The modified $(\frac{G'}{G})-$ expansion method and its applications to construct exact solutions for nonlinear PDEs

ElsayedM.E.Zayed
Department of Mathematics
Faculty of Science
Zagazig University
Egypt
e.m.e.zayed@hotmail.com

Khaled A.Gepreel
Department of Mathematics
Faculty of Science
Taif University
Kingdom of Saudi Arabia.
kagepreel@yahoo.com

Abstract: In the present article, we construct the traveling wave solutions involving parameters of some nonlinear PDEs; namely the nonlinear Klein - Gordon equations, the nonlinear reaction- diffusion equation, the nonlinear modified Burgers equation and the nonlinear Eckhaus equation by using the modified $(\frac{G'}{G})$- expansion method, where G satisfies a second order linear ordinary differential equation. When these parameters are taken special values, the solitary waves are derived from the traveling waves. The traveling waves solutions are expressed by hyperbolic, trigonometric and the rational functions.

Key–Words: Expansion method, nonlinear PDEs, exact solution.

1 Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors(see for example [1]–[45]) who are interested in nonlinear physical phenomena. Many powerful methods have been presented by those authors such as the generalized projective Riccati equation expansion method [7, 31], the sine / sinh-Gordon reduction method [32, 33], the reduction mKdV equation method [34], the tri-function method [35, 36], the homogeneous balance method [9], the hyperbolic tangent expansion method [29], the tanh-method [8, 27], the nonlinear transform method [16], the inverse scattering transform [1], the Backlund transform [21, 23], the Hirota's bilinear method [11, 12], the generalized Riccati equation [30], the Weierstrass elliptic function method [22], the Sine-Cosine method [27, 37], the Jacobi elliptic function expansion [9, 38], the complex hyperbolic function method [3],the truncated Painleve expansion [4], the F-expansion method [25], the rank analysis method [10], the ansatz method [14, 15, 16], the exp-function expansion method [13], the sub- ODE. method [19], the $(\frac{G'}{G})-$expansion method [26], [40]–[45] and so on.

Recently, Bin et al [40] and Zayed et al [42] have obtained the exact solutions of some nonlinear PDEs using the modified $(\frac{G'}{G})$- expansion method. In the present paper, we shall use the modified $(\frac{G'}{G})$ - expansion method to find the exact solutions of some different PDEs. This method is proposed by which the traveling wave solutions of nonlinear equations are obtained. The main idea of this method is that the traveling wave solutions of nonlinear evolution equations can be expressed by a polynomial in $(\frac{G'}{G})$,where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = x - Vt$, where λ, μ and V are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equations . The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method. In the present paper, the modified $(\frac{G'}{G})-$expansion method will be applied to construct the traveling wave solutions of the nonlinear Klein - Gordon equations, the nonlinear reaction- diffusion equation, the nonlinear modified Burgers equation and the nonlinear Eckhaus equation.

2 Description of the modified $(\frac{G'}{G})$-expansion method

Suppose that we have a nonlinear PDE in the following form:

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \quad (1)$$

where $u = u(x,t)$ is an unknown function, F is a polynomial in $u = u(x,t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. Let us now give the main steps for solving Eq. (1) using the modified $(\frac{G'}{G})$-expansion method [40,42]:

Step 1. The traveling wave variable

$$u(x,t) = u(\xi) , \qquad \xi = x - Vt, \quad (2)$$

where V is a constant, permits us reducing Eq. (1) to an ODE for $u = u(\xi)$ in the form

$$P(u, -Vu', u', V^2u'', -Vu'', u'', ...) = 0, \quad (3)$$

where P is a polynomial of $u = u(\xi)$ and its total derivatives.

Step 2. Suppose that the solution of Eq. (3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$\psi(\xi) = \sum_{i=-m}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \quad (4)$$

where $G = G(\xi)$ satisfies the following second order linear differential equation in the form:

$$G'' + \lambda G' + \mu G = 0, \quad (5)$$

where α_i, λ, μ are constants to be determined later, $\alpha_m \neq 0$ or $\alpha_{-m} \neq 0$ and m is a positive integer.

Step 3. Balancing the highest derivative term with the nonlinear term in (3), we find the value of the positive integer m in (4). In some nonlinear equations the balance number m is not a positive integer. In this case, we make the following transformations [20]:

(a) When $m = \frac{q}{p}$ where $\frac{q}{p}$ is a fraction in the lowest terms, we let

$$u(\xi) = \varphi^{\frac{q}{p}}(\xi) \quad (6)$$

then substituting (6) into (3) to get a new equation in the new function $\varphi(\xi)$ with a positive integer balance number.

(b) When m is a negative number, we let

$$u(\xi) = \varphi^m(\xi), \quad (7)$$

and substituting (7) into (3) to get a new equation in the new function $\varphi(\xi)$ with a positive integer balance number.

Step 4. Substituting (4) into Eq. (3) and using Eq.(5), collecting all terms with the same order of $(\frac{G'}{G})$ together, and then equating each coefficient of the resulted polynomial to zero, yields a set of algebraic equations for α_i, V, λ and μ .

Step 5. Since the general solutions of (5) have been well known for us, then substituting α_i, V and the general solutions of (5) into (4), we have the traveling wave solutions of the nonlinear PDEs (1).

3 Applications

In this section, we apply the modified $(\frac{G'}{G})$- expansion method to construct the traveling wave solutions for some nonlinear partial differential equations, namely the nonlinear Klein- Gordon equations, the nonlinear reaction- diffusion equation, the nonlinear modified Burgers equation and the nonlinear Eckhaus equation which are very important in the mathematical physics and have been paid attention by many researchers.

3.1 The nonlinear Klein - Gordon equations with the cubic- quintic nonlinearity

Wazwaz [27, 28] investigated the nonlinear Klein - Gordon equations and found many types of exact traveling wave solutions including compact solutions, soliton solution, solitary patterns solutions and periodic solutions using the tanh- function method . These equations play an important role in many scientific applications, such as the solid state physics, the nonlinear optics, the quantum field theory and so on (see[17, 18, 24]).

3.1.1 Example 1.

We start with the following nonlinear Klein - Gordon equation in the form:

$$u_{tt} - k^2u_{xx} + \alpha u - \beta u^3 + \gamma u^5 = 0, \quad (8)$$

where α, β, k and γ are constants, provided $\gamma \neq 0$. The traveling wave variable (2) permits us converting equation (8) into the following ODE:

$$(V^2 - k^2)u'' + \alpha u - \beta u^3 + \gamma u^5 = 0. \quad (9)$$

Suppose that the solution of the ODE (9) can be expressed by a polynomial in terms of $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=-m}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \quad (10)$$

where α_i are arbitrary constants, while $G(\xi)$ satisfies the second order linear ODE (5). Considering the homogeneous balance between the highest order derivative u'' and the nonlinear term u^5 in (9), we get $m = \frac{1}{2}$. According to step 3, we use the transformation

$$u = [\psi(\xi)]^{\frac{1}{2}} \quad (11)$$

where $\psi(\xi)$ is a new function of ξ. Substituting (11) into (9), we get the new ODE:

$$\tfrac{1}{4}(V^2 - k^2)[2\psi\psi'' - \psi'^2] + \alpha\psi^2 - \beta\psi^3 + \gamma\psi^4 = 0. \quad (12)$$

Determining the balance number m of the new Eq. (12), we get $m = 1$. Consequently, we have the formal solution of Eq.(12) in the form:

$$\psi(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_{-1} \left(\frac{G'}{G}\right)^{-1} + \alpha_0. \quad (13)$$

After some calculation, we get

$$\psi'(\xi) = -\alpha_1 \left(\tfrac{G'}{G}\right)^2 - \lambda\alpha_1 \left(\tfrac{G'}{G}\right) + \alpha_{-1} - \mu\alpha_1 + \lambda\alpha_{-1} \left(\tfrac{G'}{G}\right)^{-1} + \mu\alpha_{-1} \left(\tfrac{G'}{G}\right)^{-2}, \quad (14)$$

$$\psi''(\xi) = 2\alpha_1 \left(\tfrac{G'}{G}\right)^3 + 3\lambda\alpha_1 \left(\tfrac{G'}{G}\right)^2 + \alpha_1(2\mu + \lambda^2) \left(\tfrac{G'}{G}\right) + \alpha_1\mu\lambda + \lambda\alpha_{-1} + (2\mu + \lambda^2)\alpha_{-1} \left(\tfrac{G'}{G}\right)^{-1} + 3\lambda\mu\alpha_{-1} \left(\tfrac{G'}{G}\right)^{-2} + 2\alpha_{-1}\mu^2 \left(\tfrac{G'}{G}\right)^{-3}. \quad (15)$$

and so on.

On substituting (13) -(15) into (12) collecting all terms with the same powers of $\left(\frac{G'}{G}\right)$ and setting them to zero then, we have the following system of algebraic equations:

$$\frac{3}{4}V^2\alpha_1^2 - \frac{3}{4}k^2\alpha_1^2 + \gamma\alpha_1^4 = 0,$$

$$V^2\alpha_0\alpha_1 - \beta\alpha_1^3 + 4\gamma\alpha_1^3\alpha_0 - k^2\alpha 0\alpha_1 + V^2\alpha_1^2\lambda - k^2\alpha_1^2\lambda = 0,$$

$$-k^2\alpha_{-1}^2\lambda\mu - \beta\alpha_{-1}^3 + V^2\alpha_0\alpha_{-1}\mu^2 - k^2\alpha_0\alpha_{-1}\mu^2 + 4\gamma\alpha_{-1}^3\alpha_0 + V^2\alpha_{-1}^2\lambda\mu = 0,$$

$$\frac{3}{4}V^2\alpha_{-1}^2\mu^2 + \gamma\alpha_{-1}^4 - \frac{3}{4}k^2\alpha_{-1}^2\mu^2 = 0,$$

$$-3k^2\alpha_1\lambda\alpha_{-1}\mu - 3\beta\alpha_1\alpha_{-1}^2 + 2\alpha\alpha_{-1}\alpha_0 + 4\gamma\alpha_{-1}\alpha_0^3 + 12\gamma\alpha_1\alpha_{-1}^2\alpha_0 - \frac{1}{2}k^2\alpha_0\alpha_{-1}\lambda^2 - k^2\alpha_0\alpha_{-1}\mu + V^2\alpha_0\alpha_{-1}\mu + 3V^2\alpha_1\lambda\alpha_{-1}\mu + \frac{1}{2}V^2\alpha_0\alpha_{-1}\lambda^2 - 3\beta\alpha_{-1}\alpha_0^2 = 0,$$

$$-\frac{3}{2}k^2\alpha 0\alpha_1\lambda - \frac{1}{2}k^2\alpha_1^2\mu + \frac{3}{2}V^2\alpha_1\alpha_{-1} - \frac{3}{2}k^2\alpha_1\alpha_{-1} + \frac{1}{4}V^2\alpha_1^2\lambda^2 + \alpha\alpha_1^2 + \frac{3}{2}V^2\alpha_0\alpha_1\lambda + 4\gamma\alpha_1^3\alpha_{-1} - 3\beta\alpha_1^2\alpha_0 + \frac{1}{2}V^2\alpha_1^2\mu - \frac{1}{4}k^2\alpha_1^2\lambda^2 + 6\gamma\alpha_1^2\alpha_0^2 = 0,$$

$$\frac{1}{2}V^2\alpha_{-1}^2\mu + \frac{1}{4}V^2\alpha_{-1}^2\lambda^2 - \frac{1}{4}k^2\alpha_{-1}^2\lambda^2 + 6\gamma\alpha_{-1}^2\alpha_0^2 + \alpha\alpha_{-1}^2 + 4\gamma\alpha_1\alpha_{-1}^3 - \frac{3}{2}k^2\alpha_0\alpha_{-1}\lambda\mu - \frac{1}{2}k^2\alpha_{-1}^2\mu - \frac{3}{2}k^2\alpha_{-1}\mu^2\alpha_1 - 3\beta\alpha_{-1}^2\alpha_0 + \frac{3}{2}V^2\alpha_{-1}\mu^2\alpha_1 + \frac{3}{2}V^2\alpha_0\alpha_{-1}\lambda\mu = 0,$$

$$-\frac{1}{2}k^2\alpha_0\alpha_1\lambda^2 + 3V^2\alpha_1\lambda\alpha_{-1} - 3\beta\alpha_1\alpha_0^2 - 3k^2\alpha_1\lambda\alpha_{-1} + 2\alpha\alpha_1\alpha_0 - 3\beta\alpha_1^2\alpha_{-1} - k^2\alpha_0\alpha_1\mu + \frac{1}{2}V^2\alpha_0\alpha_1\lambda^2 + 4\gamma\alpha_1\alpha_0^3 + V^2\alpha_0\alpha_1\mu + 12\gamma\alpha_1^2\alpha_{-1}\alpha_0 = 0,$$

$$-\frac{1}{4}V^2\alpha_1^2\mu^2 + \gamma\alpha_0^4 + \frac{1}{4}k^2\alpha_1^2\mu^2 + 2\alpha\alpha_1\alpha_{-1} - \beta\alpha_0^3 + \alpha\alpha_0^2 - \frac{1}{4}V^2\alpha_{-1}^2 + \frac{1}{4}k^2\alpha_{-1}^2 - \frac{1}{2}k^2\alpha_0\alpha_1\lambda\mu + 12\,\gamma\alpha_1\alpha_{-1}\alpha_0^2 + 3V^2\alpha_1\alpha_{-1}\mu + \frac{1}{2}V^2\alpha_0\alpha_{-1}\lambda + 6\gamma\alpha_1^2\alpha_{-1}^2 + \frac{3}{2}V^2\alpha_1\lambda^2\alpha_{-1} - 3k^2\alpha_1\alpha_{-1}\mu - \frac{1}{2}k^2\alpha_0\alpha_{-1}\lambda - \frac{3}{2}k^2\alpha_1\lambda^2\alpha_{-1} + \frac{1}{2}V^2\alpha_0\alpha_1\lambda\mu - 6\beta\alpha_1\alpha_{-1}\alpha_0 = 0. \quad (16)$$

On solving the above algebraic equations (16) by using the Maple or Mathematica, we have

$$\alpha_1 = \tfrac{4\alpha\lambda}{\beta M},\ \alpha_0 = \tfrac{4\alpha\lambda^2}{\beta(M},\alpha_{-1} = \tfrac{4\alpha\lambda\mu}{\beta M}, \gamma = \tfrac{3\beta^2 M}{16\alpha\lambda^2},\ V = \sqrt{k^2 - \tfrac{4\alpha}{M}}, \quad (17)$$

where $M = \lambda^2 - 4\mu \neq 0$ and $k^2 > \frac{4\alpha}{M}$.

Substituting (17) into (13) yields

$$\psi(\xi) = \frac{4\alpha\lambda}{\beta M}\left(\frac{G'}{G}\right) + \frac{4\alpha\lambda\mu}{\beta M}\left(\frac{G'}{G}\right)^{-1} + \frac{4\alpha\lambda^2}{\beta M}, \quad (18)$$

where

$$\xi = x - t\ \sqrt{k^2 - \frac{4\alpha}{M}}. \quad (19)$$

On solving Eq. (12), we deduce that

$$\frac{G'}{G} = \begin{cases} \frac{1}{2}\sqrt{M}\left(\frac{A\cosh(\frac{1}{2}\sqrt{M}\ \xi)+B\sinh(\frac{1}{2}\sqrt{M}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{M}\ \xi)+B\cosh(\frac{1}{2}\sqrt{M}\ \xi)}\right) \\ -\frac{\lambda}{2} \quad \text{if} \quad M>0, \\ \frac{1}{2}\sqrt{-M}\left(\frac{-A\sin(\frac{1}{2}\sqrt{-M}\xi)+B\cos(\frac{1}{2}\sqrt{-M}\xi)}{A\cos(\frac{1}{2}\sqrt{-M}\xi)+B\sin(\frac{1}{2}\sqrt{-M}\xi)}\right) \\ -\frac{\lambda}{2} \quad \text{if} \quad M<0, \\ \frac{B}{B\xi+A} - \frac{\lambda}{2}, \quad \text{if} \quad M=0, \end{cases} \quad (20)$$

where A and B are arbitrary constants and $M = \lambda^2 - 4\mu$. On substituting (20) into (18), we deduce the following types of traveling wave solutions of Eq. (12):

Case 1.If $M > 0$, then we have the hyperbolic solution

$$\psi = \frac{2\alpha\lambda}{\beta\sqrt{M}}\left(\frac{A\cosh(\frac{1}{2}\sqrt{M}\ \xi)+B\sinh(\frac{1}{2}\sqrt{M}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{M}\ \xi)+B\cosh(\frac{1}{2}\sqrt{M}\ \xi)}\right) + \frac{2\alpha\lambda^2}{\beta M} + \frac{8\alpha\lambda\mu}{\beta M}\times \left[\sqrt{M}\left(\frac{A\cosh(\frac{1}{2}\sqrt{M}\ \xi)+B\sinh(\frac{1}{2}\sqrt{M}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{M}\ \xi)+B\cosh(\frac{1}{2}\sqrt{M}\ \xi)}\right) - \lambda\right]^{-1}. \quad (21)$$

Case 2 . If $M < 0$, then we have the trigonometric solution

$$\psi = \frac{2\alpha\lambda}{\beta\sqrt{-M}}\left(\frac{-A\sin(\frac{1}{2}\sqrt{-M}\xi)+B\cos(\frac{1}{2}\sqrt{-M}\xi)}{A\cos(\frac{1}{2}\sqrt{-M}\xi)+B\sin(\frac{1}{2}\sqrt{M}\xi)}\right) + \frac{2\alpha\lambda^2}{\beta M} + \frac{8\alpha\lambda\mu}{-\beta M}\times \left[\sqrt{-M}\left(\frac{-A\sin(\frac{1}{2}\sqrt{-M}\xi)+B\cos(\frac{1}{2}\sqrt{-M}\xi)}{A\cos(\frac{1}{2}\sqrt{-M}\xi)+B\sin(\frac{1}{2}\sqrt{-M}\xi)}\right) - \lambda\right]^{-1}. \quad (22)$$

On substituting (21) and (22) into (11) we have the traveling wave solutions of Eq. (8).

In particular, if we set $B = 0,\ A \neq 0, \lambda > 0, \mu = 0,$ in (21) then we get

$$u(\xi) = \sqrt{\frac{2\alpha}{\beta}\left[\coth(\frac{1}{2}\lambda\xi) + 1\right]}, \quad (23)$$

while if $B \neq 0,\ \ A^2 > B^2, \lambda > 0, \mu = 0$, then we have

$$u(\xi) = \sqrt{\frac{2\alpha}{\beta}\left[\tanh(\xi_0 + \frac{1}{2}\lambda\xi) + 1\right]}, \quad (24)$$

where $\xi_0 = \tanh^{-1}(\frac{A}{B})$. Note that (23) and (24) represent the solitary wave solutions of Eq.(8). These solutions are completely physical in the case $\frac{\alpha}{\beta} > 0$ and $k^2 > \frac{4\alpha}{\lambda^2}$.

3.1.2 Example 2.

In this section, we study the following nonlinear Klein - Gordon equation:

$$u_{tt} - k^2 u_{xx} + \alpha u - \beta u^n + \gamma u^{2n-1} = 0, \quad n > 2, \quad (25)$$

where α, β, γ and k are constants.

The traveling wave variable (9) permit us converting Eq. (25) into ODE in the form:

$$(V^2 - k^2)u'' + \alpha u - \beta u^n + \gamma u^{2n-1} = 0, \quad n > 2. \quad (26)$$

Suppose that the solution of the ODE (26) can be expressed by polynomial in term of $(\frac{G'}{G})$ in the same form (10) where $G(\xi)$ satisfies (5). Considering the homogeneous balance between the highest order derivative u'' and the nonlinear term u^{2n-1} in (26), we get $m = \frac{1}{n-1}$. According to step 3, we use the transformation

$$u = [\psi(\ \xi)]^{\frac{1}{n-1}} \quad (27)$$

where $\psi(\ \xi)$ is a new function of ξ. Substituting (27) into (26), we get the new ODE

$$(V^2 - k^2)\left[\frac{2-n}{(n-1)^2}\psi'^2 + \frac{1}{(n-1)}\psi\psi''\right] + \alpha\psi^2 - \beta\psi^3 + \gamma\psi^4 = 0. \quad (28)$$

Determining the balance number m of the new Eq. (28), we get $m = 1$. Thus, the solutions of Eq. (28) have the same form (13). Consequently,using the Maple or Mathematica we get the following results:

$$\alpha_1 = \frac{(n+1)\lambda\alpha}{\beta M}, \alpha_0 = \frac{(n+1)\lambda^2\alpha}{\beta M}, \alpha_{-1} = \frac{(n+1)\mu\lambda\alpha}{\beta M}, \quad \gamma = \frac{n\beta^2 M}{\alpha\lambda^2(n+1)^2}, V = \sqrt{k^2 - \frac{(n-1)^2\alpha}{M}}, \quad (29)$$

where $M = \lambda^2 - 4\mu \neq 0$ and $k^2 > \frac{(n-1)^2\alpha}{M}$. Substituting (29) into (13) yields

$$\psi(\xi) = \frac{(n+1)\lambda\alpha}{\beta M}\left(\frac{G'}{G}\right) + \frac{(n+1)\mu\lambda\alpha}{\beta M}\left(\frac{G'}{G}\right)^{-1} + \frac{(n+1)\lambda^2\alpha}{\beta M}, \quad (30)$$

where

$$\xi = x - t\ \sqrt{k^2 - \frac{(n-1)^2\alpha}{M}}. \quad (31)$$

From (20) and (30), we deduce the following types of traveling wave solutions of Eq.(28):

Case 1. If $M > 0$,then we have the hyperbolic solution

$$\psi = \frac{(n+1)\lambda\alpha}{2\beta\sqrt{M}}\left(\frac{A\cosh(\frac{1}{2}\sqrt{M}\ \xi)+B\sinh(\frac{1}{2}\sqrt{M}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{M}\ \xi)+B\cosh(\frac{1}{2}\sqrt{M}\ \xi)}\right) + \frac{(n+1)\lambda^2\alpha}{2\beta M} + \frac{2(n+1)\mu\lambda\alpha}{\beta M^{3/2}}\times \left[\left(\frac{A\cosh(\frac{1}{2}\sqrt{M}\ \xi)+B\sinh(\frac{1}{2}\sqrt{M}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{M}\ \xi)+B\cosh(\frac{1}{2}\sqrt{M}\ \xi)}\right) - \frac{\lambda}{\sqrt{M}}\right]^{-1}. \quad (32)$$

Case 2 . If $M < 0$, then we have the trigonometric solution

$$\psi = \frac{(n+1)\lambda\alpha}{2\beta\sqrt{-M}}\left(\frac{-A\sin(\frac{1}{2}\sqrt{-M}\xi)+B\cos(\frac{1}{2}\sqrt{-M}\xi)}{A\cos(\frac{1}{2}\sqrt{-M}\xi)+B\sin(\frac{1}{2}\sqrt{-M}\xi)}\right) + \frac{(n+1)\lambda^2\alpha}{2\beta M} - \frac{2(n+1)\mu\lambda\alpha}{\beta\sqrt{(-M)^3}}\times \left[\left(\frac{-A\sin(\frac{1}{2}\sqrt{-M}\xi)+B\cos(\frac{1}{2}\sqrt{-M}\xi)}{A\cos(\frac{1}{2}\sqrt{-M}\xi)+B\sin(\frac{1}{2}\sqrt{-M}\xi)}\right) - \frac{\lambda}{\sqrt{-M}}\right]^{-1}. \quad (33)$$

On substituting (32) and (33) into (27) we have the traveling wave solutions of Eq. (25).

In particular, if we set $B = 0, A \neq 0, \lambda > 0, \mu = 0,$ in (32) then we get the solitary wave solutions of Eq.(25) as follows:

$$u(\xi) = \left\{ \frac{(n+1)\alpha}{2\beta} \left[\coth \frac{\lambda}{2}\xi + 1 \right] \right\}^{\frac{1}{n-1}}, \qquad (34)$$

while if $B \neq 0,\ A^2 > B^2, \lambda > 0, \mu = 0$, then we get

$$u(\xi) = \left\{ \frac{(n+1)\alpha}{2\beta} \left[\tanh(\xi_0 + \frac{1}{2}\lambda\xi) + 1 \right] \right\}^{\frac{1}{n-1}}, \qquad (35)$$

where $\xi_0 = \tanh^{-1}(\frac{A}{B})$. These solutions are completely physical in the case $\frac{\alpha}{\beta} > 0$ and $k^2 > \frac{(n-1)^2\alpha}{\lambda^2}$.

3.1.3 Example 3.

In this section, we study the following nonlinear Klein - Gordon equation:

$$u_{tt} - k^2 u_{xx} + \alpha u - \beta u^{1-n} + \gamma u^{n+1} = 0,\ n > 2, \qquad (36)$$

where k, α, β and γ are constants.

The traveling wave variable (9) permits us converting Eq.(36) into ODE in the form:

$$(V^2 - k^2)u'' + \alpha u - \beta u^{1-n} + \gamma u^{n+1} = 0. \qquad (37)$$

Considering the homogeneous balance between the highest order derivative u'' and the nonlinear term u^{n+1} in (37), we get $m = \frac{2}{n}$. According to step 3, we take the transformation

$$u = [\psi(\xi)]^{\frac{2}{n}} \qquad (38)$$

where $\psi(\xi)$ is a new function of ξ. Substituting (38) into (37), we get

$$\begin{aligned}&(V^2 - k^2)\left[2(2-n)\psi'^2 + 2n\psi\psi''\right] + \alpha n^2\psi^2 \\ &-\beta n^2\psi^3 + \gamma n^2\psi^4 = 0,\end{aligned} \qquad (39)$$

Determining the balance number m of the new Eq. (39), we get $m = 1$. Thus, the solutions of Eq. (39) has the same form (13). Consequently, using the Maple or Mathematica we get the following results:

Case 1.

$$\begin{aligned}&\alpha_{-1} = 4\sqrt{\tfrac{\beta}{\alpha(2-n)M}},\ \alpha_0 = 2\lambda\sqrt{\tfrac{\beta}{\alpha(2-n)M}}, \\ &\gamma = \tfrac{(n^2-4)\alpha^2}{16\beta},\quad V = \sqrt{k^2 + \tfrac{n^2\alpha}{2M}},\ \alpha_1 = 0.\end{aligned} \qquad (40)$$

Case 2.

$$\begin{aligned}&\alpha_1 = 4\sqrt{\tfrac{\beta}{\alpha(2-n)M}}, \alpha_0 = 2\lambda\sqrt{\tfrac{\beta}{\alpha(2-n)M}}, \\ &\gamma = \tfrac{(n^2-4)\alpha^2}{16\beta},\ V = \sqrt{k^2 + \tfrac{n^2\alpha}{2M}}, \alpha_{-1} = 0,\end{aligned} \qquad (41)$$

where $M = \lambda^2 - 4\mu \neq 0$, $k^2 + \frac{n^2\alpha}{2M} > 0$ and $n > 2$.

We just list some exact solutions corresponding to case 1 to illustrate the effectiveness of the modified $(\frac{G'}{G})-$ expansion method. Substituting (40) into (13) yields

$$\psi = 4\sqrt{\frac{\beta}{\alpha(2-n)M}} \left(\frac{G'}{G}\right)^{-1} + 2\lambda\sqrt{\frac{\beta}{\alpha(2-n)M}}, \qquad (42)$$

where

$$\xi = x - t\sqrt{k^2 + \frac{n^2\alpha}{2M}}. \qquad (43)$$

From (20) and (42), we deduce the following types of traveling wave solutions of Eq.(36):

Family 1. If $M > 0$ and $\frac{\beta}{\alpha} < 0$ then we have the hyperbolic solution

$$\begin{aligned}\psi &= 2\lambda\sqrt{\tfrac{\beta}{\alpha(2-n)M}} + \tfrac{8}{M}\sqrt{\tfrac{\beta}{\alpha(2-n)}} \times \\ &\left[\left(\frac{A\cosh(\frac{1}{2}\sqrt{M}\ \xi) + B\sinh(\frac{1}{2}\sqrt{M}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{M})\ \xi) + B\cosh(\frac{1}{2}\sqrt{M}\ \xi)}\right) - \frac{\lambda}{\sqrt{M}}\right]^{-1}.\end{aligned} \qquad (44)$$

Family 2. If $M < 0$ and $\frac{\beta}{\alpha} > 0$ then we have the trigonometric solution

$$\begin{aligned}\psi &= 2\lambda\sqrt{\tfrac{\beta}{\alpha(2-n)M}} - \tfrac{8}{M}\sqrt{\tfrac{\beta}{\alpha(n-2)}} \times \\ &\left[\left(\frac{-A\sin(\frac{1}{2}\sqrt{-M}\xi) + B\cos(\frac{1}{2}\sqrt{-M}\xi)}{A\cos(\frac{1}{2}\sqrt{-M}\xi) + B\sin(\frac{1}{2}\sqrt{-M}\xi)}\right) - \frac{\lambda}{\sqrt{-M}}\right]^{-1}.\end{aligned} \qquad (45)$$

On substituting (44) and (45) into (38), we have the traveling wave solutions of Eq. (36).

In particular, if $B = 0, A \neq 0, \lambda > 0, \mu = 0$, then we obtain the solitary wave solutions of Eq.(36) as follows:

$$u = \left\{ 2\sqrt{\frac{\beta}{\alpha(2-n)}} + \frac{8}{\lambda^2}\sqrt{\frac{\beta}{\alpha(2-n)}} \left[\coth\frac{\lambda}{2}\xi - 1\right]^{-1} \right\}^{\frac{2}{n}} \qquad (46)$$

while if $B \neq 0,\ A^2 > B^2, \lambda > 0, \mu = 0$, then we have

$$u = \left\{ \sqrt{\frac{\beta}{\alpha(2-n)}} \left(2 + \frac{8}{\lambda^2}\left[\tanh(\xi_0 + \frac{1}{2}\lambda\xi) - 1\right]^{-1}\right) \right\}^{\frac{2}{n}} \qquad (47)$$

where $\xi_0 = \tanh^{-1}(\frac{A}{B})$. These solutions are completely physical in the case $\frac{\beta}{\alpha} < 0$ and $k^2 > -\frac{n^2\alpha}{2\lambda^2}$.

3.2 Example 4. The nonlinear reaction- diffusion equation

In this section we study the following nonlinear reaction-diffusion equation [44]:

$$u_t = (\alpha u^{p-1}u_x)_x - \beta u + \gamma u^p, \qquad (48)$$

where $u(x,t)$ describes a population density, α,β, γ are constant coefficients with $\gamma > 0$ and $p \neq 1$. The traveling wave variable $u(x,t) = u(\xi)$ and $\xi = k(x - Vt)$ permits us converting Eq. (48) into the ODE:

$$kVu' + \alpha^2k^2[(p-1)u^{p-2}u'^2 + u^{p-1}\ u''] - \beta u + \gamma u^p = 0. \qquad (49)$$

Considering the homogeneous balance between the highest order linear derivative u' and the nonlinear term $u^{p-1}\ u''$ in (49), we get $m=\frac{1}{1-p}$. According to step 3, we take the transformation

$$u=[\psi(\ \xi)]^{\frac{1}{1-p}}\ , \tag{50}$$

where $\psi(\ \xi)$ is a new function of ξ. Substituting (50) into (49), we obtain the new ODE

$$\begin{aligned}&kV(1-p)\psi^2\psi'+\alpha^2k^2\left[(2p-1)\psi'^2+(1-p)\ \psi\psi''\right]\\&-\beta(1-p)^2\psi^3+\gamma(1-p)^2\psi^2=0.\end{aligned} \tag{51}$$

Determining the balance number m of the new Eq. (51), we get $m=1$. Thus, the solutions of Eq. (51) has the same form (13). Consequently, using the Maple or Mathematica we get the following results:

Case 1.

$$\begin{aligned}&\alpha_1=\frac{\gamma}{\beta\sqrt{M}},\quad \alpha_0=\frac{\gamma}{2\beta}\left[\frac{\lambda}{\sqrt{M}}+1\right],\\&V=-\frac{\beta\alpha}{\gamma}\sqrt{\frac{-\gamma}{p}},\ \ k=\frac{(p-1)}{\alpha}\sqrt{\frac{-\gamma}{pM}},\alpha_{-1}=0.\end{aligned} \tag{52}$$

Case 2.

$$\begin{aligned}&\alpha_{-1}=\frac{\gamma\mu}{\beta\sqrt{M}},\quad \alpha_0=\frac{\gamma}{2\beta}\left[\frac{\lambda}{\sqrt{M}}+1\right],\\&V=\frac{\beta\alpha}{\gamma}\sqrt{\frac{-\gamma}{p}},k=\frac{(p-1)}{\alpha}\sqrt{\frac{-\gamma}{pM}},\ \ \alpha_1=0,\end{aligned} \tag{53}$$

where $M=\lambda^2-4\mu>0$ and $\frac{\gamma}{p}<0$.

We just list some exact solutions corresponding to case 1 to illustrate the effectiveness of the extended $(\frac{G'}{G})-$ expansion method. Substituting (52) into (13) yields

$$\psi(\xi)\ =\frac{\gamma}{\beta\sqrt{M}}\left(\frac{G'}{G}\right)+\frac{\gamma}{2\beta}\left[\frac{\lambda}{\sqrt{M}}+1\right], \tag{54}$$

where

$$\xi=\frac{(p-1)}{\alpha}\sqrt{\frac{-\gamma}{pM}}\left[x+\ \frac{\beta\alpha t}{\gamma}\sqrt{\frac{-\gamma}{p}}\right]. \tag{55}$$

From (20) and (54), we deduce for $M>0$ that the hyperbolic solution has the form

$$\psi=\frac{\gamma}{2\beta}\left(\frac{A\cosh(\frac{1}{2}\sqrt{M}\xi)+B\sinh(\frac{1}{2}\sqrt{M}\xi)}{A\sinh(\frac{1}{2}\sqrt{M}\xi)+B\cosh(\frac{1}{2}\sqrt{M}\xi)}\right)+\frac{\gamma}{2\beta} \tag{56}$$

On substituting (56) into (50), we have the traveling wave solution of Eq. (48). Note when $M=\lambda^2-4\mu\leq 0$, the $\left(\frac{G'}{G}\right)$- expansion method is no longer effective. This implies that Eq.(48) has no such type solution. In particular, if $B=0, A\neq 0,\ \lambda>0,\mu=0,$ then we get the solitary wave solutions of Eq.(48) as follows:

$$u(\xi)\ =\left\{\frac{\gamma}{2\beta}[\coth\frac{\lambda}{2}\xi+1]\right\}^{\frac{1}{1-p}} \tag{57}$$

while if $B\neq 0,\ \ A^2>B^2,\lambda>0,\mu=0,$ then we have

$$u(\xi)\ =\left\{\frac{\gamma}{2\beta}\left[\tanh(\xi_0+\frac{1}{2}\lambda\xi)+1\right]\right\}^{\frac{1}{1-p}} \tag{58}$$

where $\xi_0=\tanh^{-1}(\frac{A}{B})$. These solutions are completely physical in the case $\frac{\gamma}{\beta}>0$ and $\frac{\gamma}{p}<0$.

3.3 Example 5. The nonlinear modified Burgers equation

In this section we study the following nonlinear modified Burgers equation [45]:

$$u_t+u^2u_x+\alpha u_{xx}=0, \tag{59}$$

where α is a positive constant.

The modified Burgers equation (MBE) is also called the nonlinear advection–diffusion equation. It retains the strong nonlinear aspects of the governing equation in many practical transport problems such as nonlinear waves in a medium with low-frequency pumping or absorption, ion reflection at quasi-perpendicular shocks, turbulence transport, wave processes in thermo-elastic medium, transport and dispersion of pollutants in rivers and sediment transport. The traveling wave variable $u(x,t)=u(\xi)$ and $\xi=k(x-Vt)$ permits us converting Eq. (59) into the ODE in the form:

$$-Vu'+u^2u'+\alpha ku''=0. \tag{60}$$

Considering the homogeneous balance between the highest order linear derivative u'' and the nonlinear term $u^2\ u'$ in (60), we get $m=\frac{1}{2}$. According to step 3, we take the transformation

$$u=[\psi(\ \xi)]^{\frac{1}{2}}\ , \tag{61}$$

where $\psi(\ \xi)$ is a new function of ξ. Substituting (61) into (60), we obtain the new ODE

$$-2V\ \psi\psi'+2\psi^2\psi'+2\alpha k\psi\psi''-\alpha k\psi'^2=0. \tag{62}$$

Determining the balance number m of the new Eq. (62), we get $m=1$. Thus, the solution of Eq. (62) has the same form (13). Consequently, using the Maple or Mathematica we get the following results:

Case 1.

$$\begin{aligned}&\alpha_{-1}=-\tfrac{3}{2}\alpha\mu k,\quad \alpha_0=\tfrac{3}{4}\alpha k\left[-\lambda\pm\sqrt{M}\right],\\&V=\mp\tfrac{\alpha k}{2}\sqrt{M}\ \ ,\alpha_1=0.\end{aligned} \tag{63}$$

Case 2.

$$\begin{aligned}&\alpha_1=\tfrac{3}{2}\alpha k,\quad \alpha_0=\tfrac{3}{4}\alpha k\left[\lambda\pm\sqrt{M}\right],\\&V=\pm\tfrac{\alpha k}{2}\sqrt{M},\quad \alpha_{-1}=0.\end{aligned} \tag{64}$$

where $M=\lambda^2-4\mu>0$.

We just list some exact solutions corresponding to case 1 to illustrate the effectiveness of the extended $(\frac{G'}{G})-$ expansion method. Substituting (63) into (13) yields

$$\psi(\xi)\ =-\frac{3}{2}\alpha\mu k\ \left(\frac{G'}{G}\right)^{-1}+\frac{3}{4}\alpha k\left[-\lambda\pm\sqrt{M}\right], \tag{65}$$

where

$$\xi=k\left[x\pm\frac{\alpha kt}{2}\sqrt{M}\right]\ . \tag{66}$$

From (20) and (65) we deduce for $M > 0$, that the hyperbolic solution has the form

$$\psi = \frac{3}{4}\alpha k\left[-\lambda \pm \sqrt{M}\right] - \frac{3\alpha\mu k}{\sqrt{M}} \times \left[\left(\frac{A\cosh(\frac{1}{2}\sqrt{M}\,\xi)+B\sinh(\frac{1}{2}\sqrt{M}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{M}\,\xi)+B\cosh(\frac{1}{2}\sqrt{M}\,\xi)}\right) - \frac{\lambda}{\sqrt{M}}\right]^{-1}, \quad (67)$$

On submitting (67) into (61), we have the traveling wave solutions of Eq. (59). Note when $M = \lambda^2 - 4\mu \leq 0$, the $\left(\frac{G'}{G}\right)$- expansion method is no longer effective. This implies that Eq.(59) has no such type solution.

3.4 Example 6. The nonlinear Eckhaus equation

In this section we study the nonlinear Eckhaus equation [2, 5, 6, 43] which takes the following form:

$$i\,u_t + u_{xx} + 2u\,(|u|^2)_x + |u|^4\,u = 0, \quad i = \sqrt{-1}\,. \quad (68)$$

This equation is of nonlinear Schrodinger type. Eq. (68) was found in [5] as an asymptotic multiscale reduction of certain classes of nonlinear partial differential equations. In [6], many of the properties of the Eckhaus equation were investigated. The traveling wave variable

$$u(x,t) = U(\xi)e^{i(\alpha x+\beta)t},\ \ \xi = k(x - 2\alpha t), \quad (69)$$

permits us converting Eq. (68) into the ODE:

$$-(\beta + \alpha^2)U + k^2U'' + 4kU^2U' + U^5 = 0. \quad (70)$$

where α, β, k are constants.

Considering the homogeneous balance between the highest order linear derivative U'' and the nonlinear term U^2U' in (70), we get $m = \frac{1}{2}$. According to step 3, we take the transformation

$$U = [\psi(\,\xi)]^{\frac{1}{2}}\,. \quad (71)$$

where $\psi(\,\xi)$ is a new function of ξ . Substituting (71) into (70), we obtain the new ODE

$$2k^2\,\psi\psi'' - k^2\psi'^2 - 4(\beta+\alpha^2)\psi^2 + 8k\psi^2\psi' + 4\psi^4 = 0. \quad (72)$$

Determining the value of m in the new Eq. (72), we get $m = 1$. Thus, the solutions of Eq. (72) has the same form (13). Consequently, using the Maple or Mathematica we get the following results:

Case 1.

$$\alpha_{-1} = -\frac{1}{2}\mu k, \quad \alpha_0 = \frac{k}{4}\left[-\lambda \pm \sqrt{M}\right], \quad \beta = \frac{1}{4}[k^2M - 4\alpha^2],\ \ \alpha_1 = 0\,, \quad (73)$$

Case 2.

$$\alpha_1 = \frac{1}{2}k, \quad \alpha_0 = \frac{k}{4}\left[\lambda \pm \sqrt{M}\right], \quad \beta = \frac{1}{4}[k^2M - 4\alpha^2], \quad \alpha_{-1} = 0\,, \quad (74)$$

where $M = \lambda^2 - 4\mu > 0$.

We just list some exact solutions corresponding to case 1 to illustrate the effectiveness of the extended $(\frac{G'}{G})-$ expansion method. Substituting (73) into (13) yields

$$\psi(\xi) = -\frac{1}{2}\mu k\left(\frac{G'}{G}\right)^{-1} + \frac{k}{4}\left[-\lambda \pm \sqrt{M}\right]. \quad (75)$$

From (20) and (75) we deduce for $M > 0$ that the hyperbolic solution has the form

$$U = \left\{-\frac{\mu k}{\sqrt{M}}\left[\left(\frac{A\cosh(\frac{1}{2}\sqrt{M}\xi)+B\sinh(\frac{1}{2}\sqrt{M}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{M}\xi)+B\cosh(\frac{1}{2}\sqrt{M}\xi)}\right) - \frac{\lambda}{\sqrt{M}}\right]^{-1} - \frac{k}{4}(-\lambda \pm \sqrt{M})\right\}^{\frac{1}{2}}, \quad (76)$$

while if $M = \lambda^2 - 4\mu = 0$, then we have the rational solution

$$U(\xi) = \left[-\frac{1}{8}\,\lambda^2 k\,\left(\frac{B}{B\xi + A} - \frac{\lambda}{2}\right)^{-1} - \frac{k\lambda}{4}\right]^{\frac{1}{2}} \quad (77)$$

On substituting (76) and (77) into (69), we have the traveling wave solutions of Eq. (68). Note that (77) is completely physical if $\left[\frac{B}{B\xi+A} - \frac{\lambda}{2}\right] > -\frac{2}{\lambda}, \lambda > 0, k > 0$ and $\mu \geq 1$.

4 Conclusion

In this work, the modified (G'/G)- expansion method has been successfully applied to find exact solutions of some nonlinear PDEs, via the Klein-Gordon equations, the reaction-diffusion equation, the modified Burgers equation and the Eckhaus equation. We have shown that this method is direct,concise and effective, and can be applied to other nonlinear PDEs in the mathematical physics. The ansatz proposed in this paper is more general than the ansatz proposed in Wang et al [26]. If we set the parameters in the proposed method to special values, Wang et al method can be recovered by this method. Therefore, our method is more powerful than Wang et al's method.

References:

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, nonlinear Evolution Equations and Inverse Scattering Transform*, Cambridge Univ. Press, Cambridge 1991.

[2] M. J. Ablowitz, C. D. Ahrens and S. Lillo, On a Quasi integrable discrete Eckhaus equation, *J. Nonlinear Math.Phys.*, 12 (2005), pp.1–12.

[3] C. L. Bai and H. Zhao, Generalized method to construct the solitonic solutions to (3+1)- dimensional nonlinear equation, *Phys. Lett. A*, 354 (2006), pp.428–436.

[4] F. Cariello and M. Tabor, Similarity reductions from extended Painlevé expansions for nonintegrable evolution equations, *Physica D*, 53 (1991), pp.59–70.

[5] F. Calogero and W. Eckhaus, Nonlinear evolution equations, rescalings, model PDES and their integrability I., Inv Probl 3 (1987), 229–62.

[6] F. Calogero and S. Lillo, The Eckhaus PDE $i\ u_t + u_{xx} + 2u\ (|u|^2)_x + |u|^4\ u = 0$, *Inv Probl.*, 4 (1987), No.4, pp.633–82.

[7] R. Conte and M. Musette, Link between solitary waves and projective Riccati equations, *J. Phys. A: Math. Gen.*, 25 (1992), pp.5609–5625.

[8] E. G. Fan, Extended tanh-function method and its applications to nonlinear equations, *Phys. Lett. A*, 277 (2000), pp.212–218.

[9] E. G. Fan, Multiple traveling wave solutions of nonlinear evolution equations using a unified algebraic method, *J. Phys.A. Math. Gen.*, 35 (2002), pp.6853–6872.

[10] X. Feng, Exploratory approach to explicit solution of nonlinear evolution equations, *Int. J. Theor. Phys.*, 39 (2000), pp.207–222.

[11] R. Hirota, Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices, *J. Math. Phys.*, 14 (1973), pp.810–816.

[12] R. Hirota and J. Satsuma, Soliton solutions of a coupled Korteweg-de Vries equation, *Phys. Lett. A*, 85 (1981), pp.407–408.

[13] J. H. He and X. H. Wu, Exp-function method for nonlinear wave equations, *Chaos, Solitons and Fractals*, 30 (2006), pp.700–708.

[14] J. L. Hu, Explicit solutions to three nonlinear physical models, *Phys. Lett. A*, 287 (2001), pp.81–89.

[15] J. L. Hu, A new method for finding exact traveling wave solutions to nonlinear partial differential equations, *Phys. Lett. A*, 286 (2001), pp.175–179.

[16] J. L. Hu, A new method of exact traveling wave solution for coupled nonlinear differential equations, *Phys. Lett. A*, 322 (2004), pp.211–216.

[17] E. Infeld and G. Rowlands, *Nonlinear wave soliton, Chaos*, Cambridge Univ. Press, Cambridge, 2000.

[18] B. Jang, Exact traveling wave solutions of nonlinear Klein Gordon equations, *Chaos, Solitons and Fractals*, 41 (2009), pp.646–654.

[19] X. Z. Li and M. L. Wang, A sub-ODE method for finding exact solutions of a generalized KdV-mKdV equation with higher order nonlinear terms, *Phys.Lett. A*, 361 (2007), pp.115–118.

[20] B. Li, Y. Chan and H. Zhang, Explicit exact solutions for new general two- dimensional KdV-type and two-dimensional KdV Burgers type equations with nonlinear terms of any order, *J. Phys. A Math. Gen.*, 35 (2002), pp.8253–8265.

[21] M. R. Miura,*Backlund Transformation*, Springer-Verlag, Berlin 1978.

[22] A. V. Porubov, Periodical solution to the nonlinear dissipative equation for surface waves in a convecting liquid, *Phys. Lett. A*, 221 (1996), pp.391–394.

[23] C. Rogers and W. F. Shadwick, *Backlund Transformations*, Academic Press, New York 1982.

[24] Sirendaoreji, Auxiliary equation method and new solutions of Klien Gordon equation, *Chaos, Solitons and Fractals*, 31 (2007), pp.943–950.

[25] M. Wang and X. Li, Extended F-expansion and periodic wave solutions for the generalized Zakharov equations, *Phys. Lett. A*, 343 (2005), pp.48–54.

[26] M. L. Wang, X. Z. Li and J. L. Zhang, The $(\frac{G'}{G})$-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics, *Phys. Lett. A*, 372 (2008), pp.417–423.

[27] M. Wazwaz, The tanh and sine- cosine method for compact and noncompact solutions of nonlinear Klein Gordon equation, *Appl. Math. Comput.*, 167 (2005), pp.1179–1195.

[28] M. Wazwaz, Compacton soliton and periodic solution for some forms of nonlinear Klein Gordon equations, *Chaos, Solitons and Fractals*, 28 (2006), pp.1005–1013.

[29] L. Yang, J. Liu and K. Yang, Exact solutions of nonlinear PDE, nonlinear transformations and reduction of nonlinear PDE to a quadrature, *Phys. Lett. A*, 278 (2001), pp.267–270.

[30] Z. Y. Yan, New explicit traveling wave solutions for two new integrable coupled nonlinear evolution equations, *Phys.Lett. A*, 292 (2001), pp.100–106.

[31] Z. Y. Yan, Generalized method and its application in the higher-order nonlinear Schrodinger equation in nonlinear optical fibres, *Chaos, Solitons & Fractals*, 16 (2003), pp.759–766.

[32] Z. Y. Yan, The new constructive algorithm and symbolic computation applied to exact solutions of nonlinear wave equations, *Phys. Lett. A*, 331 (2004), pp.193–200.

[33] Z. Y. Yan, Jacobi elliptic function solutions of nonlinear wave equations via the new sinh-Gordon equation expansion method, J. Phys. A: Math. Gen., 36 (2003), 1916-1973.

[34] Z. Y. Yan, A reduction mKdV method with symbolic computation to constract new doubly- periodic solutions for nonlinear wave equations, *Int. J. Mod. Phys. C*, 14 (2003), pp.661–672.

[35] Z. Y. Yan, The new tri-function method to multiple exact solutions of nonlinear wave equations, *Physica Scripta*, 78 (2008), 035001.

[36] Z. Y. Yan, Periodic, solitary and rational wave solutions of the 3D extended quantum Zakharov–Kuznetsov equation in dense quantum plasmas,*Physics Letters A*, 373 (2009), pp.2432–2437.

[37] Z. Y. Yan and H. Q. Zhang, New explicit solitary wave solutions and periodic wave solutions for Whitham–Broer–Kaup equation in shallow water, *Phys. Lett. A*, 285 (2001), pp.355–362.

[38] Z. Y. Yan, Abundant families of Jacobi elliptic functions of the (2+1)-dimensional integrable Davey-Stawartson-type equation via a new method, *Chaos, Solitons and Fractals*, 18 (2003), pp.299–309.

[39] Z. Y. Yan and H. Q. Zhang, New explicit and exact traveling wave solutions for a system of variant Boussinesq equations in mathematical physics, *Phys. Lett. A*, 252 (1999), pp.291–296.

[40] Z. Yu-Bin and L. Chao, Application of modified $(\frac{G'}{G})$ expansion method to traveling wave solutions for Whitham- Kaup- Like equation, *Commu. Theor. Phys.*, 51 (2009), pp.664–670.

[41] E. M. E. Zayed and K. A. Gepreel, The $(\frac{G'}{G})-$ expansion method for finding traveling wave solutions of nonlinear PDEs in mathematical physics, *J. Math. Phys.*, 50 (2009), 013502–013513.

[42] E. M. E. Zayed and S. Al-Joudi, Applications of an improved $(\frac{G'}{G})-$ expansion method to nonlinear PDEs in mathematical physics, AIP Conference Proceeding, *Amer. Institute of Phys.*, 1168 (2009), pp.371–376.

[43] E. M. E. Zayed, New traveling wave solutions for higher dimensional nonlinear evolution equations using a generalized $(\frac{G'}{G})$- expansion method, *J. Phys. A: Math. Theoretical*, 42 (2009), 195202–195214.

[44] E. M. E. Zayed, The $(\frac{G'}{G})-$ expansion method and its applications to some nonlinear evolution equations in the mathematical physics, *J. Appl. Math. Computing*, 30 (2009), pp.89–103.

[45] H. Zhang, New application of $(\frac{G'}{G})-$ expansion, *Commun. Nonlinear Sci. Numer. Simulat.*, 14(2009), pp.3220–3225.