;&\exp\left\{-\beta P\left[\mathrm{E}(r) + g\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi}\right] + \frac{\beta^2P^2}{2}[\mathrm{Var}(r)\right. \\ &\left.+g^2\boldsymbol{\phi}^{\mathrm{T}}(\mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V} + \mathbf{D})\boldsymbol{\phi} + 2g\boldsymbol{\rho}^{\mathrm{T}}\boldsymbol{\phi}]\right\}. \end{aligned} \tag{11}$$

Substituting (11) into (9) yields

$$\begin{aligned} &\mathrm{E}\left[-\frac{1}{\beta}\exp(-\beta\pi)\right] \\ &= -\frac{1}{\beta}\exp\left\{-\beta r_f gP + \beta r_f gP\mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\phi} - \beta P\mathrm{E}(r)\right. \\ &-\beta gP\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi} + \frac{\beta^2P^2}{2}\mathrm{Var}(r) + \frac{\beta^2g^2P^2}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V}\boldsymbol{\phi} \\ &\left.+\frac{\beta^2g^2P^2}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{D}\boldsymbol{\phi} + \beta^2gP^2\boldsymbol{\rho}^{\mathrm{T}}\boldsymbol{\phi}\right\}. \end{aligned} \tag{12}$$

Since $-\frac{1}{\beta} < 0$ and $\exp(x)$ is strictly increasing, then problem (5) is equivalent to

$$\begin{aligned} &\min_{\boldsymbol{\phi}\in\Phi}\left(-\beta r_f gP + \beta r_f gP\mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\phi} - \beta P\mathrm{E}(r)\right. \\ &-\beta gP\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi} + \frac{\beta^2P^2}{2}\mathrm{Var}(r) + \frac{\beta^2g^2P^2}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V}\boldsymbol{\phi} \\ &\left.+\frac{\beta^2g^2P^2}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{D}\boldsymbol{\phi} + \beta^2gP^2\boldsymbol{\rho}^{\mathrm{T}}\boldsymbol{\phi}\right). \end{aligned} \tag{13}$$

Furthermore, (13) reduces to

$$\begin{aligned} \max_{\boldsymbol{\phi}\in\Phi} \;&\left(\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi} - r_f\mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\phi} - \frac{\beta gP}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V}\boldsymbol{\phi}\right. \\ &\left.-\frac{\beta gP}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{D}\boldsymbol{\phi} - \beta P\boldsymbol{\rho}^{\mathrm{T}}\boldsymbol{\phi}\right). \end{aligned} \tag{14}$$

Substituting (8) into (14), we can reformulate (14) as

$$\begin{aligned} &\max_{\boldsymbol{\phi}\in\Phi}\left(\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi} - r_f\mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\phi} - \frac{\beta gP}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V}\boldsymbol{\phi}\right. \\ &\left.-\frac{\beta gP}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{D}\boldsymbol{\phi} - \beta P\mathbf{Cov}_{(r,\mathbf{f})}^{\mathrm{T}}\mathbf{V}\boldsymbol{\phi} - \beta P\mathbf{Cov}_{(r,\boldsymbol{\epsilon})}^{\mathrm{T}}\boldsymbol{\phi}\right). \end{aligned}$$

Considering the uncertainty of $\boldsymbol{\mu}$ and $\mathbf{V}$, we formulate the robust portfolio selection problem for an insurer, i.e.,

$$\max_{\boldsymbol{\phi}\in\Phi} \left\{ \min_{(\boldsymbol{\mu},\mathbf{V})\in S} \left\{ \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi} - \frac{\beta g P}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V}\boldsymbol{\phi} -\beta P \mathbf{Cov}^{\mathrm{T}}_{(r,\mathbf{f})}\mathbf{V}\boldsymbol{\phi} \right\} - \frac{\beta g P}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{D}\boldsymbol{\phi} - r_f \mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\phi} -\beta P \mathbf{Cov}^{\mathrm{T}}_{(r,\boldsymbol{\epsilon})}\boldsymbol{\phi} \right\}, \tag{15}$$

where S is the uncertainty set for $(\boldsymbol{\mu}, \mathbf{V})$.

The insurer's robust investment model (15) considers the bond and the covariance of the underwriting rate of return and the rate of return on stock portfolio, which is different from the general robust portfolio selection problems. Moreover, we see that the optimal portfolio is independent of the mean and the variance of the underwriting rate of return, but relates to the vector of covariance $\boldsymbol{\rho}$. Furthermore, (15) tells us that in contrast to classical robust portfolio selection problems, for a robust portfolio model of an insurer, we should consider the worst case of $\beta P \mathbf{Cov}^{\mathrm{T}}_{(r,\mathbf{f})}\mathbf{V}\boldsymbol{\phi}$.

In the following, we construct the joint ellipsoidal uncertainty structure for $(\boldsymbol{\mu}, \mathbf{V})$. Suppose that the market data consists of rates of returns on stocks $\{\mathbf{r}_0^t : t = 1, \ldots, p\}$ and the corresponding rates of factor returns $\{\mathbf{f}^t : t = 1, \ldots, p\}$ for p trading periods. Let $\mathbf{y}_i = (r_i^1, r_i^2, \ldots, r_i^p)^{\mathrm{T}}, i = 1, \ldots, n$ be the rates of returns on stock i over p periods, and

$$\mathbf{B} := (\mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^p) \in \mathbb{R}^{m\times p}, \quad \mathbf{A} = (\mathbf{1_p}, \mathbf{B}^{\mathrm{T}})$$
$$\mathbf{x}_i := (\mu_i, V_{1i}, V_{2i}, ..., V_{mi})^{\mathrm{T}},$$

where $\mathbf{1_p} \in \mathbb{R}^p$ is the all-one vector. Furthermore, let $\overline{\mathbf{x}}_i$ be the least-square estimate of $\mathbf{x}_i$. Because the columns of $\mathbf{A}$ are linearly independent and $p \gg m$ in practice, it is assumed that $\mathbf{A}$ has the full column rank. It follows that

$$\overline{\mathbf{x}}_i = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{y}_i.$$

Define

$$s_i^2 = \frac{\|\mathbf{y}_i - \mathbf{A}\overline{\mathbf{x}}_i\|^2}{p-m-1}, \tag{16}$$

where $\|\cdot\|$ denotes the Euclidean norm. Let $\tilde{c}(\omega)$ be a critical value for a standard normal variable Z, i.e., $P(Z \le \tilde{c}(\omega)) = \omega$ and

$$\mu_F = \frac{p-m-1}{p-m-3},$$
$$\sigma_F = \sqrt{\frac{2(p-m-1)^2(p-2)}{(m+1)(p-m-3)^2(p-m-5)}}. \tag{17}$$

Define $c(\omega)$ by

$$c(\omega) = (m+1)\left(\tilde{c}(\omega)\sigma_F\sqrt{n} + n\mu_F\right).$$

Then the joint ellipsoidal uncertainty set with ω-confidence level is in the form of

$$S_{\boldsymbol{\mu},\mathbf{V}} \equiv S_{\boldsymbol{\mu},\mathbf{V}}(\omega) = \left\{ (\boldsymbol{\mu},\mathbf{V}) : \sum_{i=1}^{n} \frac{(\overline{\mathbf{x}}_i - \mathbf{x}_i)^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{A})(\overline{\mathbf{x}}_i - \mathbf{x}_i)}{s_i^2} \le c(\omega) \right\}, \tag{18}$$

and when n is relatively large (saying a couple dozen), we have

$$P\left(\sum_{i=1}^{n} \frac{(\overline{\mathbf{x}}_i - \mathbf{x}_i)^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{A})(\overline{\mathbf{x}}_i - \mathbf{x}_i)}{s_i^2} \le c(\omega) \right) \approx \omega, \tag{19}$$

(see Section 3 of [18] for more details). Since $(\boldsymbol{\mu}, \mathbf{V}) \in S_{\boldsymbol{\mu},\mathbf{V}}$, we rewrite (15) as

$$\begin{aligned} \max_{\boldsymbol{\phi}} \quad & \left\{ \min_{(\boldsymbol{\mu},\mathbf{V})\in S_{\boldsymbol{\mu},\mathbf{V}}} \left\{ \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi} - \frac{\beta g P}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V}\boldsymbol{\phi} -\beta P \mathbf{Cov}^{\mathrm{T}}_{(r,\mathbf{f})}\mathbf{V}\boldsymbol{\phi} \right\} - \frac{\beta g P}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{D}\boldsymbol{\phi} - r_f \mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\phi} - \beta P \mathbf{Cov}^{\mathrm{T}}_{(r,\boldsymbol{\epsilon})}\boldsymbol{\phi} \right\} \\ s.t. \quad & \boldsymbol{\phi} \in \Phi. \end{aligned} \tag{20}$$

3 Solution to robust investment problem for an insurer

In this section, the robust investment problem (20) for an insurer is reformulated as a cone programming problem, and then we can use the optimization solvers (e.g., SeDuMi proposed by Sturm [19] and SDPT3 by Tütüncü et al. [20]) to solve it. By introducing auxiliary variables ν and t, problem (20) can be reformulated as

$$\begin{aligned} \max_{\boldsymbol{\phi},\nu,t} \quad & \left(\nu - \frac{\beta g P}{2} t - r_f \mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\phi} - \beta P \mathbf{Cov}^{\mathrm{T}}_{(r,\boldsymbol{\epsilon})}\boldsymbol{\phi} \right) \\ s.t. \quad & \min_{(\boldsymbol{\mu},\mathbf{V})\in S_{\boldsymbol{\mu},\mathbf{V}}} \left\{ \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi} - \frac{\beta g P}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V}\boldsymbol{\phi} -\beta P \mathbf{Cov}^{\mathrm{T}}_{(r,\mathbf{f})}\mathbf{V}\boldsymbol{\phi} \right\} \ge \nu \\ & \boldsymbol{\phi}^{\mathrm{T}}\mathbf{D}\boldsymbol{\phi} \le t \\ & \boldsymbol{\phi} \in \Phi. \end{aligned} \tag{21}$$

Next we aim to reformulate the inequality

$$\min_{(\boldsymbol{\mu},\mathbf{V})\in S_{\boldsymbol{\mu},\mathbf{V}}} \left\{\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi} - \frac{\beta g P}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V}\boldsymbol{\phi} - \beta P \mathbf{Cov}^{\mathrm{T}}_{(r,\mathbf{f})}\mathbf{V}\boldsymbol{\phi}\right\} \geq \nu \quad (22)$$

as linear matrix inequalities. First, we give two lemmas that will be used subsequently.

Lemma 1. *($\mathscr{S}$-procedure) Let $F_i(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}_i\mathbf{x} + 2\mathbf{b}_i^{\mathrm{T}}\mathbf{x} + c_i$ be quadratic functions of $\mathbf{x} \in \mathbb{R}^n$ for $i = 0, ..., p$. Then $F_0(\mathbf{x}) \leq 0$ for all $\mathbf{x}$ such that $F_i(\mathbf{x}) \leq 0, i = 1, ..., p$, if there exists $\tau_i \geq 0$ such that*

$$\sum_{i=1}^{p}\tau_i\begin{pmatrix} c_i & \mathbf{b}_i^{\mathrm{T}} \\ \mathbf{b}_i & \mathbf{A}_i \end{pmatrix} - \begin{pmatrix} c_0 & \mathbf{b}_0^{\mathrm{T}} \\ \mathbf{b}_0 & \mathbf{A}_0 \end{pmatrix} \succeq \mathbf{0}.$$

Moreover, if $p = 1$ then the converse holds whenever there exists $\mathbf{x}_0$ such that $F_1(\mathbf{x}_0) < 0$.

For the $\mathscr{S}$-procedure and its applications, one can refer to [21]. Before proceeding further, we need to recall the definition of the standard Kronecker product from [22], denoted by $\otimes$.

Definition 2. *Given $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $\mathbf{B} = [b_{ij}] \in \mathbb{R}^{p\times q}$, the standard Kronecker product $\mathbf{A}\otimes\mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ is defined by*

$$\mathbf{A}\otimes\mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{pmatrix}.$$

In the following lemma, we list one simple property of the standard Kronecker product from [23].

Lemma 3. *If $\mathbf{H} \succeq \mathbf{0}$ and $\mathbf{K} \succeq \mathbf{0}$, then $\mathbf{H}\otimes\mathbf{K} \succeq \mathbf{0}$.*

The following lemma reformulates constraint (22) as a collection of linear matrix inequalities (LMIs).

Lemma 4. *Let $S_{\boldsymbol{\mu},\mathbf{V}}$ be defined in (18) for $\omega > 0$. Then, the inequality (22) is equivalent to the following LMIs*

$$\begin{pmatrix} \tau\mathbf{M} - \beta g P\mathbf{S}\otimes\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{F}\end{pmatrix} & \tau\mathbf{h} + \mathbf{q} \\ \tau\mathbf{h}^{\mathrm{T}} + \mathbf{q}^{\mathrm{T}} & \tau\eta - 2\nu \end{pmatrix} \succeq \mathbf{0},$$
$$\begin{pmatrix} 1 & \boldsymbol{\phi}^{\mathrm{T}} \\ \boldsymbol{\phi} & \mathbf{S} \end{pmatrix} \succeq \mathbf{0}, \quad \tau \geq 0, \quad (23)$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{\mathbf{A}^{\mathrm{T}}\mathbf{A}}{s_1^2} & & \\ & \ddots & \\ & & \frac{\mathbf{A}^{\mathrm{T}}\mathbf{A}}{s_n^2} \end{pmatrix} \in \mathbb{R}^{[(m+1)n]\times[(m+1)n]}, \quad (24)$$

$$\eta = \sum_{i=1}^{n}\overline{\mathbf{x}}_i^{\mathrm{T}}\left(\frac{\mathbf{A}^{\mathrm{T}}\mathbf{A}}{s_i^2}\right)\overline{\mathbf{x}}_i - c(\omega), \quad (25)$$

$$\mathbf{h} = \begin{pmatrix} -\frac{\mathbf{A}^{\mathrm{T}}\mathbf{A}\overline{\mathbf{x}}_1}{s_1^2} \\ \vdots \\ -\frac{\mathbf{A}^{\mathrm{T}}\mathbf{A}\overline{\mathbf{x}}_n}{s_n^2} \end{pmatrix} \in \mathbb{R}^{(m+1)n}, \quad (26)$$

$$\mathbf{q} = \begin{pmatrix} \phi_1 \\ -\beta P\phi_1\mathbf{Cov}_{(r,\mathbf{f})} \\ \vdots \\ \phi_n \\ -\beta P\phi_n\mathbf{Cov}_{(r,\mathbf{f})} \end{pmatrix} \in \mathbb{R}^{(m+1)n}. \quad (27)$$

Proof: Given any $(\nu, \beta, g, P, \boldsymbol{\phi}) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$, we define

$$H(\boldsymbol{\mu},\mathbf{V}) = -\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\phi} + \frac{\beta g P}{2}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{F}\mathbf{V}\boldsymbol{\phi} + \beta P\mathbf{Cov}^{\mathrm{T}}_{(r,\mathbf{f})}\mathbf{V}\boldsymbol{\phi} + \nu.$$

As $\mathbf{x}_i = (\mu_i, V_{1i}, V_{2i}, \ldots, V_{mi})^{\mathrm{T}}$, $i = 1, \ldots, n$, we view $H(\boldsymbol{\mu},\mathbf{V})$ as a function of $\mathbf{x} = (\mathbf{x}_1^{\mathrm{T}}, \ldots, \mathbf{x}_n^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{(m+1)n}$. Then we easily see that

$$\frac{\partial H}{\partial \mathbf{x}_i} = \begin{pmatrix} -\phi_i \\ \beta g P\phi_i\mathbf{F}\mathbf{V}\boldsymbol{\phi} + \beta P\phi_i\mathbf{Cov}_{(r,\mathbf{f})} \end{pmatrix},$$
$$\frac{\partial^2 H}{\partial \mathbf{x}_i\partial \mathbf{x}_j} = \begin{pmatrix} 0 & 0 \\ 0 & \beta g P\phi_i\phi_j\mathbf{F} \end{pmatrix}, i, j = 1, \ldots, n. \quad (28)$$

Using (28) and performing the Taylor series expansion for $H(\boldsymbol{\mu},\mathbf{V})$ at $\mathbf{x} = \mathbf{0}$, we obtain

$$H(\boldsymbol{\mu},\mathbf{V}) = \frac{1}{2}\sum_{i,j=1}^{n}\mathbf{x}_i^{\mathrm{T}}\begin{pmatrix} 0 & 0 \\ 0 & \beta g P\phi_i\phi_j\mathbf{F} \end{pmatrix}\mathbf{x}_j + \sum_{i=1}^{n}\begin{pmatrix} -\phi_i \\ \beta P\phi_i\mathbf{Cov}_{(r,\mathbf{f})} \end{pmatrix}^{\mathrm{T}}\mathbf{x}_i + \nu. \quad (29)$$

Since $\mathbf{A}$ has the full column rank, from (19), we get

$$\omega > 0 \Rightarrow c(\omega) > 0. \quad (30)$$

In view of (18), $S_{\boldsymbol{\mu},\mathbf{V}}$ can be written as

$$S_{\boldsymbol{\mu},\mathbf{V}} = \Bigg\{(\boldsymbol{\mu},\mathbf{V}) : \sum_{i=1}^{n}\mathbf{x}_i^{\mathrm{T}}\left(\frac{\mathbf{A}^{\mathrm{T}}\mathbf{A}}{s_i^2}\right)\mathbf{x}_i -2\sum_{i=1}^{n}\left(\frac{\mathbf{A}^{\mathrm{T}}\mathbf{A}\overline{\mathbf{x}}_i}{s_i^2}\right)^{\mathrm{T}}\mathbf{x}_i + \sum_{i=1}^{n}\overline{\mathbf{x}}_i^{\mathrm{T}}\left(\frac{\mathbf{A}^{\mathrm{T}}\mathbf{A}}{s_i^2}\right)\overline{\mathbf{x}}_i -c(\omega) \le 0\Bigg\}. \tag{31}$$

For $\omega > 0$, (30) implies that $\mathbf{x} = \overline{\mathbf{x}}$ strictly satisfies the inequality given in (31). According to (29), (31) and Lemma 1, we conclude that $H(\boldsymbol{\mu},\mathbf{V}) \le 0$ for all $(\boldsymbol{\mu},\mathbf{V}) \in S_{\boldsymbol{\mu},\mathbf{V}}$ if and only if there exists a $\tau \ge 0$ such that

$$\tau\begin{pmatrix} \mathbf{M} & \mathbf{h} \\ \mathbf{h}^{\mathrm{T}} & \eta \end{pmatrix} - \begin{pmatrix} \mathbf{K} & -\mathbf{q} \\ -\mathbf{q}^{\mathrm{T}} & 2\nu \end{pmatrix} \succeq \mathbf{0}, \tag{32}$$

where $\mathbf{M}$, η, $\mathbf{h}$ and $\mathbf{q}$ are defined in (24), (25), (26) and (27) respectively, and $\mathbf{K}$ is given by

$$\mathbf{K} = (\mathbf{K}_{ij}) \in \mathbb{R}^{[(m+1)n]\times[(m+1)n]},$$
$$\mathbf{K}_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & \beta g P \phi_i \phi_j \mathbf{F} \end{pmatrix} \in \mathbb{R}^{(m+1)\times(m+1)},$$
$$i, j = 1, \ldots, n.$$

In terms of Definition 2, we have

$$\mathbf{K} = \beta g P(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}) \otimes \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{F} \end{pmatrix}.$$

This together with Lemma 3 and the fact $\mathbf{F} \succeq 0$ shows that (32) holds if and only if there exists a $\tau \ge 0$ satisfying,

$$\begin{pmatrix} \tau\mathbf{M} - \beta g P\mathbf{S} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{F} \end{pmatrix} & \tau\mathbf{h} + \mathbf{q} \\ \tau\mathbf{h}^{\mathrm{T}} + \mathbf{q}^{\mathrm{T}} & \tau\eta - 2\nu \end{pmatrix} \succeq \mathbf{0},$$
$$\mathbf{S} \succeq \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}.$$

Using the Schur Complement Lemma, $\mathbf{S} \succeq \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}$ can be written as

$$\begin{pmatrix} 1 & \boldsymbol{\phi}^{\mathrm{T}} \\ \boldsymbol{\phi} & \mathbf{S} \end{pmatrix} \succeq \mathbf{0}.$$

Hence, it follows that $H(\boldsymbol{\mu},\mathbf{V}) \le 0$ for all $(\boldsymbol{\mu},\mathbf{V}) \in S_{\boldsymbol{\mu},\mathbf{V}}$ if and only if (23) holds. It is easy to find that the inequality (22) holds if and only if $H(\boldsymbol{\mu},\mathbf{V}) \le 0$ for all $(\boldsymbol{\mu},\mathbf{V}) \in S_{\boldsymbol{\mu},\mathbf{V}}$. Then the proof is completed. □

In the following theorem, the robust investment problem (20) is reformulated as a cone programming problem.

Theorem 5. *Let $S_{\boldsymbol{\mu},\mathbf{V}}$ be given by (18) for $\omega > 0$. Then the robust investment problem (20) for an insurer is equivalent to the following cone programming problem,*

$$\max_{\boldsymbol{\phi},\nu,t,\tau,\mathbf{S}} \nu - \frac{\beta g P}{2}t - r_f\mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\phi} - \beta P\mathbf{Cov}_{(r,\boldsymbol{\epsilon})}^{\mathrm{T}}\boldsymbol{\phi}$$
$$s.t. \quad \begin{pmatrix} \tau\mathbf{M} - \beta g P\mathbf{S} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{F} \end{pmatrix} & \tau\mathbf{h} + \mathbf{q} \\ \tau\mathbf{h}^{\mathrm{T}} + \mathbf{q}^{\mathrm{T}} & \tau\eta - 2\nu \end{pmatrix} \succeq \mathbf{0}$$
$$\begin{pmatrix} 1 & \boldsymbol{\phi}^{\mathrm{T}} \\ \boldsymbol{\phi} & \mathbf{S} \end{pmatrix} \succeq \mathbf{0}, \quad \begin{pmatrix} 1+t \\ 1-t \\ 2\mathbf{D}^{\frac{1}{2}}\boldsymbol{\phi} \end{pmatrix} \in \mathcal{L}^{n+2}$$
$$0 < \mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\boldsymbol{\phi} < 1, \; \boldsymbol{\phi} \ge \mathbf{0}, \tau \ge 0, \tag{33}$$

where $\mathbf{M}$, η, $\mathbf{h}$ and $\mathbf{q}$ are defined in (24), (25), (26) and (27) respectively, $\mathbf{1}_{\mathbf{n}} \in \mathbb{R}^n$ is the all-one vector, $\mathbf{D}^{\frac{1}{2}}\mathbf{D}^{\frac{1}{2}} = \mathbf{D}$ and $\mathcal{L}^n$ denotes the n-dimensional second-order cone given by

$$\mathcal{L}^n = \left\{\mathbf{z} \in \mathbb{R}^n : z_1 \ge \sqrt{\sum_{i=2}^{n} z_i^2}\right\}.$$

Proof: Since $\mathbf{D} = \mathrm{diag}(\mathbf{d}) \succeq \mathbf{0}$, the constraints $\boldsymbol{\phi}^{\mathrm{T}}\mathbf{D}\boldsymbol{\phi} \le t$ can be transformed into a second-order cone constraint as follows:

$$\boldsymbol{\phi}^{\mathrm{T}}\mathbf{D}\boldsymbol{\phi} \le t \Leftrightarrow \quad 4\boldsymbol{\phi}^{\mathrm{T}}\mathbf{D}^{\frac{1}{2}}\mathbf{D}^{\frac{1}{2}}\boldsymbol{\phi} \le (1+t)^2 - (1-t)^2$$
$$\Leftrightarrow \quad \begin{pmatrix} 1+t \\ 1-t \\ 2\mathbf{D}^{\frac{1}{2}}\boldsymbol{\phi} \end{pmatrix} \in \mathcal{L}^{n+2}. \tag{34}$$

The above result (34), Lemma 4 and constraint (2) imply that (21) is equivalent to (33). The conclusion immediately follows from this result and the fact that (20) is equivalent to (21). □

Theorem 5 shows that the robust investment problem for an insurer under a joint uncertainty can be reformulated as a cone programming problem. Then, problem (20) can be solved by optimization solvers. In the case where there is no bond and the underwriting rate of return is regarded as being independent of the stock return, our model is reduced to the following general robust portfolio selection model:

$$\max_{\phi,\nu,t,\tau,\mathbf{S}} \nu - \frac{\beta g P}{2} t$$
$$s.t. \begin{pmatrix} \tau \mathbf{M} - \beta g P \mathbf{S} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{F} \end{pmatrix} & \tau \mathbf{h} + \hat{\mathbf{q}} \\ \tau \mathbf{h}^{\mathrm{T}} + \hat{\mathbf{q}}^{\mathrm{T}} & \tau\eta - 2\nu \end{pmatrix} \succeq \mathbf{0}$$
$$\begin{pmatrix} 1 & \phi^{\mathrm{T}} \\ \phi & \mathbf{S} \end{pmatrix} \succeq \mathbf{0}, \begin{pmatrix} 1+t \\ 1-t \\ 2\mathbf{D}^{\frac{1}{2}}\phi \end{pmatrix} \in \mathcal{L}^{n+2}$$
$$\mathbf{1}_{\mathbf{n}}^{\mathrm{T}}\phi = 1,\ \phi \ \geq\ \mathbf{0},\ \tau \ \geq\ 0, \tag{35}$$

where $\hat{\mathbf{q}} = (\phi_1, \mathbf{0}, \ldots, \phi_n, \mathbf{0})^{\mathrm{T}} \in \mathbb{R}^{(m+1)n}$.

4 Numerical analysis

In this section, we present some computational results for raw market data. These computational experiments aim to obtain the optimal portfolio for an insurer and to compare the performance of the robust investment strategies for an insurer with that of the general robust portfolio selection strategies. We conduct all computation using SeDuMi V1.1 with Matlab 6.5.1 and R 2.7.2.

Firstly, the hypotheses are verified by using real market data. Through quantile-quantile plots, we can regard $(r, \mathbf{r}_0^{\mathrm{T}}\phi)$ as a bivariate normal distributed vector. By correlation analysis, the underwriting rate of return r is correlated with the rate of return on each stock.

We choose 12 industries from finance.cn.yahoo.com in 2009 according to real market data and n=47 well-performed stocks are chosen for investment from these industries (see Table 1). We fix 15 factors through principal component analysis of asset returns, i.e., m=15. The data sequence consists of daily asset returns from January 3, 2005 to August 26, 2009, which were the most recent data when they were collected.

The following is a complete description of the experimental procedures. The entire data sequence is divided into investment periods of length $p = 90$ days. For each investment period t, the rates of returns on factors are calculated by principal component analysis on the rates of stock returns. The rates of returns on these factors are used to estimate the factor covariance matrix $\mathbf{F}$. The variance of the residual rate of return d_i is set to $d_i = s_i^2$, where s_i^2 is given in (16). The risk-free rate r_f is set according to the daily bank interest rate of each period. We set $P = 500000$, $g = 0.8$ and $\beta = 0.02$. The data sequence of underwriting rate r is computed basing on the business statistics of some

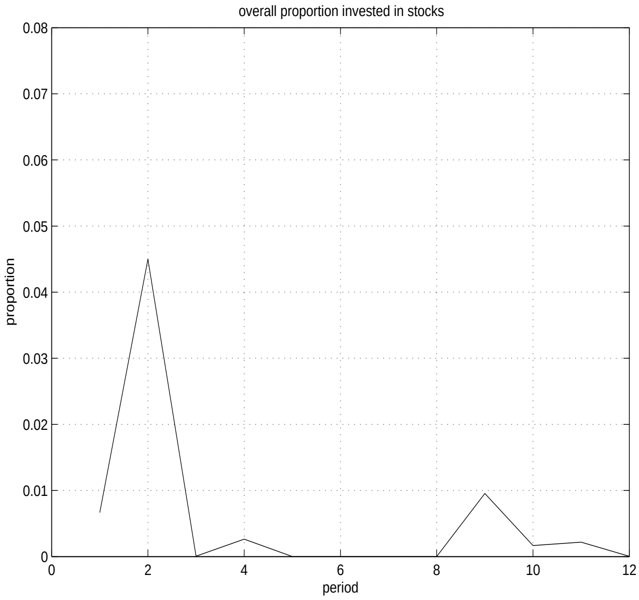

Figure 1: overall proportion invested in 47 stocks for $\omega = 0.95$

insurance companies in one year. Denote by I the premium income and by C the benefit paid in one year. Then

$$r = \frac{I - C}{360I}.$$

The historical data of r are used to estimate $\mathbf{Cov}_{(r,\boldsymbol{\epsilon})}$ and $\mathbf{Cov}_{(r,\mathbf{f})}$. Given an $\omega > 0$, the joint uncertainty set $S_{\boldsymbol{\mu},\mathbf{V}}$ is built as (18). The robust portfolio for an insurer (resp. general robust portfolio) is computed by solving model (33) (resp. model (35)).

Since a block of data of length $p = 90$ is required to estimate the parameters for the robust model, the first investment period labeled $t = 1$ starts from $(p + 1)$th day. The time period January 3, 2005-August 25, 2009 contains 13 periods of length $p = 90$, so there are 12 investment periods in total. A portfolio ϕ^t is held constant for the period t and rebalance to portfolio ϕ^{t+1} in period $t + 1$.

In Figure 1, We see that for an insurer, the overall proportion invested in the 47 stocks is small, which implies the safety orientation of the investment for an insurer.

The diversification number of a portfolio is defined as the number of its components that are above 10^{-3}. In comparison with the robust portfolio for a general investor, Figure 2 shows that our robust portfolio is highly non-diversified. This fact is consistent with the intuition. The insurer should consider the claim paid, so she/he is much more conservative than the general investor and will choose less stocks for investment in order to reduce risk. We also find that in periods $t = 5, t = 6, t = 7, t = 8$ and $t = 12$, the proportions invested in stocks by the insurer are nearly zero, which is due to the fall of stock market. This fact implies that insurers are more sensitive than general investors on fluctuations of stock prices. Again, this result is consistent with intuition. Figure 3 plots the average diversification numbers for different confidence levels respectively. Since in periods $t = 6, t = 7, t = 8$ and $t = 12$, the proportions invested in stocks are so small, we only consider the other periods and define the average diversification number as

$$\frac{1}{8}\left(\sum_{t=1}^{5} I(\boldsymbol{\phi}^t) + \sum_{t=9}^{11} I(\boldsymbol{\phi}^t)\right),$$

where $I(\boldsymbol{\phi}^t)$ denotes the diversification number of the portfolio $\boldsymbol{\phi}^t$.

Figure 4 and Figure 5 compare the cost of implementing two robust investment strategies. The transaction cost is quantified by

$$\|\boldsymbol{\phi}^t - \boldsymbol{\phi}^{t-1}\|_1,$$

and the average transaction cost of portfolios $\{\boldsymbol{\phi}^t\}_{t=1}^{12}$ is quantified by

$$\frac{1}{11}\sum_{t=2}^{12} \|\boldsymbol{\phi}^t - \boldsymbol{\phi}^{t-1}\|_1,$$

where $\|\cdot\|_1$ denotes the 1-norm. For confidence level $\omega = 0.95$, we see that the transaction cost for an insurer is much less than that of a general investor from Figure 4. More precisely, the 12 periods' average transaction cost of the general investor is about 61 times more than that of the insurer. In Figure 5, we plot the average transaction cost for different ω. Without loss of generality, the periods $t = 6, t = 7, t = 8$ and $t = 12$ are eliminated.

Figure 3 and Figure 5 show that the performances of both robust portfolios are not much sensitive with respect to ω. In particular, the average transaction cost of the insurer's robust strategy has hardly changed as ω increases.

5 Conclusion

This paper studies the robust investment problem for an insurer. In terms of the characteristics of the insurer's investment problem, underwriting results and a risk-free asset are considered in our formulation. Based on the joint uncertainty set for the model parameters, the robust investment problem for an insurer is established. This robust portfolio selection model is reformulated as a cone programming problem and then can be solved through optimization solvers.

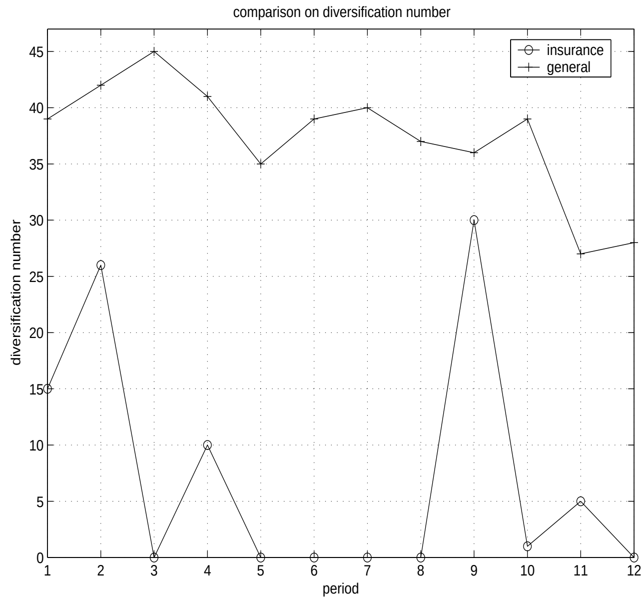

Figure 2: comparison on diversification number for $\omega = 0.95$

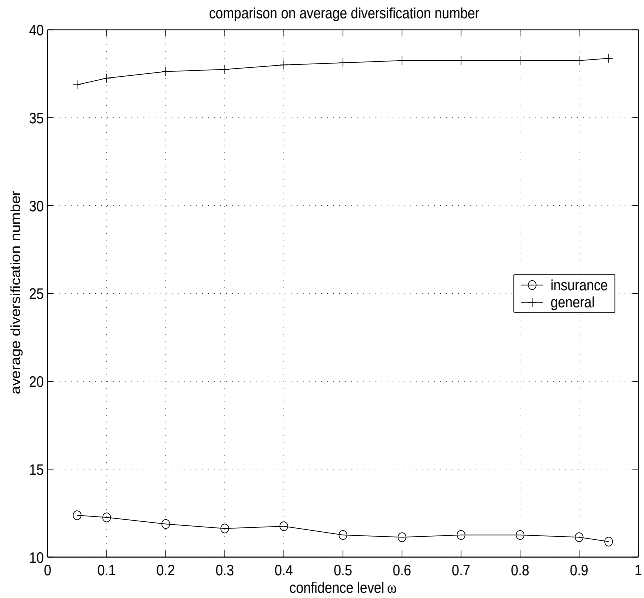

Figure 3: comparison on average diversification number

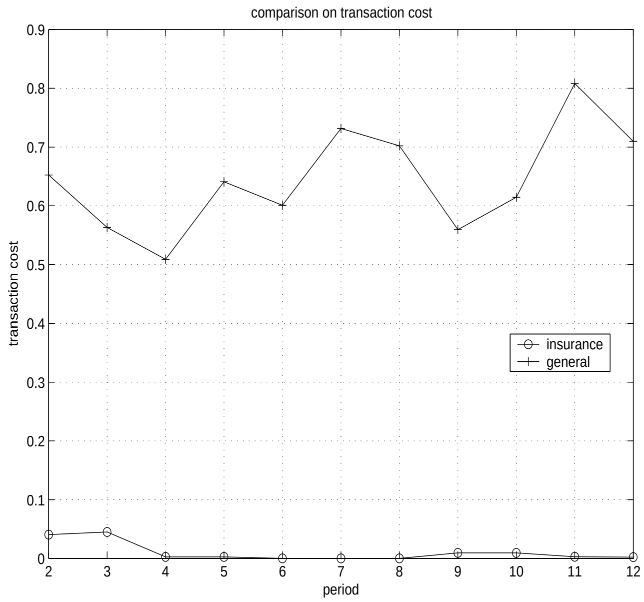

Figure 4: comparison on transaction cost for $\omega = 0.95$

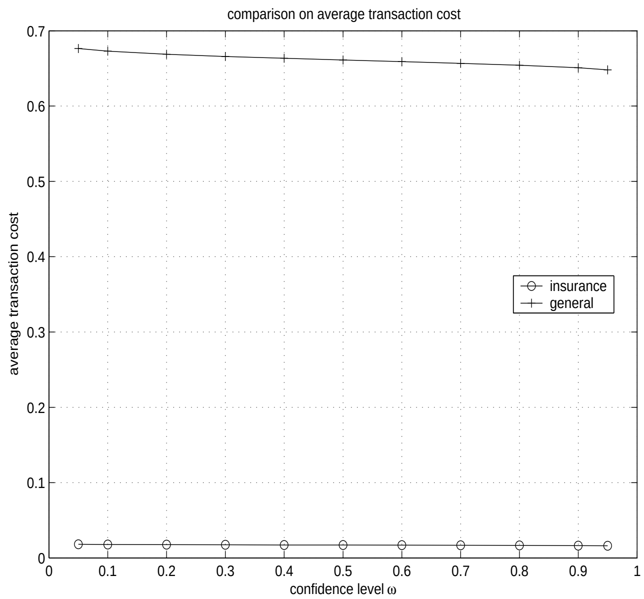

Figure 5: comparison on average transaction cost

The computational results show that the optimal strategy of an insurer is to invest a small amount of money in stocks. In view of the results on the diversification number of stocks and the transaction cost, the investment of an insurer is much more conservative than a general investor, which implies that the insurer is more concerned with safety than profitability of an investment strategy.

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Appendix

Table 1: Stocks

Aerospace and Defense	
AIR	AAR corporation
BA	Boeing Corp.
HON	Honeywell Intl.
LMT	Lockheed Martin
UTX	United Technologies
Auto Manufactures	
F	Ford Motor Co.
TM	Toyota Motor Cp ADR
Biotechnology and Drug Manufacturers	
AMGN	Amgen Co
GILD	Gilead Science
JNJ	Johnson & Johnson
PFE	Pfizer
Chemicals	
APD	Air Products & Chem
DD	DuPon
DOW	Dow Chemical
EMN	Eastman Chemical Co.
Communication Equipment	
DTV	Directv
GLW	Corning Inc.
MOT	Motorola
NOK	Nokia
QCOM	Qualcomm

Computer Software	
ADBE	Adobe Systems Inc.
ARBA	Ariba
MSFT	Microsoft
ORCL	Oracle Corp.
Discount	
WMT	Wal-Mart Stores Inc.
Diversified Computer Systems	
HPQ	Hewlett-Packard Co.
IBM	Intl. Business Machines
IGT	Intl. Game Technology
JAVA	Sun Microsys
SGMS	Scientific Games Corp.
Major Integrated Oil & Gas	
BP	BP Plc.
CVX	Chevron Corp.
XOM	Exxon Mobil Corp.
Semiconductor-Broad Line	
ADI	Analog Devices Inc.
INTC	Intel Corp.
STM	STMicroelectronics Ads
TXN	Texas Instruments
Telecom Services	
BCE	BCE Inc.
CHT	Chunghwa Telecom Co. Ltd.
CTL	Centurytel Inc.
T	AT&T
VZ	Verizon Communications Inc.
Utilities (Gas & Electric)	
D	Dominion Resources Inc.
DUK	Duke Energy Corporation
EXC	Exelon Corp.
PEG	Public Service Enterprise Group
SO	Southern

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