# Population Dynamics: A Geometrical Approach of Some Epidemic Models

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*Abstract:* Recently, the behavior of different epidemic models and their relation both to different types of geometries and to some biological models has been revisited. Path equations representing the behavior of epidemic models and their corresponding deviation vectors are examined. A comparison between paths and their deviation vectors in Riemannian and Finslerian Geometries is presented.

*Key–Words:* Epidemic model, Path equation, Geometrical method

## 1 Introduction

In this study, we are going to describe a specific susceptive-infective (SI)-model of epidemics [1] using Antonelli's approach of describing two species in ecology geometrically [2]. This approach has been successfully used in case of wild rabbit disease as well as coral-starfish equilibrium[5]. Also, allometric relationship between net production biomass and the amount of secondary compound plants produced to defend against herbivores can be geometrized as well as corresponding interaction condiments are taken to be constant. The aim of this work is to apply Antonelli's idea of imposing geometrical paths defined in metric spaces extracted from stochastic allometry space in Riemannian geometry to obtain a better nonlinear regression formula with geometrical origin [2] . The importance of allometry space, due to Antonelli's approach, is to obtain a space having a positive definite metric with negative curvature acting as a tool to examine the behavior of growth curves. Thus, any additive terms associated with geometrical structures are useful for adjusting a proper path describing the behavior of any epidemic curve. Since 1990s Antonelli's approach has been extended to include other types of geometries such as Finsler geometry rather than Riemannian[3]. Consequently, Antonelli and Bradbury (1997) have used Gompertz growth and allometric relationships to build a dynamical theory of ecology, evolution and development in colonial organisms using Finslerian Geometry. Accordingly, geometrization of SI-model can be considered as an introductory step to the geometric version of SIR-model and others. Moreover, this approach has been extended to include Finslerian Geometry due to its richness of imposing many interacting factors geometrically rather than the Riemannian one [5]. The importance of geometrizing such a model is to construct the path equations and their corresponding path deviation equations to know the behavior of the growth curves and their effect after a slight perturbation on them. This can be seen by studying the stability conditions from examining the motion of path deviation equations. In the present work, we are going to utilize the nonlinear version of allometry for obtaining the relevant equations of epidemics in SI-models using different types of geometries with some details on its extension to include SIR models and others by increasing the dimension of the manifold. This type of extending dimensions to examine various epidemic models will be examined in our future work.

## 2 Historical Background

### 2.1 SI-Model of Epidemics

It is well known that the simplest model of describing population growth stems from the famous Lotka-Volterra model which can be expressed as follows [6]:

$$\frac{dS(t)}{dt} = -\alpha_1 S(t)I(t), \tag{1}$$

and

$$\frac{dI(t)}{dt} = \alpha_1 S(t)I(t), \tag{2}$$

where $S(t)$ the susceptive class $I(t)$, the infected class and $\alpha_1$ is a parameter. Equations (1) and (2) can be related together using the following condition:

$$S(t) + I(t) = 1. \tag{3}$$

The above model has been modified, to accommodate some other factors based on some oscillations either internally or externally. In this study, it is sufficient to display the amended version owing to this extra factor to take the following form [7]

$$\frac{dS(t)}{dt} = -\alpha_1 S(t)I(t) + \alpha_2 S(t) \tag{4}$$

and

$$\frac{dI(t)}{dt} = \alpha_1 S(t)I(t) + \alpha_2 I(t) \tag{5}$$

where $\alpha_2$ is another a parameter with a constant coefficient. This model has been applied using deterministic and stochastic approaches in the following subsections.

## 2.2 Deterministic Approach of Epidemics

**SI-Model of Epidemics:**

The concept of deterministic models is essentially depending on the concept of threshold of an epidemic, may be expressed in the following way

$$\frac{dS(t)}{dt} = -\lambda S(t)I(t) - \delta S(t) + \gamma I(t) + \delta \tag{6}$$

and

$$\frac{dI(t)}{dt} = \lambda S(t)I(t) + (\gamma + \delta)I(t) \tag{7}$$

where $\lambda$ , $\frac{1}{\delta}$, and $\frac{1}{\gamma}$ stand for average number of contacts per infective per day, and average period of infectivity respectively.

Substituting (2) into equations (6) and (7) one obtains

$$\frac{dI(t)}{dt} = \lambda I(t)^2 + (\lambda - \gamma - \delta)I(t). \tag{8}$$

The above set of equations can be applied to describe the evolution of some epidemics such as plague, malaria, meningitis. The significance of expressing any epidemic model in terms of differential equations is due to large number of the population size.

Accordingly, the continuous variables play for this task. From the above description, it is possible to consider an example for some fatal diseases by suggesting the following set of equations

$$\frac{dS(t)}{dt} = -\lambda S(t)I(t) + (\gamma + \delta)I(t) \tag{9}$$

and

$$\frac{dI(t)}{dt} = \lambda S(t)I(t) - (\gamma + \delta + \eta)I(t), \tag{10}$$

where $\frac{1}{(\gamma+\delta)}$ and $\eta$ denote the death-adjusted period of infectivity and the rate of daily death respectively.

If one takes $\lambda = 1$,$\gamma = 0.2$, $\delta = 0.0001$, $\eta = 0.2$ [8], then equations (9) and (10) become

$$\frac{dS(t)}{dt} = -\ S(t)I(t) + (0.2001)I \tag{11}$$

and

$$\frac{dI(t)}{dt} = S(t)I(t) - 0.4001I(t). \tag{12}$$

**SIR-Model of Epidemics:**

If there is a chance of recovery or removal by death, another model can be expressed such as the susceptive-infective-recovered (SIR) model as described in the following set of equations

$$\frac{dS(t)}{dt} = -\lambda S(t)I(t) - \delta S(t) + \gamma I(t) + \delta, \tag{13}$$

$$\frac{dI(t)}{dt} = \lambda S(t)I(t) + (\gamma + \delta)I(t) \tag{14}$$

and

$$\frac{dR(t)}{dt} = \gamma I(t) \tag{15}$$

provided that:

$$S(t) + I(t) + R(t) = 1, \tag{16}$$

where $R(t)$ is the removed class of susceptible-infection caused by isolation, immunity or even death. The above set of equations can be expressed only ($S(t)$ and $I(t)$,

$$\frac{dI(t)}{dS(t)} = -1 + \frac{\gamma}{\lambda S(t)}. \tag{17}$$

It is well known that $\frac{\gamma}{\lambda}$ represents the infectious contact number.

In case of SIR model having a temporary immunity without changing the threshold phenomenon. Equation (13) can be modified by imposing such a parameter to express the daily loss immunity rate $(\alpha)$, to become

$$\frac{dS(t)}{dt} = -\lambda S(t)I(t) - \delta S(t) + \gamma I(t) + \delta + \alpha R(t). \tag{18}$$

Thus, for two practical examples of SIR models can be obtained if one can substitute (i) $\alpha = 0.02$, $\delta =$

0.0001, $\lambda = 0.1$ with taking into consideration that the infectious contact number $= 0.2$

$$\frac{dS(t)}{dt} = -0.1S(t)I(t) - 0.0001S(t) + \gamma I(t) + \delta + \alpha R(t).$$

(ii) $\alpha = 0.02$, $\delta = 0.0001$ , $\lambda = 0.4$ with taking into consideration that the infectious contact number $= 2$(see, [8]).

### 2.3 Stochastic Approach of Epidemic Model

A simple example of a stochastic version of epidemic model can be described in the following way

$$\frac{dN(t)}{dt} = -mN(t), \tag{19}$$

where $m$ is a positive rate constant and $N(t)$ denotes the size of a population whose rate constant is based on a specific type of probability distribution defined by a bivariate gamma distribution [9]. Accordingly, SI-model, can be expressed stochastically as follows

$$\frac{dS(t)}{dt} = -mS(t) \tag{20}$$

and

$$\frac{dI(t)}{dt} = mI(t). \tag{21}$$

### 2.4 The Concept of Allometry

In 1936, Sir Julian Huxley introduced the concept of allometry, or the experimental study of relative growth of parts of animals, via log -log plots of measurements of morphological characteristics of individuals in a Euclidean geometry. In these plots resulted straight lines via statistical method of least squares. Around the same time Sir Joseph Needham made some experimental work to show that straight lines allometries.

In 1944, J. Kittredge used the same concept to estimate the crown biomass of trees in forest stand by measuring the trunk girth, or diameter at breast height. Again some experiments of botanist J. Harper in 1962 who found Gompertz curves the best ones to fit for describing the growth of simple aquatic plants.

By 1965 Laird studied on vertebrate growth using Huxley's allometric law as well. Several applications of concept of allometry have been discussed in detail. Recently, Karl Niklas (1994) has recorded a variety growing plants and flowers satisfying the allometric concept as well [10].

It is well known that, Antonelli and Voorhees (1974) have suggested the following metric in order to define geometrically the behavior of growth curves using allometric space [4].

$$g_{ij} = e^{-2\alpha_k x^k}\delta_{ij}, \tag{22}$$

and its affine connection is given by

$$\Gamma^i_{.jk} = \frac{1}{2} g^{il}(g_{lk,j} + g_{jl,k} - g_{jk,l}), \tag{23}$$

where $g^{ij}$ is the the matrix inverse of $g_{ij}$. Suppose $\alpha_i$ is a constant vector to define the Christoffel symbol in the following way:

$$\Gamma^i_{.ii} = -\alpha_i = constant \tag{24}$$

$$\Gamma^i_{.jj} = \alpha_i = constant, i \neq j \tag{25}$$

$$\Gamma^i_{.ji} = 0, i \neq j. \tag{26}$$

Thus the metric condition

$$g_{ij;k} = 0 \tag{27}$$

becomes

$$e^{-2\alpha_m x^m}\delta_{ij_k} - e^{-2\alpha_m x^m}\delta_{in}\Gamma^{nj}_{k} - e^{-2\alpha_m x^m}\delta_{nj}\Gamma^n_{ki} = 0, \tag{28}$$

$$-2\alpha_k\delta_{ij} - e^{-2\alpha_m x^m}\delta_{in}\Gamma^{nj}_{k} - e^{-2\alpha_m x^m}\delta_{nj}\Gamma^n_{ki} = 0, \tag{29}$$

and

$$\gamma_k\delta_{ij} = \delta_{in}\Gamma^n_{kj} + \delta_{jn}\Gamma^n_{ik}. \tag{30}$$

We can find out that the Riemann-Christoffel Curvature

$$R^c_{abd} = \Gamma^c_{ad,b} - \Gamma^c_{ab,d} + \Gamma^m_{ad}\Gamma^c_{mb} - \Gamma^m_{ab}\Gamma^c_{md}, \tag{31}$$

has reduced to

$$R^c_{abd} = \Gamma^m_{ad}\Gamma^c_{mb} - \Gamma^m_{ab}\Gamma^c_{md}. \tag{32}$$

Substituting (25) and (26) into equations (31) and (32), one obtains the non vanishing components of the curvature tensor,

$$R^i_{jij} = \sum_{k\neq i,j}(\alpha_k)^2, i \neq j \tag{33}$$

$$R^i_{jjl} = \alpha_i\alpha_j, i \neq j \neq l. \tag{34}$$

# 3 Geometerization of Epidemics

## 3.1 Equations of Epidemics in Riemannian version of SI-Model

It is well known that path equations of any growth curves can be obtained if the variation principle is applied on Lagrangian functions. This trend of imposing biological action principle has been established prior to Euler by Mauphritis cf.[10]

$$L = g_{ab}U^aU^b[10] \tag{35}$$

However, in this approach, we are going to obtain path and path deviation equations from one single lagrangian using the Bazanski Lagrangian [13]:

$$L = g_{ab}U^a\frac{D\Psi^b}{Dt} \tag{36}$$

where $a, b = 1, 2, 3, ..n$ and $\frac{D\Psi^\alpha}{Dt}$ is the covariant derivative with respect to a parameter $t$.

Taking the variation with respect to the deviation vector $\Psi^c$ and the tangent vector $U^c$ respectively one obtains path equation

$$\frac{dU^c}{dt} + \Gamma^c_{ab}U^aU^b = 0 \tag{37}$$

and path deviation equation

$$\frac{D^2\Psi^c}{Dt^2} = R^c_{abd}U^aU^b\Psi^d \tag{38}$$

Thus, the path and path deviation equations of the SI-model can be obtained from the following Bazanski Lagrangian [13]

$$L = g_{\mu\nu}U^\mu_{SI}\frac{D\Psi^\nu_{SI}}{Dt} \tag{39}$$

where $U^\mu_{SI} = (S, I)$ and $\Psi^\nu_{SI} = (\Psi_S, \Psi_I)$.

Accordingly, if we take the variation with respect to the deviation vector $\Psi^\sigma$to get the following components of path equation

$$\frac{dS}{dt} + \Gamma^1_{11}S^2 + \Gamma^1_{22}I^2 + 2\Gamma^1_{12}SI = 0 \tag{40}$$

and

$$\frac{dI}{dt} + \Gamma^2_{11}S^2 + \Gamma^2_{22}I^2 + 2\Gamma^2_{12}SI = 0. \tag{41}$$

And taking the variation with respect to velocity vector $U^\sigma$ to get the corresponding components of path deviation equation [14] :

$$\begin{aligned}\frac{D^2\Psi_S}{Dt^2} &= R^1_{112}S^2\Psi_I + R^1_{121}SI\Psi_S \\ &\quad +R^1_{212}SI\Psi_I + R^1_{221}I^2\Psi_S\end{aligned} \tag{42}$$

and

$$\begin{aligned}\frac{D^2\Psi_I}{Dt^2} &= R^2_{112}S^2\Psi_I + R^2_{121}SI\Psi_S \\ &\quad +R^2_{212}SI\Psi_I + R^2_{221}I^2\Psi_S.\end{aligned} \tag{43}$$

Consequently, we can suggest the path and path deviation equations of a modified $SI$ model can be following Lagrangian:

$$L = g_{\mu\nu}U^\mu_{SI}\frac{D\Psi^\nu_{SI}}{Dt} + \lambda\Psi_\mu U^\mu \tag{44}$$

to give,

$$\frac{dU^a_{SI}}{dt} + \Gamma^a_bcU^b_{SI}U^c_{SI} = \lambda_aU^a_{SI}, \tag{45}$$

which can be expressed by components as follows

$$\frac{dS}{dt} + \Gamma^1_{11}S^2 + 2\Gamma^1_{12}SI + \Gamma^1_{22}I^2 = \bar{\lambda}S \tag{46}$$

and

$$\frac{dI}{dt} + \Gamma^2_{11}S^2 + 2\Gamma^2_{12}SI + \Gamma^2_{22}I^2 = \bar{\lambda}I, \tag{47}$$

i.e.

$$\frac{dS}{dt} + 2\bar{\beta}SI - \bar{\alpha}I^2 + \bar{\alpha}S^2 = \bar{\lambda}S \tag{48}$$

and

$$\frac{dI}{dt} + 2(\bar{\alpha} - \bar{\beta}L)SI - (\bar{\beta} + \bar{\alpha}L)S + (\bar{\alpha} - \bar{\beta}L)I^2 = \bar{\lambda}I. \tag{49}$$

And their corresponding path deviation equations become as follows:

$$\frac{D^2\Psi^a_{SI}}{Dt^2} + \bar{\lambda}\frac{D\Psi^a_{SI}}{Dt} = R^a_{bcd}U^b_{SI}U^c_{SI}\Psi^d_{SI} \tag{50}$$

i.e.

$$\begin{aligned}\frac{D^2\Psi_S}{Dt^2} + \bar{\lambda}\frac{D\Psi_S}{Dt} &= R^1_{112}S^2\Psi_I + R^1_{121}SI\Psi_S \\ &\quad + R^1_{212}SI\Psi_I + R^1_{221}S^2\Psi_S\end{aligned} \tag{51}$$

and

$$\begin{aligned}\frac{D^2\Psi_I}{Dt^2} + \bar{\lambda}\frac{D\Psi_I}{Dt} &= R^2_{112}S^2\Psi_I + R^2_{121}SI\Psi_S \\ &\quad +R^2_{212}SI\Psi_I + R^2_{221}I^2\Psi_S.\end{aligned} \tag{52}$$

## 3.2 Berwald type

The metric tensor of Berwald type in Finslerian geometry can be described as follows [12]

$$g_y = \dot{\partial}_i\dot{\partial}_j(\frac{1}{2}F^2) \tag{53}$$

such a metric tensor is called a Minkowoski space with Finslerian norm $ds = f(dx^1, dx^2)$. Thus, its corresponding geodesic can be described as

$$\frac{d^2x^i}{dt^2} + \hat{\Gamma}^i_{jk}(x,y)\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \quad (54)$$

such that

$$\hat{\Gamma}^i_{jh} = \frac{1}{2}g^{im}(\delta_h g_{jm} + \delta_j g_{mh} - \delta_m g_{jh}), \quad (55)$$

where $\delta_h$ is a partial derivative with respect to non-linear connection.

$$\delta_h = \partial_h - \Gamma^i_{jh}(x)\frac{\partial}{\partial y^h}, \quad (56)$$

$$\hat{C}^i_{.jk} = \frac{1}{2}g^{il}(\frac{\partial g_{lk}}{\partial y^j} + \frac{\partial g_{jl}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^l}) \quad (57)$$

The Berwald type has some associated curvatures are defined in the following way [11]:

$$B^i_{kjh} = R^i_{kjh} + G^r_j D^i_{rhk} - G^r_k D^i_{rhj} \quad (58)$$

where

$$G^i_j = \hat{\Gamma}^i_{jk}y^j - \frac{1}{2}g^{im}\hat{\Gamma}^l_{jk}\frac{\partial g_{ml}}{\partial y^l}y^j y^k \quad (59)$$

and

$$D^i_{kjh} = \frac{\partial^3 H^i_2}{\partial y^j \partial y^k \partial y^l}, \quad (60)$$

such that

$$\hat{\Gamma}^i_{\alpha_1\alpha_2\alpha_3....\alpha_m} = \frac{1}{m!}\frac{\partial^m H_{(m)}}{\partial y^{\alpha_1}\partial y^{\alpha_2}\partial y^{\alpha_3}.....\partial y^{\alpha_m}} \quad (61)$$

### 3.3 SI-Model in Finsler Geometry

In 1991 Antonelli developed such a metric for 2-dimensional Berwald space with locally constant coefficients to become [3]

$$F^2 = e^{2\alpha_i X^i(L^2+1)}((\dot{X}^1)^2 + (\dot{X}^2)^2), i = 1, 2 \quad (62)$$

where $X^1$ and $X^2$ are Cartesian coordinates on $R^2$ and $L$ acts as a perturbation parameter. The above relation can be related to Riemananian geometry by relaxing the term $L$

$$\bar{g}_{ij}(X, \dot{X}) = e^{2\alpha_i X^i(L^2+1)}g_{ij} \quad (63)$$

The above system of equations together with their corresponding deviation vector equations can be obtained from taking the action principle to the following Lagrangian.

$$L_{BF} = g_{ij}(x,y)\dot{X}^i\frac{\hat{D}\hat{\Psi}^j}{\hat{D}t} \quad (64)$$

such that

$$\frac{\hat{D}\hat{\Psi}^i}{\hat{D}t} = \frac{d\hat{\Psi}^i}{dt} + \hat{\Gamma}^i_{jh}\hat{\Psi}^j U^h + C^i_{jh}\hat{\Psi}^j V^h, \quad (65)$$

where $V^h = \frac{\delta y}{\delta t}$.

Accordingly, the geodesic equation may be as follows

$$\frac{d^2x^i}{dt^2} + \hat{\Gamma}^i_{jh}(x,y)y^j y^h = 0, \quad (66)$$

i.e.

$$\frac{dS}{dt} + 2(\bar{\beta} + \bar{\alpha}L)SI + (\bar{\beta}L - \bar{\alpha})I^2 + (\bar{\alpha} - \bar{\beta}L)S^2 = 0 \quad (67)$$

and

$$\frac{dI}{dt} + 2(\bar{\alpha} - \bar{\beta}L)SI - (\bar{\beta} + \bar{\alpha}L)S^2 + (\bar{\alpha} - \bar{\beta}L)I^2 = 0 \quad (68)$$

and its deviation equation

$$\frac{\hat{D}^2\hat{\Psi}^a}{\hat{D}t^2} = B^a_{bcd}U^b U^c \hat{\Psi}^d \quad (69)$$

become

$$\begin{aligned} \frac{\hat{D}^2\hat{\Psi}_S}{\hat{D}t^2} &= B^1_{112}S^2\hat{\Psi}_I + B^1_{121}SI\hat{\Psi}_S \\ &+ B^1_{212}SI\hat{\Psi}_I + B^1_{221}I^2\hat{\Psi}_S, \end{aligned} \quad (70)$$

and

$$\begin{aligned} \frac{\hat{D}^2\hat{\Psi}_I}{\hat{D}t^2} &= B^2_{112}S^2\hat{\Psi}_I + B^2_{121}tSI\hat{\Psi}_S \\ &+ B^2_{212}SI\hat{\Psi}_I + B^2_{221}I^2\hat{\Psi}_S. \end{aligned} \quad (71)$$

Similarly, applying Antonelli's method to define Volterra's equation in ecology [3] , we obtain the its corresponding part of the SI-model in Berwald space

$$\frac{dS}{dt} + \hat{\Gamma}^1_{11}S^2 + 2\hat{\Gamma}^1_{12}SI + \hat{\Gamma}^1_{22}I^2 = \bar{\lambda}S \quad (72)$$

and

$$\frac{dI}{dt} + \hat{\Gamma}^2_{11}S^2 + 2\hat{\Gamma}^2_{12}SI + \hat{\Gamma}^2_{22}I^2 = \bar{\lambda}I. \quad (73)$$

The coefficients of its affine connection become:

$$\hat{\Gamma}^1_{11} = \alpha_1 - \alpha_2 L, \quad (74)$$

$$\hat{\Gamma}^1_{22} = -(\alpha_1 - \alpha_2 L), \quad (75)$$

$$\hat{\Gamma}^1_{12} = \hat{\Gamma}^1_{21} = -(\alpha_2 + \alpha_1 L), \tag{76}$$

$$\hat{\Gamma}^2_{12} = \hat{\Gamma}^1_{21} = -(\alpha_1 - \alpha_2 L), \tag{77}$$

$$\hat{\Gamma}^2_{11} = -(\alpha_2 + \alpha_1 L) \tag{78}$$

$$\hat{\Gamma}^2_{22} = -(\alpha_2 + \alpha_1 L). \tag{79}$$

Thus, the components of path equations are expressed as follows:

$$\frac{dS}{dt} + 2(\bar{\beta} + \bar{\alpha}L)SI + (\bar{\beta}L - \bar{\alpha})(I)^2 + (\bar{\alpha} - \bar{\beta}L)(S)^2 = \bar{\lambda}S \tag{80}$$

and

$$\frac{dI}{dt} + 2(\bar{\alpha} - \bar{\beta}L)SI - (\bar{\beta} + \bar{\alpha}L)S^2 + (\bar{\alpha} - \bar{\beta}L)I^2 = \bar{\lambda}I. \tag{81}$$

Equations (80) and (81) are obtained by taking the variation with respect the corresponding deviation vector of the following modified Bazanski Lagrangian

$$L = \bar{g}_{ab}\dot{X}^a \frac{\hat{D}\hat{\Psi}^b}{\hat{D}t} + \bar{\lambda}\Psi_a \dot{X}^a. \tag{82}$$

Also, after some multiplications, we can find their corresponding deviation equations by taking the variation with respect to $\dot{X}^c$ to become:

$$\begin{aligned} \frac{\hat{D}^2\Psi_S}{\hat{D}t^2} + \lambda\frac{\hat{D}\hat{\Psi}_S}{Dt} &= B^1_{112}S^2\hat{\Psi}_I + B^1_{121}SI\hat{\Psi}_S \\ &+ B^1_{212}SI\hat{\Psi}_I + B^1_{221}I^2\hat{\Psi}_S \end{aligned} \tag{83}$$

and

$$\begin{aligned} \frac{\hat{D}^2\hat{\Psi}_I}{\hat{D}t^2} + \lambda\frac{\hat{D}\hat{\Psi}_I}{Dt} &= B^2_{112}S^2\hat{\Psi}_I + B^2_{121}SI\hat{\Psi}_S \\ &+ B^2_{212}SI\hat{\Psi}_I + B^2_{221}I^2\hat{\Psi}_S. \end{aligned} \tag{84}$$

### 3.4 Antonelli-Finsler Metric Function

In order to generalize the previous metric in (63) Antonelli has modified Finsler metric to become [12]:

$$F = e^{\phi}\dot{(}\sum_{i=1}^{n}(\dot{x}^i)^m)^{\frac{1}{m}}, \tag{85}$$

where $m \geq 2$ and $n = 2$ . Thus their corresponding components of its affine connection are given as :

$$\hat{\Gamma}^1_{11} = \sigma_1 - \frac{1}{9}\beta_1(\frac{I}{S})^{\frac{4}{3}}, \tag{86}$$

$$\hat{\Gamma}^1_{21} = \frac{4}{9}\beta_1(\frac{I}{S})^{\frac{4}{3}} + \frac{1}{2}\gamma_{12}, \tag{87}$$

$$\hat{\Gamma}^1_{22} = \frac{2}{9}\beta_1(\frac{S}{I})^{\frac{2}{3}}, \tag{88}$$

$$\hat{\Gamma}^2_{22} = \sigma_2 - \frac{1}{9}\beta_2(\frac{S}{I})^{\frac{4}{3}}. \tag{89}$$

Consequently, the Bazanski method of obtaining path and path deviation equations becomes in the following way:

[i] The components of Path Equations:

$$\frac{dS}{dt} + \sigma_1 S^2 + \sigma_2(\frac{m}{m-1})SI + \frac{\sigma_1}{m-1}(\frac{I}{S})^{m-2}I^2 = \lambda_1 S \tag{90}$$

and

$$\frac{dI}{dt} + \sigma_2 I^2 + \sigma_1(\frac{m}{m-1})SI + \frac{\sigma_2}{m-1}(\frac{\dot{I}}{\dot{S}})^{m-2}I^2 = \lambda_2 I. \tag{91}$$

[ii] The components of Path Deviation Equations:

$$\begin{aligned} \frac{\hat{D}^2\Psi_S}{\hat{D}t^2} + \lambda\frac{\hat{D}\hat{\Psi}_S}{Dt} &= \bar{B}^1_{112}S^2\Psi_I + \bar{B}^1_{121}SI\Psi_S \\ &+ \bar{B}^1_{212}SI\Psi_I + \bar{B}^1_{221}S * 2\Psi_S \end{aligned} \tag{92}$$

and

$$\begin{aligned} \frac{\hat{D}^2\hat{\Psi}_I}{\hat{D}t^2} + \lambda\frac{\hat{D}\hat{\Psi}_I}{Dt} &= \bar{B}^2_{112}S^2\Psi_I + \bar{B}^2_{121}SI\Psi_S \\ &+ \bar{B}^2_{212}SI\Psi_I + \bar{B}^2_{221}S * 2\Psi_S, \end{aligned} \tag{93}$$

where $\bar{B}^a_{bcd} = \hat{\Gamma}^c_{ad,b} - \hat{\Gamma}^c_{ab,d} + \hat{\Gamma}^m_{ad}\hat{\Gamma}^c_{mb} - \hat{\Gamma}^m_{ab}\hat{\Gamma}^c_{md}$.

Thus, it is well known to find from Antonelli-Finsler SI-model the appearance of an interaction between $S\&I$ which is vitally important when $m > 2$ as well as for cases of increasing dimensions to examine the possibility to geometrize SIR models.

## 4 Geomertization of SIR Model

In a similar way, we can extend our study to examine SIR model using the Bazanski method in each Riemannian and Finslerian to become:

(i) For Riemannian Geometry

$$\frac{DS}{Dt} = 0, \tag{94}$$

$$\frac{DI}{Dt} = \alpha I, \tag{95}$$

and

$$\frac{DR}{Dt} = 0. \tag{96}$$

The above equation can easily be obtained by assuming the following Lagrangian:

$$\begin{aligned} L = g_{\mu\nu}{U_{SIR}}^{\mu}\frac{D{\Psi_{SIR}}^{\nu}}{Dt} + \alpha_{(\mu)}\Psi_{SIR\nu}{U_{SIR}}^{\nu}, \\ \mu\nu = 1, 2, 3 \end{aligned} \tag{97}$$

where $\alpha_{(\mu)}$ is an arbitrary constant.

From this perspective, we can develop the SIR model in its geometric version as follows

$$\frac{DU^{\mu}_{SIR}}{Dt} = \alpha_{(\mu)} U^{\mu}_{SIR}, \tag{98}$$

and their corresponding deviation equations become:

$$\begin{aligned} &\frac{D^2 \Psi^{\mu}_{SIR}}{Dt^2} + \alpha_{(\mu)} \frac{D\Psi^{\mu}_{SIR}}{Dt} \\ &= R^{\mu}_{\nu\rho\sigma} U^{\nu}_{SIR} U^{\rho}_{SIR} \Psi^{\sigma}_{SIR}. \end{aligned} \tag{99}$$

(ii) For Finslerian Geometry (Berwald Type):

In a similar way to equations [94-98] , we can obtain the following equations:

$$\frac{\hat{D}S}{\hat{D}t} = 0, \tag{100}$$

$$\frac{\hat{D}I}{\hat{D}t} = \bar{\alpha} I, \tag{101}$$

and

$$\frac{\hat{D}R}{\hat{D}t} = 0. \tag{102}$$

Equations [100-102] are obtained by taking the action of the following Largangian:

$$\begin{aligned} L = g_{\mu\nu} \hat{U}^{\mu}_{SIR} \frac{\hat{D}\hat{\Psi}^{\nu}_{SIR}}{\hat{D}t} + \alpha_{(\mu)} \hat{\Psi}_{\nu SIR} \hat{U}^{\nu}_{SIR}, \\ \mu, \nu = 1, 2, 3. \end{aligned} \tag{103}$$

We also can find the corresponding deviation equations to become:

$$\begin{aligned} &\frac{\hat{D}^2 \hat{\Psi}^{\mu}_{SIR}}{\hat{D}t^2} + \alpha_{(\mu)} \frac{\hat{D}\hat{\Psi}^{\mu}_{SIR}}{\hat{D}t} \\ &= B^{\mu}_{\nu\rho\sigma} U^{\nu}_{SIR} U^{\rho}_{SIR} \hat{\Psi}^{\sigma}{}_{SIR}. \end{aligned} \tag{104}$$

For a complete description of this model will be examined in our future work.

## 5 Discussion and Concluding Remarks

The paper deals with geometrizing some epidemic models using their corresponding path equations. Also, we have obtained their corresponding path deviation equations from one single Lagrangian for different types of geometries based under a positive definite metric with constant affine connections and a negative curvature. In this study we have begun with a geometrized version of the concept of allometry, to examine epidemic curves by finding their path and path deviation equations . The paper has also opened the window to impose non conventional types of geometries in future work to describe more complex version of SI or SIR models . One of good results is increasing the dimensions may describe other additive factors different from SIR-model to be geometrically expressed e.g. susceptive-exposed-infective-recovered model (SEIR) which can be expressed in 4-dimensional manifold with negative curvature .

This study can be extended to obtain such an appropriate version of epidemic curves functioning on long time periods without imposing too many parameters as in case of the traditional regression analysis. It is not the optimal case to get an exact form of epidemic curve, but it is a trend to apply such a concept of geometrization to understand nature.

Finally, in our future work we will be in need to develop these geometries to include some new parameters affecting the SI-model such as transmission rate [15]. Also, SIR model can also be extended to include some interactions like population fertility or immigration parameters to be considered in demographic using partial differential equations of McKendrick- von Forester [16]. Accordingly, the above mentioned new parameters in both SI and SIR models, may be defined geometrically in terms of non linear connections of the Finslerian approach .

## Appendix

**Derivation of Geodesic and geodesic deviation using the Bazanski Lagrangian**

Let

$$L = g_{\mu\nu} U^{\alpha} \frac{D\Psi^{\nu}}{Dt} \tag{A.1}$$

i.e.

$$L = g_{\mu\nu} U^{\alpha} \left(\frac{d\Psi^{\nu}}{dt} + \Gamma^{\alpha}_{\mu\nu} U^{\mu} \Psi^{\nu}\right)$$

where $U^{\alpha} = \frac{dx^{\alpha}}{d\tau}$ is defined to a tangent vector of a curve whose parameter is $\tau$.

(i) <u>Geodesic Equation</u>

Applying the action principle by taking the variation on $(A.1)$ with respect to $\dot{\Psi}^{\sigma}$ to get

$$\begin{aligned} \frac{\partial L}{\partial \dot{\Psi}^{\sigma}} &= g_{\mu\nu} \delta^{\nu}_{\sigma} U^{\mu}, \\ \frac{\partial L}{\partial \dot{\Psi}^{\sigma}} &= U_{\sigma}, \\ \frac{d}{d\tau} \frac{\partial L}{\partial \dot{\Psi}^{\sigma}} &= \dot{U}_{\sigma}. \end{aligned} \tag{A.2}$$

Also, one obtains

$$\begin{aligned} \frac{\partial L}{\partial \Psi^{\sigma}} &= g_{\mu\nu} \Gamma^{\nu}_{\alpha\beta} \delta^{\alpha}_{\sigma} U^{\beta} U^{\mu} \\ &= g_{\mu\nu} \Gamma^{\nu}_{\sigma\beta} U^{\beta} U_{\nu}. \end{aligned}$$

Thus the Euler-Lagrange equation becomes

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\Psi}^{\sigma}} - \frac{\partial L}{\partial \Psi^{\sigma}} = 0, \qquad (A.3)$$

which can be written in the following form

$$\frac{dU_{\sigma}}{d\tau} - \Gamma^{\nu}_{\sigma\beta}U^{\beta}U_{\nu} = 0$$

i.e.

$$\frac{DU_{\sigma}}{D\tau} = 0,$$

which becomes

$$\frac{Dg_{\sigma\mu}U^{\mu}}{D\tau} = 0,$$

$$U^{\mu}\frac{Dg_{\mu\nu}}{D\tau} + g_{\sigma\mu}\frac{DU^{\mu}}{D\tau} = 0.$$

It is well known that in Riemannian geometry, the covariant derivative of $g_{\mu\nu}$ vanishes identically.

Thus, one obtains

$$g_{\sigma\mu}\frac{DU^{\mu}}{D\tau} = 0. \qquad (A.4)$$

Multiplying both sides by $g^{\epsilon\sigma}$, one can find

$$g^{\epsilon\sigma}g_{\sigma\mu}\frac{DU^{\mu}}{D\tau} = 0,$$

$$\delta^{\epsilon}_{mu}\frac{DU^{\mu}}{D\tau} = 0,$$

$$\frac{DU^{\mu}}{D\tau} = 0, \qquad (A.5)$$

which is the well known equation of geodesic.

(ii) **Geodesic Deviation Equation**

The Lagrangian $(A.1)$ can be written as

$$L = U^{\mu}\frac{Dg_{\mu\nu}\Psi^{\nu}}{D\tau}$$

Consequently, by taking the variation on $(A.1)$ with respect to $U^{\sigma}$ to get

$$\begin{aligned}\frac{\partial L}{\partial U^{\sigma}} &= \dot{\Psi}_{\sigma} - \Gamma^{\rho}_{\sigma\lambda}\Psi_{\rho}U^{\lambda} - \Gamma^{\rho}_{\mu\sigma}U^{\mu}\Psi_{\rho} \\ &= \frac{D\Psi_{\sigma}}{D\tau} - \Gamma^{\rho}_{\mu\sigma}U^{\mu}\Psi_{\rho},\end{aligned} \qquad (A.6)$$

$$\begin{aligned}\frac{d}{d\tau}\frac{\partial L}{\partial U^{\sigma}} &= \frac{d}{d\tau}\frac{D\Psi_{\sigma}}{D\tau} - \Gamma^{\rho}_{\mu\sigma,\nu}\Psi_{\rho}U^{\nu}U^{\mu} \\ &\quad -\Gamma^{\rho}_{\mu\sigma}\frac{dU^{\mu}}{d\tau}\Psi_{\rho} - \Gamma^{\rho}_{\mu\sigma}\frac{d\Psi_{\sigma}}{d\tau}U^{\mu}.\end{aligned} \qquad (A.7)$$

Also, it is well known that

$$\frac{D\Psi_{\rho}}{D\tau} = \frac{d\Psi_{\rho}}{d\tau} - \Gamma^{\lambda}_{\rho\delta}U^{\delta}\Psi_{\rho},$$

and from geodesic equation

$$\frac{dU^{\mu}}{d\tau} = -\Gamma^{\mu}_{\rho\delta}U^{\rho}U^{\delta}$$

(A.7) becomes

$$\begin{aligned}\frac{d}{d\tau}\frac{\partial L}{\partial U^{\sigma}} &= \frac{d}{d\tau}\frac{D\Psi_{\sigma}}{D\tau} - \Gamma^{\rho}_{\mu\sigma,\nu}U^{\nu}U^{\mu}\Psi_{\rho} \\ &\quad -\Gamma^{\rho}\mu\sigma(\frac{dU^{\mu}}{d\tau}\Gamma^{\mu}_{\lambda\delta}U^{\lambda}U^{\delta})\Psi_{\rho} \\ &\quad -\Gamma^{\rho}_{\mu\sigma}(\frac{D\Psi_{\rho}}{D\tau} + \Gamma^{\lambda}_{\rho\delta}\Psi_{\lambda}U^{\delta})U^{\mu}\end{aligned}$$

and

$$\frac{\partial L}{\partial x^{\sigma}} = -\Gamma^{\rho}_{\mu\lambda,\sigma}U^{\mu}U^{\lambda}\Psi_{\rho}.$$

Thus, the Euler-Lagrange equation becomes

$$\frac{d}{d\tau}\frac{\partial L}{\partial U^{\sigma}} - \frac{\partial L}{\partial x^{\sigma}} = 0, \qquad (A.8)$$

which gives

$$\begin{aligned}&\frac{d}{d\tau}\frac{D\Psi_{\sigma}}{D\tau} - \Gamma^{\rho}_{\mu\sigma}\frac{D\Psi_{\rho}}{D\tau}U^{\mu} - \Gamma^{\rho}_{\mu\sigma,\nu}\Psi_{\rho}U^{\mu}U^{\nu} \\ &\quad -\Gamma^{\rho}_{\mu\sigma}\Gamma^{\lambda}_{\rho\delta}\Psi_{\lambda}U^{\mu}U^{\delta}\Psi_{\lambda}U^{\mu}U^{\delta} \\ &\quad +\Gamma^{\rho}_{\mu\sigma}\Gamma^{\mu}_{\lambda\delta}\Psi_{\rho}U^{\lambda}U^{\delta}\Psi_{\lambda}U^{\mu}U^{\delta} - \Gamma^{\rho}\mu\sigma\frac{DU^{\mu}}{D\tau}\Psi_{\rho} \\ &\quad +\Gamma^{\rho}_{\mu\lambda,\sigma}\Psi_{\rho}U^{\lambda}U^{\mu} = 0.\end{aligned}$$

Consequently, the above equation reduces to

$$\begin{aligned}&\frac{d}{d\tau}\frac{D\Psi_{\sigma}}{D\tau} - \Gamma^{\rho}_{\mu\sigma}\frac{D\Psi_{\rho}}{D\tau}U^{\mu} - \Gamma^{\rho}_{\mu\sigma,\nu}\Psi_{\rho}U^{\mu}U^{\nu} \\ &\quad +\Gamma^{\rho}_{\mu\lambda,\sigma}\Psi_{\rho}U^{\lambda}U^{\mu} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\mu}_{\lambda\delta}\Psi_{\rho}U^{\lambda}U^{\delta}\Psi_{\lambda}U^{\mu}U^{\delta} \\ &-\Gamma^{\rho}_{\mu\sigma,\nu}\Psi_{\rho}U^{\mu}U^{\nu} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\lambda}_{\rho\delta}\Psi_{\lambda}U^{\mu}U^{\delta}\Psi_{\lambda}U^{\mu}U^{\delta} = 0.\end{aligned}$$

Thus, after some manipulations, one obtains the geodesic deviation equations in following way

$$\frac{D^2\Psi^{\alpha}}{D\tau^2} = R^{\alpha}_{\nu\rho\mu}U^{\mu}U^{\nu}\Psi^{\rho}. \qquad (A.9)$$

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