# A stability result for a generalized trigonometric-quadratic functional equation with one unbounded function

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*Abstract:* A generalized trigonometric-quadratic functional equation of the form

$$\mathcal{F}(x+y)+\mathcal{G}(x-y)=2\mathcal{H}(x)\mathcal{K}(y)+\mathcal{L}(x)+\mathcal{M}(y)$$

over the domain of an abelian group and the range of the complex field is considered. Its stability is established based on the assumption that the function $\mathcal{K}$ is unbounded. Subject to certain natural conditions, explicit shapes of the functions $\mathcal{H}$ and $\mathcal{K}$ are determined. Several existing related results are derived as direct consequences.

*Key–Words:* Quadratic functional equation, trigonometric functional equation, stability, unboundedness, abelian group, additive function.

## 1 Introduction

In 1940, Ulam, [1], raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, Hyers, [2], proved that if $f : E_1 \to E_2$ is a mapping satisfying

$$\|f(x+y)-f(x)-f(y)\| \leqslant \delta$$

for all $x, y \in E_1$, where $E_1$ and $E_2$ are Banach spaces and $\delta$ is a given positive number, then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x)-T(x)\| \leqslant \delta$$

for all $x \in E_1$. If $f(x)$ is a real continuous function of $x$ over $\mathbb{R}$, and

$$|f(x+y)-f(x)-f(y)| \leqslant \delta,$$

it was showed by Hyers and Ulam, [3], that there exists a constant $k$ such that

$$|f(x)-kx| \leq 2\delta.$$

A (generalized) quadratic functional equation is a functional equation of the form

$$f_1(x+y)+f_2(x-y)=f_3(x)+f_4(y), \qquad (1)$$

which is so named because the quadratic function $x^2$ is a solution of a particular case, viz.,

$$f(x+y)+f(x-y)=2f(x)+2f(y), \qquad (2)$$

and any solution of (2) is referred to as a *quadratic* function. It is well known (cf. [4]) that a function $f : E_1 \to E_2$ between vector spaces is quadratic if and only if there exists a unique symmetric function $B : E_1 \times E_2 \to E_2$, which is additive in $x$ for each fixed $y$, such that $f(x) = B(x, x)$ for any $x \in E_1$.

The (Hyers-Ulam) stability of the quadratic functional equation was first proved by Skof (cf. [5]) in 1983 for functions from a normed space to a Banach space. In 1984, Cholewa (cf. [5]) showed that Skof's theorem is also valid if the normed space is replaced by an abelian group. His result says that: let $(\mathbb{G}, +)$ be an abelian group and $E$ a Banach space. If $f : \mathbb{G} \to E$ satisfies

$$\|f(x+y) + f(x-y) - 2f(x) + 2f(y)\| \le \delta \qquad (x, y \in \mathbb{G})$$

for some $\delta \ge 0$, then there exists a unique quadratic function $Q : \mathbb{G} \to E$ such that

$$\|f(x) - Q(x)\| \le \frac{\delta}{2} \qquad (x \in \mathbb{G}).$$

Later in [6] (see also [7]), Czerwik extended Cholewa's theorem by relaxing the control function and by considering functions from a normed space to a Banach space. His result reads: let $E_1$ be a normed space and $E_2$ a Banach space.

**I.** If $f : E_1 \to E_2$ satisfies

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \delta + \theta\left(\|x\|^p + \|y\|^p\right) \quad (x, y \in E_1 \setminus \{0\})$$

for some nonnegative real numbers $\delta, \theta, p < 2$, then there exists a unique quadratic function $Q : E_1 \to E_2$ such that

$$\|f(x) - Q(x)\| \le \frac{\delta + c}{3} + \frac{2\theta}{4 - 2^p}\|x\|^p \qquad (x \in E_1 \setminus \{0\}).$$

**II.** If $f : E_1 \to E_2$ satisfies

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \theta\left(\|x\|^p + \|y\|^p\right) \quad (x, y \in E_1)$$

for some $\theta \ge 0, p > 2$, then there exists a unique quadratic function $Q : E_1 \to E_2$ such that

$$\|f(x) - Q(x)\| \le \frac{2\theta}{2^p - 4}\|x\|^p \quad (x \in E_1).$$

If $p = 2$, then the result is no longer valid.

In another paper [8], Czerwik investigated the stability problem of the 'partially pexiderized'quadratic functional equation

$$f_1(x+y) + f_1(x-y) = f_2(x) + f_2(y).$$

In 2000, Jung, [9], generalized the result of Czerwik [6] using ideas from [10] and [11]. As an application, he used this result together with a theorem from [12] to determine the stability of the quadratic equation of pexider type.

**Theorem 1.1.** *Let $E_1$ be a normed space, $E_2$ a Banach space, and let $\varphi : E_1 \times E_2 \to [0, \infty)$ be a given function with the properties*

***(i)*** *$\varphi(y, x) = \varphi(x, y)$;*

***(ii)*** *$\varphi(x, -y) = \varphi(x, y)$;*

***(iii)*** *there is an integer $k \ge 2$ such that*

$$\sum_{i=0}^{\infty} \frac{1}{k^i}\varphi\left(k^i x, k^i y\right) < \infty$$

*or*

$$\sum_{i=0}^{\infty} k^{2i}\varphi\left(\frac{x}{k^i}, \frac{y}{k^i}\right) < \infty \quad (x, y \in E_1).$$

*If $f_1, f_2, f_3$ and $f_4$ are functions from $E_1$ to $E_2$ and satisfy*

$$\|f_1(x+y) + f_2(x-y) - f_3(x) - f_4(y)\| \le \varphi(x, y) \qquad (x, y \in E_1), \tag{3}$$

*then there exist a quadratic function $Q : E_1 \to E_2$ and additive functions $A_1, A_2 : E_1 \to E_2$ such that, for all $x \in E_1$, we have*

$$\begin{aligned}
&\|f_1(x) - Q(x) - A_1(x) - A_2(x) - f_1(0)\| \\
&\le \frac{1}{2k^2}\Phi_k(x, x) + \frac{1}{2k}\Phi'_k(x, x) + \frac{1}{2k}\Phi''_k(x, x) \\
&+ 3\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{5}{2}\varphi(x, 0) + \frac{11}{2}\varphi(0, 0), \\
&\|f_2(x) - Q(x) - A_1(x) - A_2(x) - f_2(0)\| \\
&\le \frac{1}{2k^2}\Phi_k(x, x) + \frac{1}{2k}\Phi'_k(x, x) + \frac{1}{2k}\Phi''_k(x, x) \\
&+ 3\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{5}{2}\varphi(x, 0) + \frac{7}{2}\varphi(0, 0), \\
&\|f_3(x) - 2Q(x) - 2A_1(x) - f_3(0)\| \\
&\le \frac{1}{k^2}\Phi_k(x, x) + \frac{1}{k}\Phi'_k(x, x) + 2\varphi(x, 0) + 2\varphi(0, 0), \\
&\|f_4(x) - 2Q(x) - 2A_2(x) - f_4(0)\| \\
&\le \frac{1}{k^2}\Phi_k(x, x) + \frac{1}{k}\Phi''_k(x, x),
\end{aligned}$$

*where* $\Phi_k(x,y)$, $\Phi'_k(x,y)$, $\Phi''_k(x,y)$ *are well-determined functions depending on* $\varphi(x,y)$.

*Moreover, if* $f_3(tx)$ *and* $f_4(tx)$ *are continuous in* $t \in \mathbb{R}$ *for each* $x \in E_1$, *then the function* $Q$ *satisfies*

$$Q(tx) = t^2 Q(x) \qquad (x \in E_1)$$

*and* $A_1, A_2$ *are linear.*

In the next year, Jung and Sahoo, [13], used ideas from [9] and [14] to prove a similar stability result of a quadratic functional equation:

**Theorem 1.2.** *If functions* $f_1, f_2, f_3, f_4 : E_1 \to E_2$ *satisfy the inequality*

$$\|f_1(x+y) + f_2(x-y) - f_3(x) - f_4(y)\| \le \varepsilon$$

*for some* $\epsilon \ge 0$ *and for all* $x, y \in E_1$ *then there exists a unique quadratic function* $Q : E_1 \to E_2$ *and exactly two addition functions* $A_1, A_2 : E_1 \to E_2$ *such that*

$$\|f_1(x) - Q(x) - A_1(x) - A_2(x) - f_1(0)\| \le \frac{137}{3}\varepsilon$$
$$\|f_2(x) - Q(x) - A_1(x) - A_2(x) - f_2(0)\| \le \frac{125}{3}\varepsilon$$
$$\|f_3(x) - 2Q(x) - 2A_1(x) - f_3(0)\| \le \frac{136}{3}\varepsilon$$
$$\|f_4(x) - 2Q(x) - 2A_2(x) - f_4(0)\| \le \frac{124}{3}\varepsilon$$

*for all* $x \in E_1$. *Moreover, if* $f_3(tx)$ *and* $f_4(tx)$ *are continuous in* $t \in \mathbb{R}$ *for each* $x \in E_1$, *then the function* $Q$ *satisfies*

$$Q(tx) = t^2 Q(x) \qquad (x \in E_1)$$

*and* $A_1, A_2$ *are linear.*

A (generalized) trigonometric functional equation is a functional equation of the form

$$F(x+y) - G(x-y) = 2H(x)K(y), \qquad (4)$$

which is so named because the two best known trigonometric functions, sine and cosine, are solutions of two special cases of this equation. In [15], the authors investigated the stability of (4) where $F, G, H, K$ are nonzero functions from an abelian group $(\mathbb{G}, +)$ to the complex field $\mathbb{C}$. To state this result, recall that by an additive (respectively, exponential) function $A$ (respectively, $E$) we refer to a function $A$ (respectively, $E$) satisfying the additive (respectively, exponential) Cauchy functional equation

$$A(x+y) = A(x) + A(y)$$

respectively,

$$E(x+y) = E(x)E(y)$$

for all $x, y$ belonging to the domain of $A$ (respectively, $E$). The main result in [15] is:

**Theorem 1.3.** ***I.*** *Let* $F, G, H$ *and* $K$ *be nonzero functions from an abelian group* $(\mathbb{G}, +)$ *to the complex field* $\mathbb{C}$ *and* $\psi : \mathbb{G} \to [0, \infty)$. *Suppose that* $F, G, H, K$ *satisfy*

$$|F(x+y) - G(x-y) - 2H(x)\ K(y)| \le \psi(x) \qquad (x, y \in \mathbb{G}). \qquad (5)$$

*Then either*
***(I.i)*** $K$ *is bounded, or*
***(I.ii)*** *there is a sequence* $\{y_n\} \subset \mathbb{G}$ *such that the limit*

$$\ell_K(y) := \lim_{n\to\infty} \frac{K(y_n + y) + K(y_n - y)}{K(y_n)}$$

*exists for each* $y \in \mathbb{G}$, *and* $H$ *satisfies the functional equation*

$$H(x+y) + H(x-y) = H(x)\ell_K(y) \qquad (x, y \in \mathbb{G}).$$

*Assume* ***(I.ii)*** *holds.*
***(I.iia)*** *If* $K$ *satisfies the functional equation*

$$K(x+y) + K(x-y) = 2K(x)K(y),$$

*then* $H, K$ *are solutions of the functional equation*

$$f(x+y) + f(x-y) = 2f(x)g(y),$$

*and are given by*

$$K(x) = \frac{E(x) + E^*(x)}{2},$$
$$H(x) = \frac{k\,(E(x) + E^*(x))}{2} + c\,(E(x) - E^*(x)),$$

*where* $k, c \in \mathbb{C}$, $E$ *is an exponential function and* $E^*(x) = 1/E(x)$;
***(I.iib)*** *If* $H(0) = 0$ *and* $\mathbb{G}$ *is a 2-divisible abelian group, then* $H$ *satisfies the functional equation*

$$H\left(\frac{x+y}{2}\right)^2 - H\left(\frac{x-y}{2}\right)^2 = H(x)H(y)$$

*and is of the form*

$$H(x) = A(x)$$

*or*

$$H(x) = c\,(E(x) - E^*(x))$$

*where $A$ is an additive function, $c, E$ and $E^*$ are as in (II.iia).*

***II.*** *If $F, G, H, K$ satisfy*

$$|F(x+y) - G(x-y) - 2H(x)K(y)| \leq \psi(y), \tag{6}$$

*then*
***(II.i)*** *$H$ is bounded, or*
***(II.ii)*** *there is a sequence $\{y_n\} \subset \mathbb{G}$ such that the limit*

$$\ell_H(y) := \lim_{n\to\infty} \frac{H(y_n+y) + H(y_n-y)}{H(y_n)}$$

*exists for each $y \in \mathbb{G}$, and $K$ satisfies the functional equation*

$$K(x+y) + K(x-y) = K(x)\ell_H(y) \qquad (x, y \in \mathbb{G}).$$

*Assume **(II.ii)** holds.*
***(II.iia)*** *If $H$ satisfies the functional equation*

$$H(x+y) + H(x-y) = 2H(x)H(y),$$

*then $H, K$ are solutions of the functional equation*

$$f(x+y) + f(x-y) = 2f(x)g(y)$$

*and are given by*

$$H(x) = \frac{E(x) + E^*(x)}{2},$$
$$K(x) = \frac{k\left(E(x) + E^*(x)\right)}{2} + c\left(E(x) - E^*(x)\right),$$

*where $k, c \in \mathbb{C}$, $E$ is an exponential function and $E^*(x) = 1/E(x)$;*

***(II.iib)*** *If $K(0) = 0$, and $\mathbb{G}$ is a 2-divisible abelian group, then $K$ satisfies the functional equation*

$$K\left(\frac{x+y}{2}\right)^2 - K\left(\frac{x-y}{2}\right)^2 = K(x)K(y)$$

*and is given by*

$$K(x) = A(x),$$

*or*

$$H(x) = c\left(E(x) - E^*(x)\right),$$

*where $A$ is an additive function, $c, E$ and $E^*$ are as in (II.iia).*

Later in the same year, [16], a generalized trigonometric functional equation with either the function $H$ or $K$ being bounded, was investigated to complement Theorem 1.3, where such function boundedness was not treated. The result so obtained is:

**Theorem 1.4.** *Let $(\mathbb{G}, +)$ be an abelian group, $\alpha \in \mathbb{G}, \psi : \mathbb{G} \times \mathbb{G} \to [0, \infty)$ and let*

$$\varphi_r(y, x) := \varphi(x, y) + \varphi(-x, -y).$$

*If $F, G, H, K$ satisfy*

$$|F(x+y) - G(x-y) - 2H(x)\ K(y)| \leq \varphi(x, y) \qquad (x, y \in \mathbb{G}). \tag{7}$$

*Then*

$$\begin{aligned}|H(\alpha)K(x) &- H(x)K(\alpha) \\ &+ H(-\alpha)K(-x) - H(-x)K(-\alpha)| \\ &\leq \frac{1}{2}\left\{\varphi_r(x, \alpha) + \varphi_r(\alpha, x)\right\};\end{aligned} \tag{8}$$

*in particular, if $\alpha = 0$, then*

$$\begin{aligned}|H(0)\{K(x) &+ K(-x)\} - \{H(x) + H(-x)\}K(0)| \\ &\leq \frac{1}{2}\left\{\varphi_r(x, 0) + \varphi_r(0, x)\right\}.\end{aligned} \tag{9}$$

Theorem 1.4 shows that the boundedness of one function, either $H$ or $K$, essentially implies the boundedness of the even part of the other, viz., we have

**Corollary 1.5.** *Let the notation be as in Theorem 1.4, and let $M_1, M_2$ be positive real numbers. Assume that $F, G, H, K$ satisfy (7).*
***I.*** *If $|H(x)| \leq M_1 \quad (x \in \mathbb{G})$, then*

$$\begin{aligned}|H(\alpha)K(x) &+ H(-\alpha)K(-x)| \\ &\leq \frac{1}{2}\left\{\varphi_r(x, \alpha) + \varphi_r(\alpha, x)\right\} \\ &\quad + M_1\left\{|K(\alpha)| + |K(-\alpha)|\right\}.\end{aligned}$$

***II.*** *If $|K(x)| \leq M_2 \quad (x \in \mathbb{G})$, then*

$$\begin{aligned}|K(\alpha)H(x) &+ K(-\alpha)H(-x)| \\ &\leq \frac{1}{2}\left\{\varphi_r(x, \alpha) + \varphi_r(\alpha, x)\right\} \\ &\quad + M_2\left\{|H(\alpha)| + |H(-\alpha)|\right\}.\end{aligned}$$

If we specialize the control function $\varphi$ in Theorem 1.4 to be functions of a single variable ($x$ or $y$ ) or a constant, we get

**Corollary 1.6.** *I. Let $\psi : \mathbb{G} \to [0, \infty)$, let $M_1, M_2$ be positive real numbers, and let*

$$\psi_r(x) := \psi(x) + \psi(-x).$$

*If $F, G, H, K$ satisfy*

$$|F(x+y) - G(x-y) - 2H(x)K(y)| \leq \psi(x) \text{ or } \psi(y) \qquad (x, y \in \mathbb{G}),$$

*then*

$$|H(\alpha)K(x) - H(x)K(\alpha) + H(-\alpha)K(-x) - H(-x)K(-\alpha)| \leq \frac{1}{2}\{\psi_r(x,\alpha) + \psi_r(\alpha,x)\}.$$

*Moreover, if* $|H(x)| \leq M_1 \quad (x \in \mathbb{G})$*, then*

$$|H(\alpha)K(x) + H(-\alpha)K(-x)| \leq \frac{1}{2}\{\psi_r(x,\alpha) + \psi_r(\alpha,x)\} + M_1\{|K(\alpha)| + |K(-\alpha)|\};$$

*while if* $|K(x)| \leq M_2 \quad (x \in \mathbb{G})$*, then*

$$|K(\alpha)H(x) + K(-\alpha)H(-x)| \leq \frac{1}{2}\{\psi_r(x,\alpha) + \psi_r(\alpha,x)\} + M_2\{|H(\alpha)| + |H(-\alpha)|\}.$$

***II.*** *Let* $\delta$ *be a positive real number. If* $F, G, H, K$ *satisfy*

$$|F(x+y) - G(x-y) - 2H(x)\ K(y)| \leq \delta \quad (x, y \in \mathbb{G}), \quad (10)$$

*then*

$$|H(\alpha)K(x) - H(x)K(\alpha) + H(-\alpha)K(-x) - H(-x)K(-\alpha)| \leq 2\delta.$$

*Moreover, if* $|H(x)| \leq M_1 \quad (x \in \mathbb{G})$*, then*

$$|H(\alpha)K(x) + H(-\alpha)K(-x)| \leqslant 2\delta + M_1\{|K(\alpha)| + |K(-\alpha)|\};$$

*while if* $|K(x)| \leq M_2 \quad (x \in \mathbb{G})$*, then*

$$|K(\alpha)H(x) + K(-\alpha)H(-x)| \leqslant 2\delta + M_2\{|H(\alpha)| + |H(-\alpha)|\}.$$

Motivated by the above works, it is natural to ask whether there is a similar stability result along the same line as in Theorem 1.3 if the generalized trigonometric and quadratic functional equations are combined together. We give here a positive answer to this question. Our generalized trigonometric-quadratic functional equation takes the form

$$\mathcal{F}(x+y) + \mathcal{G}(x-y) = 2\mathcal{H}(x)\mathcal{K}(y) + \mathcal{L}(x) + \mathcal{M}(y). \quad (11)$$

As elaborated in Section 4.2.4, pp.196-201 of [4], a general differentiable solution of (11) can be found by a method due to Levi-Civita, and the solution functions are mostly exponential polynomials.

There are cases that can be put under the above two results (Theorems 1.3 and 1.1) which we describe in the following two remarks.

- Taking the domain as an abelian group $(\mathbb{G}, +)$ and the complex field $\mathbb{C}$ as the range. If $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}, \mathcal{L}, \mathcal{M}$ satisfy
$$|\mathcal{F}(x+y) + \mathcal{G}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y) - \mathcal{L}(x) - \mathcal{M}(y)| \leq \psi(x) \text{ or } \psi(y) \quad (12)$$
with both $\mathcal{L}$ and $\mathcal{M}$ being bounded functions, then the inequality (12) can be put under the form (5) or (6), and the stability results of parts I and II in Theorem 1.3 apply.
- Taking the domain as a normed space $E_1$ and a Banach space $E_2$, with norm $\|\cdot\|$, as the range.
If $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}, \mathcal{L}, \mathcal{M}$ satisfy
$$\|\mathcal{F}(x+y) + \mathcal{G}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y) - \mathcal{L}(x) - \mathcal{M}(y)\| \leq \phi(x,y) \quad (13)$$
with either $\mathcal{H}$ or $\mathcal{K}$ being bounded functions, then the inequality (13) can be put under the form (3) and the stability results in Theorem 1.1 apply.

Since Theorems 1.3 and 1.1 have different domains and ranges, to put both of them into one perspective, we have adopted to take, in our main result and throughout the rest of the paper, an abelian group $(\mathbb{G}, +)$ as the domain and the complex field $\mathbb{C}$ as the range of all functions involved, except the control function $\phi$ whose range is taken to be the nonnegative real numbers $[0, \infty)$. We now state our main result.

**Theorem 1.7.** *Let* $(\mathbb{G}, +)$ *be an abelian group, and* $\phi : \mathbb{G} \to [0, \infty)$.
***A.*** *If* $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}, \mathcal{L}, \mathcal{M} : \mathbb{G} \to \mathbb{C}$ *satisfy*

$$|\mathcal{F}(x+y) + \mathcal{G}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y) - \mathcal{L}(x) - \mathcal{M}(y)| \leq \phi(x) \qquad (x, y \in \mathbb{G}), \quad (14)$$

*and* $\mathcal{K}$ *is not bounded, then there is a sequence* $\{y_n\} \subset \mathbb{G}$ *such that the following two limits exist for all* $x, y \in \mathbb{G}$,

$$\mathcal{H}(x) = \lim_{n\to\infty} \frac{\mathcal{F}(x+y_n) + \mathcal{G}(x-y_n) - \mathcal{M}(y_n)}{2\mathcal{K}(y_n)},$$

$$\Im_{\mathcal{K},\mathcal{M}}(x,y) := \lim_{n\to\infty} \frac{1}{2\mathcal{K}(y_n)}\{2\mathcal{H}(x)\left(\mathcal{K}(y_n+y) + \mathcal{K}(y_n-y)\right) + \mathcal{M}(y_n+y) + \mathcal{M}(y_n-y) - 2\mathcal{M}(y_n)\}$$

*and the function* $\mathcal{H}$ *satisfies the functional equation*

$$\mathcal{H}(x+y) + \mathcal{H}(x-y) = \Im_{\mathcal{K},\mathcal{M}}(x,y) \quad (15)$$

***B1.*** *Assuming part* ***A****, if either the limit*

$$\ell_{\mathcal{K}}(y) := \lim_{n\to\infty} \frac{\mathcal{K}(y_n + y) + \mathcal{K}(y_n - y)}{\mathcal{K}(y_n)}$$

*or the limit*

$$\ell_{\mathcal{M},\mathcal{K}}(y) := \lim_{n\to\infty} \frac{\mathcal{M}(y_n + y) + \mathcal{M}(y_n - y) - 2\mathcal{M}(y_n)}{2\mathcal{K}(y_n)}$$

*exists for all $y \in \mathbb{G}$, then the functional equation (15) simplifies to*

$$\mathcal{H}(x + y) + \mathcal{H}(x - y) = \mathcal{H}(x)\ell_{\mathcal{K}}(y) + \ell_{\mathcal{M},\mathcal{K}}(y) \quad (x, y \in \mathbb{G}). \tag{16}$$

***B2.*** *Assuming part* ***A****, if $\mathcal{K}$ satisfies the equation*

$$\mathcal{K}(x + y) + \mathcal{K}(x - y) = 2\mathcal{K}(x)\mathcal{K}(y),$$

*then the limit*

$$\ell_{\mathcal{M},\mathcal{K}}(y) := \lim_{n\to\infty} \frac{\mathcal{M}(y_n+y) + \mathcal{M}(y_n-y)-2\mathcal{M}(y_n)}{2\mathcal{K}(y_n)}$$

*exists for each $y \in \mathbb{G}$, and the function $\mathcal{H}$ satisfies the functional equation*

$$\mathcal{H}(x + y) + \mathcal{H}(x-y) = 2\mathcal{H}(x)\mathcal{K}(y)+\ell_{\mathcal{M},\mathcal{K}}(y).$$

***C1.*** *Suppose parts* ***A*** *and* ***B1*** *hold.*
***I.*** *When $\mathbb{G} = \mathbb{R}$, if $\mathcal{H}$, $\ell_{\mathcal{K}}$, $\ell_{\mathcal{M},\mathcal{K}}$ are differentiable, and either*

$$\ell'_{\mathcal{K}}(0) \neq 0 \text{ or } \ell'_{\mathcal{M},\mathcal{K}}(0) \neq 0,$$

*then*

$$\mathcal{H}(x) \equiv h \in \mathbb{C}$$

*is a constant function.*
*Furthermore,*
***(Ia)*** *if $\ell'_{\mathcal{K}}(0) \neq 0$ and $h = 0$ then*

$$\ell_{\mathcal{M},\mathcal{K}}(y) \equiv 0,$$

*and $\ell_{\mathcal{K}}$ is an arbitrary function;*
***(Ib)*** *if $\ell'_{\mathcal{K}}(0) \neq 0$ and $h \neq 0$ then*

$$\ell_{\mathcal{M},\mathcal{K}}(y) = (2 - \ell_{\mathcal{K}}(y))h,$$

*and is an arbitrary function;*
***(Ic)*** *the case where $\ell'_{\mathcal{M},\mathcal{K}}(y) \neq 0$ and $h = 0$ never occurs;*
***(Id)*** *if $\ell'_{\mathcal{M},\mathcal{K}}(y) \neq 0$ and $h \neq 0$ then*

$$\ell_{\mathcal{M},\mathcal{K}}(y) = (2 - \ell_{\mathcal{K}}(y))h,$$

*and $\ell_{\mathcal{K}}$ is an arbitrary function.*

***II.*** *When $\mathbb{G}$ is a 2-divisible abelian group, if*

$$\ell_{\mathcal{K}}(y) \equiv c \in \mathbb{C}$$

*is a constant function, then*
***(IIa)*** *for $c = 2$, we have*

$$\mathcal{H}(x) = B(x, x) + A(x) + r, \ell_{\mathcal{M},\mathcal{K}}(x) = 2B(x, x),$$

*where $B : \mathbb{G} \times \mathbb{G} \to \mathbb{C}$ is a symmetric biadditive function, $A : \mathbb{G} \to \mathbb{C}$ is an additive function and $r \in \mathbb{C}$;*
***(IIb)*** *for $c \neq 2$, we have*

$$\mathcal{H}(x) \equiv r, \ell_{\mathcal{M},\mathcal{K}}(x) \equiv r(2 - c) \qquad (r \in \mathbb{C})$$

*being two constant functions.*

***C2.*** *Suppose parts A and B2 hold.*
***I.*** *If $\mathcal{M}$ is quadratic (i.e.,*

$$\mathcal{M}(x + y) + \mathcal{M}(x - y) = 2\mathcal{M}(x) + 2\mathcal{M}(y))$$

*or if $\mathcal{M}$ is additive, then the functions $\mathcal{H}$ and $\mathcal{K}$ satisfies the functional equation*

$$\mathcal{H}(x + y) + \mathcal{H}(x - y) = 2\mathcal{H}(x)\mathcal{K}(y)$$

*and are given by*

$$\mathcal{K}(x) = \frac{E(x) + E^*(x)}{2},$$
$$\mathcal{H}(x) = \frac{k\,(E(x) + E^*(x))}{2} + c\,(E(x) - E^*(x)),$$

*where $k, c \in \mathbb{C}, E$ is an exponential function and $E^*(x) = 1/E(x)$.*

For related results, see also [17], [18], [19], [20].

## 2 Some Preliminaries

Since the proof of Theorem 1.7 leads us to solve certain special cases of the functional equation (11), for convenience, we solve these special cases in this section. Our approach is similar to the one in [21] (see also [22], [23], [24]).

**Proposition 2.1.** *Let $\mathcal{H}, \mathcal{K}, \ell : \mathbb{R} \to \mathbb{C}$ be differentiable functions. If $\mathcal{K}'(0) \neq 0$ or $\ell'(0) \neq 0$ and if $\mathcal{H}, \mathcal{K}, \ell$ satisfy*

$$\mathcal{H}(x + y) + \mathcal{H}(x - y) = \mathcal{H}(x)\mathcal{K}(y) + \ell(y) \quad (x, y \in \mathbb{R}), \tag{17}$$

*then*

$$\mathcal{H}(x) = h \in \mathbb{C}.$$

*Moreover,*
***(i)*** *if $\mathcal{K}'(0) \neq 0$ and $h = 0$ then*

$$\ell(y) \equiv 0,$$

*and $\mathcal{K}$ is an arbitrary function;*
***(ii)*** *if $\mathcal{K}'(0) \neq 0$ and $h \neq 0$ then*

$$\ell(y) = (2 - \mathcal{K}(y))h,$$

*and $\mathcal{K}$ is an arbitrary function;*
***(iii)*** *the case where $\ell'(0) \neq 0$ and $h = 0$ never occurs;*
***(iv)*** *if $\ell'(0) \neq 0$ and $h \neq 0$, then*

$$\ell(y) = (2 - \mathcal{K}(y))h,$$

*and $\mathcal{K}$ is an arbitrary function.*

*Proof.* Differentiating (17) with respect to $x$ and to $y$, we obtain

$$\mathcal{H}'(x+y) + \mathcal{H}'(x-y) = \mathcal{H}'(x)\mathcal{K}(y) \quad (18)$$

$$\mathcal{H}'(x+y) - \mathcal{H}'(x-y) = \mathcal{H}(x)\mathcal{K}'(y) + \ell'(y). \quad (19)$$

Adding and subtracting (18) and (19), we get

$$2\mathcal{H}'(x+y) = \mathcal{H}'(x)\mathcal{K}(y) + \mathcal{H}(x)\mathcal{K}'(y) + \ell'(y) \quad (20)$$

$$2\mathcal{H}'(x-y) = \mathcal{H}'(x)\mathcal{K}(y) - \mathcal{H}(x)\mathcal{K}'(y) - \ell'(y). \quad (21)$$

Putting $y = 0$ in (20) and (21), we respectively get

$$2\mathcal{H}'(x) = \mathcal{H}'(x)\mathcal{K}(0) + \mathcal{H}(x)\mathcal{K}'(0) + \ell'(0) \quad (22)$$

$$2\mathcal{H}'(x) = \mathcal{H}'(x)\mathcal{K}(0) - \mathcal{H}(x)\mathcal{K}'(0) - \ell'(0). \quad (23)$$

Adding (22) and (23) leads to

$$4\mathcal{H}'(x) = 2\mathcal{H}'(x)\mathcal{K}(0). \quad (24)$$

We now consider two separate cases.
• If $\mathcal{K}(0) \neq 2$, by (24), we get

$$\mathcal{H}'(x) \equiv 0.$$

• If $\mathcal{K}(0) = 2$, by (22), we get

$$\mathcal{H}(x)\mathcal{K}'(0) = -\ell'(0)$$

and since $\mathcal{K}'(0) \neq 0$ or $\ell'(0) \neq 0$, we see that $\mathcal{H}$ is constant.
In both cases, we deduce that

$$\mathcal{H}(x) = h \in \mathbb{C},$$

a constant function. There are four possibilities:
**(i)** If $\mathcal{K}'(0) \neq 0$ and $h = 0$ then (17) yields

$$\ell(y) \equiv 0,$$

and $\mathcal{K}$ is an arbitrary function.
**(ii)** If $\mathcal{K}'(0) \neq 0$ and $h \neq 0$ then (17) yields

$$\ell(y) = (2 - \mathcal{K}(y))h,$$

and $\mathcal{K}$ is an arbitrary function.
**(iii)** If $\ell'(0) \neq 0$ and $h = 0$, then (17) yields

$$\ell(y) \equiv 0$$

which contradicts $\ell'(0) \neq 0$, and so this situation does not occur.
**(iv)** If $\ell'(0) \neq 0$ and $h \neq 0$, then (17) yields

$$\ell(y) = (2 - \mathcal{K}(y))h,$$

and $\mathcal{K}$ is an arbitrary function. □

Note that the general solution of (1) can be found in Theorem 4.25, pp.238-240 of [5], which we now quote.

**Lemma 2.2.** *Let $\mathbb{G}$ be a 2-divisible group and let $\mathbb{F}$ be a commutative field of characteristic $\neq 2$. The general solution of (1) with $f_1, f_2$ satisfying*

$$f_i\,(x+y+z) = f_i\,(x+z+y) \;\; (i = 1,2; x,y,z \in \mathbb{G})$$

*is given by*

$$f_1\,(x) = B\,(x,\ x) - A_1\,(x) + A_2\,(x) + b_1,$$
$$f_2\,(x) = B\,(x,\ x) - A_1\,(x) - A_2\,(x) + b_2,$$
$$f_3\,(x) = 2B\,(x,\ x) - 2A_1\,(x) + b_3,$$
$$f_4\,(x) = 2B\,(x,\ x) + 2A_2\,(x) + b_4,$$

*with*

$$b_1 + b_2 = b_3 + b_4,$$

*where $B : \mathbb{G} \times \mathbb{G} \to \mathbb{F}$ is a symmetric bi-additive function and $A_i : \mathbb{G} \to \mathbb{F}$ $(i = 1,2)$ are additive.*

Making use of Lemma 2.2, we have:

**Proposition 2.3.** *Let $\mathcal{H}, \ell$ be functions from a 2-divisible abelian group $\mathbb{G}$ to a field $\mathbb{F}$ of characteristic $\neq 2$ and $c \in \mathbb{F}$. Assume that*

$$\mathcal{H}(x+y) + \mathcal{H}(x-y) = c\mathcal{H}(x) + \ell(y) \quad (x,y \in \mathbb{G}).$$

• *If $c = 2$, then*

$$\mathcal{H}(x) = B(x,x) + A(x) + r, \ell(x) = 2B(x,x),$$

*where $B : \mathbb{G} \times \mathbb{G} \to \mathbb{F}$ is a symmetric bi-additive function, $A : \mathbb{G} \to \mathbb{F}$ is an additive function and $r \in \mathbb{F}$.*

• *If $c \neq 2$, then*

$$\mathcal{H}(x) \equiv r, \quad \ell(x) \equiv r(2-c) \qquad (r \in \mathbb{F})$$

*are constant functions.*

*Proof.* Substituting

$$f_1(x) = f_2(x) = \mathcal{H}(x),\ f_3(x) = c\mathcal{H}(x),\ f_4(y) = \ell(y)$$

in Lemma 2.2, we get

$$\mathcal{H}(x) = B(x,x) - A_1(x) + A_2(x) + b_1 \qquad (25)$$

$$\mathcal{H}(x) = B(x,x) - A_1(x) - A_2(x) + b_2 \qquad (26)$$

$$c\mathcal{H}(x) = 2B(x,x) - 2A_1(x) + b_3 \qquad (27)$$

$$\ell(x) = 2B(x,x) + 2A_2(x) + b_4 \qquad (28)$$

with

$$b_1 + b_2 = b_3 + b_4,$$

where $B$ is a symmetric bi-additive function and $A_1, A_2$ are additive functions. Equating (25) and (26), we get

$$2A_2(x) = b_2 - b_1,$$

a constant function. Since $A_2$ is additive, we must have $A_2(x) \equiv 0$ and so $b_1 = b_2$. Putting this information back into (25), we get

$$\mathcal{H}(x) = B(x,x) - A_1(x) + b_1.$$

Multiplying this last equation by $c$, equating with (27) and using $b_3 = 2b_1 - b_4$, we get

$$c(B(x,x) - A_1(x) + b_1) = 2B(x,x) - 2A_1(x) + 2b_1 - b_4. \qquad (29)$$

Consider now two possible cases.

• If $c = 2$, then (29) yields $b_4 = 0$ and (28) gives

$$\ell(x) = 2B(x,x).$$

• If $c \neq 2$, then (2.13) yields

$$B(x,x) - A_1(x) = \frac{b_4}{2-c} - b_1, \qquad (30)$$

and so

$$B(x+y,x+y) - A_1(x+y) = \frac{b_4}{2-c} - b_1. \qquad (31)$$

Since $B$ is symmetric bi-additive and $A_1$ is additive, the relation (31) implies

$$\frac{b_4}{2-c} - b_1 = B(x+y,x+y) - A_1(x+y)$$
$$= (B(x,x) - A_1(x)) + (B(y,y) - A_1(y)) + 2B(x,y).$$

Replacing the expression on the right-hand side using (28), we get

$$\frac{b_4}{2-c} - b_1 = \frac{b_4}{2-c} - b_1 + \frac{b_4}{2-c} - b_1 + 2B(x,y),$$

which shows that $B$ is constant, and so

$$B \equiv 0$$

(because $B$ is symmetric biadditive). Thus, (30) shows that $A_1$ is constant and so

$$A_1 \equiv 0$$

(because $A_1$ is additive). Consequently,

$$b_4 = b_1(2-c), \mathcal{H}(x) = b_1, \ell(x) = b_1(2-c).$$

□

# 3 Proof of Theorem 1.7

**A.** Since $\mathcal{K}$ is not bounded, there is a sequence $\{y_n\} \subset \mathbb{G}$ such that

$$0 \neq |\mathcal{K}(y_n)| \to \infty (n \to \infty).$$

Substituting $y = y_n$ in (14) and dividing by $2|\mathcal{K}(y_n)|$, we get

$$\left| \frac{\mathcal{F}(x+y_n) + \mathcal{G}(x-y_n) - \mathcal{M}(y_n)}{2\mathcal{K}(y_n)} - \frac{\mathcal{L}(x)}{2\mathcal{K}(y_n)} - \mathcal{H}(x) \right| \leq \frac{\phi(x)}{|2\mathcal{K}(y_n)|}.$$

Thus,

$$\lim_{n\to\infty} \frac{\mathcal{F}(x+y_n) + \mathcal{G}(x-y_n) - \mathcal{M}(y_n)}{2\mathcal{K}(y_n)} = \mathcal{H}(x). \qquad (32)$$

Replacing $y$ by $y_n \pm y$ in (14), we respectively obtain

$$|\mathcal{F}((x+y)+y_n) + \mathcal{G}((x-y)-y_n)$$
$$-2\mathcal{H}(x)\mathcal{K}(y_n+y) - \mathcal{L}(x) - \mathcal{M}(y_n+y)|$$
$$\leq \phi(x), \qquad (33)$$

and

$$|\mathcal{F}((x-y)+y_n)+\mathcal{G}((x+y)-y_n) \\ -2\mathcal{H}(x)\mathcal{K}(y_n-y)-\mathcal{L}(x)-\mathcal{M}(y_n-y)| \\ \leq \phi(x). \quad (34)$$

From (33) and (34), we have

$$\Bigg|\frac{\mathcal{F}((x+y)+y_n)+\mathcal{G}((x+y)-y_n)-\mathcal{M}(y_n)}{2\mathcal{K}(y_n)} \\ +\frac{\mathcal{F}((x-y)+y_n)+\mathcal{G}((x-y)-y_n)-\mathcal{M}(y_n)}{2\mathcal{K}(y_n)} \\ -\frac{1}{2\mathcal{K}(y_n)}\{2\mathcal{H}(x)\left(\mathcal{K}(y_n+y)+\mathcal{K}(y_n-y)\right) \\ +\mathcal{M}(y_n+y)+\mathcal{M}(y_n-y)-2\mathcal{M}(y_n)-\frac{\mathcal{L}(x)}{\mathcal{K}(y_n)}\Bigg| \\ \leqslant \frac{\phi(x)}{|\mathcal{K}(y_n)|}.$$

Using (32), we deduce that the limit

$$\Im_{\mathcal{K},\mathcal{M}}(x,y) \\ := \lim_{n\to\infty}\frac{1}{2\mathcal{K}(y_n)}\{2\mathcal{H}(x)\left(\mathcal{K}(y_n+y)+\mathcal{K}(y_n-y)\right) \\ +\mathcal{M}(y_n+y)+\mathcal{M}(y_n-y)-2\mathcal{M}(y_n)\}$$

exists and

$$\Im_{\mathcal{K},\mathcal{M}}(x,y)=\mathcal{H}(x+y)+\mathcal{H}(x-y),$$

which finishes the proof of part A.

**B1.** Because $\Im_{\mathcal{K},\mathcal{M}}(x,y)$ exists, the existence of either

$$\ell_{\mathcal{K}}(y):=\lim_{n\to\infty}\frac{\mathcal{K}(y_n+y)+\mathcal{K}(y_n-y)}{\mathcal{K}(y_n)}$$

or

$$\ell_{\mathcal{M},\mathcal{K}}(y):=\lim_{n\to\infty}\frac{\mathcal{M}(y_n+y)+\mathcal{M}(y_n-y)-2\mathcal{M}(y_n)}{2\mathcal{K}(y_n)}$$

implies that of the other, and yields at once

$$\mathcal{H}(x+y)+\mathcal{H}(x-y)=\mathcal{H}(x)\ell_{\mathcal{K}}(y)+\ell_{\mathcal{M},\mathcal{K}}(y). \quad (35)$$

**B2.** Since $\mathcal{K}(x+y)+\mathcal{K}(x-y)=2\mathcal{K}(x)K(y)$, the limit

$$\ell_{\mathcal{K}}(y):=\lim_{n\to\infty}\frac{\mathcal{K}(y_n+y)+\mathcal{K}(y_n-y)}{\mathcal{K}(y_n)}$$

exists and

$$\ell_{\mathcal{K}}(y)=2\mathcal{K}(y).$$

Thus, the limit

$$\ell_{\mathcal{M},\mathcal{K}}(y):=\lim_{n\to\infty}\frac{\mathcal{M}(y_n+y)+\mathcal{M}(y_n-y)-2\mathcal{M}(y_n)}{2\mathcal{K}(y_n)}$$

exists and

$$\mathcal{H}(x+y)+\mathcal{H}(x-y)=2\mathcal{H}(x)\mathcal{K}(y)+\ell_{\mathcal{M},\mathcal{K}}(y).$$

**C1.** Part I follows by taking

$$\mathbb{G}=\mathbb{R},\ K=\ell_{\mathcal{K}},\ \ell=\ell_{\mathcal{M},\mathcal{K}}$$

in Proposition 2.1. Part II follows similarly by taking $\mathbb{G}$ to be a 2-divisible abelian group, $\mathbb{F}=\mathbb{C}$, $\ell=\ell_{\mathcal{M},\mathcal{K}}$ and $\ell_{\mathcal{K}}=c$ in Proposition 2.3.

**C2.** If $\mathcal{M}$ is quadratic or additive, then the limit

$$\ell_{\mathcal{M},\mathcal{K}}(y):=\lim_{n\to\infty}\frac{\mathcal{M}(y_n+y)+\mathcal{M}(y_n-y)-2\mathcal{M}(y_n)}{2\mathcal{K}(y_n)}$$

exists and

$$\ell_{\mathcal{M},\mathcal{K}}(y)=0.$$

Thus, the functions $\mathcal{H}$ and $\mathcal{K}$ satisfy the functional equation

$$\mathcal{H}(x+y)+\mathcal{H}(x-y)=2\mathcal{H}(x)\mathcal{K}(y) \quad (36)$$

and the given explicit solutions are taken from [5, p. 148]. This completes the proof of Theorem 1.7.

Since the functions $\mathcal{F}$ and $\mathcal{G}$ do not appear in the conclusion of Theorem 1.7, interchanging $x$ with $y$, we have:

**Theorem 3.1.** *Let* $(\mathbb{G},+)$ *be an abelian group, and* $\phi:\mathbb{G}\to[0,\infty)$.
***A.*** *If* $\mathcal{F},\mathcal{G},\mathcal{H},\mathcal{K},\mathcal{L},\mathcal{M}:\mathbb{G}\to\mathbb{C}$ *satisfy*

$$|\mathcal{F}(x+y)+\mathcal{G}(x-y)-2\mathcal{H}(x)\mathcal{K}(y)-\mathcal{L}(x)-\mathcal{M}(y)| \\ \leq \phi(y) \qquad (x,y\in\mathbb{G}), \quad (37)$$

*and* $\mathcal{K}$ *is not bounded, then there is a sequence* $\{x_n\}\subset\mathbb{G}$ *such that the following two limits exist for all* $x,y\in\mathbb{G}$,

$$\mathcal{H}(y)=\lim_{n\to\infty}\frac{\mathcal{F}(y+x_n)+\mathcal{G}(y-x_n)-\mathcal{M}(x_n)}{2\mathcal{K}(x_n)},$$

$$\Im_{\mathcal{K},\mathcal{M}}(y,x):= \\ \lim_{n\to\infty}\frac{1}{2\mathcal{K}(x_n)}\{2\mathcal{H}(y)\left(\mathcal{K}(x_n+x)+\mathcal{K}(x_n-x)\right) \\ +\mathcal{M}(x_n+x)+\mathcal{M}(x_n-x)-2\mathcal{M}(x_n)\}$$

*and the function $H$ satisfies the functional equation*

$$\mathcal{H}(y+x)+\mathcal{H}(y-x)=\Im_{\mathcal{K},\mathcal{M}}(y,x) \qquad (38)$$

***B1.*** *Assuming part **A**, if either the limit*

$$\ell_{\mathcal{K}}(x):=\lim_{n\to\infty}\frac{\mathcal{K}(x_n+x)+\mathcal{K}(x_n-x)}{\mathcal{K}(x_n)}$$

*or the limit*

$$\ell_{\mathcal{M},\mathcal{K}}(x):=\lim_{n\to\infty}\frac{\mathcal{M}(x_n+x)+\mathcal{M}(x_n-x)-2\mathcal{M}(x_n)}{2\mathcal{K}(x_n)}$$

*exists for all $x\in\mathbb{G}$, then the functional equation (38) simplifies to*

$$\mathcal{H}(y+x)+\mathcal{H}(y-x)=\mathcal{H}(y)\ell_{\mathcal{K}}(x)+\ell_{\mathcal{M},\mathcal{K}}(x) \qquad (x,y\in\mathbb{G}). \qquad (39)$$

***B2.*** *Assuming part **A**, if $\mathcal{K}$ satisfies the equation*

$$\mathcal{K}(y+x)+\mathcal{K}(y-x)=2\mathcal{K}(y)\mathcal{K}(x),$$

*then the limit*

$$\ell_{\mathcal{M},\mathcal{K}}(x):=\lim_{n\to\infty}\frac{\mathcal{M}(x_n+x)+\mathcal{M}(x_n-x)-2\mathcal{M}(x_n)}{2\mathcal{K}(x_n)}$$

*exists for each $x\in\mathbb{G}$, and the function $\mathcal{H}$ satisfies the functional equation*

$$\mathcal{H}(y+x)+\mathcal{H}(y-x)=2\mathcal{H}(y)\mathcal{K}(x)+\ell_{\mathcal{M},\mathcal{K}}(x).$$

***C1.*** *Suppose parts **A** and **B1** hold.*
***I.*** *When $\mathbb{G}=\mathbb{R}$, if $\mathcal{H}$, $\ell_{\mathcal{K}}$, $\ell_{\mathcal{M},\mathcal{K}}$ are differentiable, and either*

$$\ell'_{\mathcal{K}}(0)\neq 0 \text{ or } \ell'_{\mathcal{M},\mathcal{K}}(0)\neq 0$$

*then*

$$\mathcal{H}(y)\equiv h\in\mathbb{C}$$

*is a constant function.*
*Furthermore,*
***(Ia)*** *if $\ell'_{\mathcal{K}}(0)\neq 0$ and $h=0$ then $\ell_{\mathcal{M},\mathcal{K}}(x)\equiv 0$, and $\ell_{\mathcal{K}}$ is an arbitrary function;*
***(Ib)*** *if $\ell'_{\mathcal{K}}(0)\neq 0$ and $h\neq 0$ then*

$$\ell_{\mathcal{M},\mathcal{K}}(x)=(2-\ell_{\mathcal{K}}(x))h$$

*and is an arbitrary function;*
***(Ic)*** *the case where $\ell'_{\mathcal{M},\mathcal{K}}(x)\neq 0$ and $h=0$ never occurs;*
***(Id)*** *if $\ell'_{\mathcal{M},\mathcal{K}}(0)\neq 0$ and $h\neq 0$ then*

$$\ell_{\mathcal{M},\mathcal{K}}(x)=(2-\ell_{\mathcal{K}}(x))h$$

*and $\ell_{\mathcal{K}}$ is an arbitrary function.*

***II.*** *When $\mathbb{G}$ is a 2-divisible abelian group, if*

$$\ell_{\mathcal{K}}(x)\equiv c\in\mathbb{C}$$

*is a constant function, then*
***(IIa)*** *for $c=2$, we have*

$$\mathcal{H}(y)=B(y,y)+A(y)+r,\ell_{\mathcal{M},\mathcal{K}}(y)=2B(y,y),$$

*where $B:\mathbb{G}\times\mathbb{G}\to\mathbb{C}$ is a symmetric biadditive function, $A:\mathbb{G}\to\mathbb{C}$ is an additive function and $r\in\mathbb{C}$;*
***(IIb)*** *for $c\neq 2$, we have*

$$\mathcal{H}(y)\equiv r,\ell_{\mathcal{M},\mathcal{K}}(y)\equiv r(2-c) \qquad (r\in\mathbb{C})$$

*being two constant functions.*

***C2.*** *Suppose parts **A** and **B2** hold.*
***I.*** *If $\mathcal{M}$ is quadratic (i.e.,*

$$\mathcal{M}(y+x)+\mathcal{M}(y-x)=2\mathcal{M}(y)+2\mathcal{M}(x))$$

*or if $\mathcal{M}$ is additive, then the functions $\mathcal{H}$ and $\mathcal{K}$ satisfy the functional equation*

$$\mathcal{H}(y+x)+\mathcal{H}(y-x)=2\mathcal{H}(y)\mathcal{K}(x)$$

*and are given by*

$$\mathcal{K}(y)=\frac{E(y)+E^*(y)}{2},$$
$$\mathcal{H}(y)=\frac{k\left(E(y)+E^*(y)\right)}{2}+c\left(E(y)-E^*(y)\right),$$

*where $k,c\in\mathbb{C},E$ is an exponential function and $E^*(x)=1/E(x)$.*

# 4 Examples

In this final section, we give several examples of Theorems 1.7 and 3.1, some of which are special cases obtained previously by other authors.

**Example 4.1.** *If we put $\mathcal{L}=\mathcal{M}\equiv 0$ in Theorem 1.7, respectively, Theorem 3.1, then we have part I and part II of Theorem 1.3, which shows that all earlier works mentioned in [15] are direct consequences of our main results above.*

**Example 4.2.** *If we put $\mathcal{L}=\mathcal{M}\equiv 0,\mathcal{F}(x)=\mathcal{G}(x)=f(x),\mathcal{H}(x)=g(x)$, and $\mathcal{K}(y)=h(y)$ in Theorem 1.7, respectively, Theorem 3.1, then we have Theorem 1, respectively, Theorem 2 of Kim's results in [24].*

**Example 4.3.** *If we put* $\mathcal{L} = \mathcal{M} \equiv 0, \mathcal{F}(x) = \mathcal{G}(x) = \mathcal{K}(x) = f(x)$*, and* $\mathcal{H}(x) = g(x)$ *in Theorem 1.7, then we have Theorem 2.1 of Kim's result in [21].*

**Example 4.4.** *If we put* $\mathcal{L} = \mathcal{M} \equiv 0, \mathcal{F}(x) = \mathcal{G}(x) = f(x), \mathcal{H}(x) = g(x)$*, and* $\mathcal{K}(y) = g(y)$ *in Theorem 3.1, then we have Theorem 3.1 of Kim's results in [21].*

**Example 4.5.** *If we put* $\mathcal{L} = \mathcal{M} \equiv 0$*, and* $\mathcal{F} = \mathcal{G} = \mathcal{H} = \mathcal{K} = f$ *in Theorem 1.7, respectively, Theorem 3.1, we get an extension of Baker's result in [25].*

**Example 4.6.** *As another simple example of Theorem 3.1 with explicit solutions, take the group* $\mathbb{G}$ *to be* $\mathbb{R}$*,* $\mathcal{F}(x) = \mathcal{G}(x) = x, \mathcal{L}(x) = c_1$ *and* $\mathcal{M}(x) = c_2$ *as two constant functions. The condition* (37) *of Theorem 3.1 becomes*

$$|2x - 2\mathcal{H}(x)\mathcal{K}(y) - c_1 - c_2| \leq \phi(y). \tag{40}$$

*Dividing by* $x$ *and taking the limit* $x \to \infty$*, we deduce that*

$$2 - 2\mathcal{K}(y) \lim_{x\to\infty} \frac{\mathcal{H}(x)}{x} = 0,$$

*so that* $\mathcal{K}(y) = c_3 \neq 0$*, a constant function. Substituting bact into* (40)*, we see that*

$$\mathcal{H}(x) = \frac{x}{c_3} + H_1(x),$$

*where* $H_1(x)$ *is a bounded function.*

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