Relatively Relaxed Proximal Point Algorithms for Generalized Maximal Monotone Mappings and Douglas-Rachford Splitting Methods

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Abstract: The theory of maximal set-valued monotone mappings provide a powerful framework to the study of convex programming and variational inequalities. Based on the notion of relatively maximal relaxed monotonicity, the approximation solvability of a general class of inclusion problems is discussed, while generalizing most of investigations on weak convergence using the proximal point algorithm in a real Hilbert space setting. A well-known method of multipliers of constrained convex programming is a special case of the proximal point algorithm. The obtained results can be used to generalize the Yosida approximation, which, in turn, can be applied to generalize first-order evolution equations to the case of evolution inclusions. Furthermore, we observe that the Douglas-Rachford splitting method for finding the zero of the sum of two monotone operators is a specialization of the proximal point algorithm as well. This allows a further generalization and unification of a wide range of convex programming algorithms.

Key–Words: Variational inclusion problems; Relatively maximal relaxed monotone mapping; Generalized resolvent

1 Introduction

Let X be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and with the norm $\|\cdot\|$ on X. We consider the variational inclusion problem: find a solution to

$$0 \in M(x), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on X.

In [16, Theorem 1], Rockafellar investigated the general weak convergence using the proximal point algorithm to the context of solving (1), by showing for M maximal monotone, that the sequence $\{x^k\}$ generated for an initial point x^0 by the proximal point algorithm

$$x^{k+1} \approx P_k(x^k) \tag{2}$$

converges weakly to a solution of (1), provided the approximation is sufficiently accurate as the iteration proceeds, where $P_k = (I + c_kM)^{-1}$ is the classical resolvent of M for a sequence $\{c_k\}$ of positive real numbers, that is bounded away from zero. We observe from (2) that x^{k+1} is an approximate solution to the variational inclusion problem: determine a solution to

$$0 \in M(x) + c_k^{-1}(x - x^k). \tag{3}$$

Next, we recall the theorem of Rockafellar[16, Theorem 1], where the strong monotonicity of M is avoided to achieve a more application-enhanced convergence analysis to the context of convex programming.

Theorem 1 *Let X be a real Hilbert space, and let $M : X \to 2^X$ be maximal monotone. For an arbitrarily chosen initial point x^0, let the sequence $\{x^k\}$ be generated by the proximal point algorithm*

$$x^{k+1} \approx P_k(x^k) \tag{4}$$

such that

$$\|x^{k+1} - P_k(x^k)\| \leq \epsilon_k,$$

where $P_k = (I + c_kM)^{-1}$ and the scalar sequences $\{\epsilon_k\}$ and $\{c_k\}$, respectively satisfy $\Sigma_{k=0}^{\infty}\epsilon_k < \infty$ and $\{c_k\}$ is bounded away from zero. Then the sequence $\{x^k\}$ converges weakly to x^*, a unique solution to (1).

On the other hand, Xu [20] modified the proximal point algorithm considered by Rockafellar [16] into form: choose arbitrarily an initial point $x^0 \in C$ such that

$$x^{n+1} = \alpha_n x^0 + (1-\alpha_n) R^M_{\rho_n}(x^n) + e_n$$

where C is a closed convex subset of X, $\alpha_n \in$ [0,1] and $\{e_n\}$ is the sequence of errors such that $\Sigma_n \|e_n\| < \infty$. Then Xu [20] showed the strong convergence of the algorithm under suitable conditions. It turned out that this algorithm was better than considered by Solodov and Svaiter [4]. More recently, Boikanyo and Morosanu [3] observed that the algorithm in [20] requires that the error sequence must be summable, which is way too strong from computational point of view. They constructed a suitable algorithm as follows:

$$x^{n+1} = (1-\alpha_n) f_n(x^n) + \alpha_n R^M_{\rho_n}(x^n) + e_n,$$

where $f_n : C \to C$ is sequence of nonexpansive mappings, and $R^M_{\rho_n} = (I + \rho_n M)^{-1}$ for $\rho_n > 0$. As a matter of fact, this algorithm is a relaxed version of the proximal point algorithm and its variants. Furthermore, they also presented a different form

$$x^{n+1} = (1-\alpha_n) f_n(x^n) + \alpha_n R^M_{\rho_n}(u) + e_n,$$

where u is any point of X (not necessarily the starting point x^0 of the proximal point algorithm).

Eckstein and Bertsekas [14] relaxed the proximal point algorithm applied in [16], widely cited in literature, and examined the approximation solvability of (1). They further applied the obtained results to the Douglas-Rachford splitting method for finding the zero of the sum of two monotone mappings, while this turned out to be a specialized case of the proximal point algorithm. Note that most of the variational problems, including minimization or maximization of functions, variational inequality problems, quasivariational inequality problems, decision and management sciences, and engineering sciences problems can be unified into form (1), and the notion of the general maximal monotonicity has played a crucially significant role by providing a powerful framework to develop and use suitable proximal point algorithms in exploring and further studying convex programming as well as variational inequalities. For more details, we recommend the reader [1]–[22].

In this communication, we examine the approximation solvability of variational inclusion problem (1) by introducing the notion of relatively maximal relaxed monotone mappings, and derive some significant results involving relatively maximal relaxed monotone mappings [2] to that setting. The notion of the relatively maximal relaxed monotonicity is based on the notion of $A-$maximal relaxed monotonicity [1] and its variants introduced and studied in [6]–[13] and is more general than the usual maximal monotonicity, especially it could not be achieved to that context, but it seems to be application-oriented. More details on relaxed and hybrid proximal point algorithms can be found in [1]–[5], [8]–[14], [16]–[18],[21, 22]. We present a generalization to a well-cited work (in literature) of Eckstein and Bertsekas [14, Theorem 3] to the case of relatively maximal $(m)-$relaxed monotone mappings with some specializations, while the obtained results generalize the result of Agarwal and Verma [2]. In a way it seems interesting that we observe that our findings do not reduce to existing results in trivial sense unless there is a real shift from the maximal relaxed monotonicity (with respect to mapping A) to maximal monotonicity. We note that our main results, Theorem 3.2 and Theorem 3.3 on the approximation solvability of (1) differ significantly from that of [14, Theorem 3] in the sense that M is without the monotonicity assumptions, while the relative maximal $(m)-$relaxed monotonicity (and relative maximal monotonicity) is applied instead of just the maximal monotonicity. However, the construction of Theorem 3.2 collapses when $A = I$, but Theorem 3.3 still holds for $A = I$ and reduces to the case of the maximal monotonicity results in [14, Theorem 3]. There exists a tremendous amount of research work on new developments and further applications of proximal point algorithms in literature to approximating solutions of variational inclusion problem of the form (1) in different space settings, especially in Hilbert as well as Banach space settings.

2 Preliminaries

In this section, first we introduce the notion of the *relatively maximal relaxed monotonicity* and then we derive some basic properties along with some auxiliary results for the problem on hand. Let X be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and with the norm $\|\cdot\|$ on X.

Definition 2 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a set-valued mapping, and $A : X \to X$ be a single-valued mapping on X. The map M is said to be:*

(i) *Monotone if*

$$\langle u^*-v^*, u-v\rangle \geq 0\,\forall\,(u,u^*),(v,v^*) \in graph(M).$$

(ii) *Strictly monotone if M is monotone and equality holds only if u=v.*

(iii) *Strongly monotone if there exists a positive constant r such that*

$$\langle u^* - v^*, u - v\rangle \geq r\|u - v\|^2$$

$\forall\,(u,u^*),(v,v^*) \in graph(M)$.

(iv) *Strongly expansive if there exists a positive constant r such that*

$$\|u^*-v^*\| \geq r\|u-v\|\,\forall\,(u,u^*),(v,v^*) \in graph(M).$$

(v) *Relaxed monotone if there exists a positive constant m such that*

$$\langle u^* - v^*, u - v\rangle \geq -m\|u - v\|^2$$

$\forall\,(u,u^*),(v,v^*) \in graph(M)$.

(vi) *Cocoercive if there exists a positive constant c such that*

$$\langle u^* - v^*, u - v\rangle \geq c\|u^* - v^*\|^2$$

$\forall\,(u,u^*),(v,v^*) \in graph(M)$.

(vii) *Monotone with respect to A if*

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq 0$$

$\forall\,(u,u^*),(v,v^*) \in graph(M)$.

(viii) *Strictly monotone with respect to A if M is monotone with respect to A and equality holds only if u=v.*

ix) *Strongly monotone with respect to A if there exists a positive constant r such that*

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq r\|u - v\|^2$$

$\forall\,(u,u^*),(v,v^*) \in graph(M)$.

(x) *Relaxed monotone with respect to A if there exists a positive constant m such that*

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq -m\|u - v\|^2$$

$\forall\,(u,u^*),(v,v^*) \in graph(M)$.

(xi) *Cocoercive with respect to A if there exists a positive constant c such that*

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq c\|u^* - v^*\|^2$$

$\forall\,(u,u^*),(v,v^*) \in graph(M)$.

Definition 3 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a set-valued mapping on X. The map M is said to be:*

(i) *Nonexapansive if*

$$\|u^* - v^*\| \leq \|u - v\|$$

$\forall\,(u,u^*),(v,v^*) \in graph(M)$.

(ii) *Lipschitz continuous if there is a positive constant s such that*

$$\|u^* - v^*\| \leq s\|u - v\|$$

$\forall\,(u,u^*),(v,v^*) \in graph(M)$.

Definition 4 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a set-valued mapping, and $A : X \to X$ be a single-valued mapping on X. The map M is said to be relatively maximal relaxed monotone (with respect to A) with a positive constant m if*

(i) *M is relatively relaxed monotone (with respect to A) with a constant m, that is,*

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq -m\|u - v\|^2$$

$\forall\,(u,u^*),(v,v^*) \in graph(M)$,

(ii) $R(I + \rho M) = X\,for\,\rho > 0$.

Definition 5 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a set-valued mapping, and $A : X \to X$ be a single-valued mapping on X. The map M is said to be relatively maximal monotone (with respect to A) if*

(i) *M is relatively monotone (with respect to A), that is,*

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq 0$$

$\forall\,(u,u^*),(v,v^*) \in graph(M)$,

(ii) $R(I + \rho M) = X\,for\,\rho > 0$.

Definition 6 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a set-valued mapping, and $A : X \to X$ be a single-valued mapping on X. Suppose that map M is relatively maximal relaxed monotone (with respect to A) with a positive constant m. Then the resolvent $R^M_{\rho,m,A} : X \to X$ is defined by*

$$R^M_{\rho,m,A}(u) = (I + \rho M)^{-1}(u)\, for\; r - \rho m > 0.$$

Proposition 7 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a relatively maximal relaxed monotone set-valued mapping with a constant $m > 0$, and let $A : X \to X$ be a strongly monotone single-valued mapping on X with a constant $r > 0$. Then the generalized resolvent $R^M_{\rho,m,A} = (I + \rho M)^{-1}$ is single-valued for $r - \rho m > 0$.*

Proof: The proof follows from the definition of the generalized resolvent. For any $z \in X$, consider $x, y \in (I + \rho M)^{-1}(z)$. Then

$$-x + z \in \rho M(x)\; and\; -y + z \in \rho M(y).$$

Since M is relatively maximal $(m)-$relaxed monotone for $m > 0$, it implies

$$-\rho m\|x-y\|^2 \leq -\langle x-y, A(x)-A(y)\rangle \leq -r\|x-y\|^2$$

$$\Rightarrow (r - \rho m)\|x - y\|^2 \leq 0$$

$$\Rightarrow x = y\; for\; r - \rho m > 0.$$

Definition 8 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a relatively maximal monotone set-valued mapping, and let $A : X \to X$ be strongly monotone with a constant $r > 0$. Then the generalized resolvent is defined by*

$$R^M_{\rho,A}(u) = (I + \rho M)^{-1}(u)\, for\, r > 0.$$

Proposition 9 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a relatively maximal monotone set-valued mapping, and let $A : X \to X$ be a strongly monotone single-valued mapping on X with a constant $r > 0$. Then the generalized resolvent $R^M_{\rho,A} = (I + \rho M)^{-1}$ is single-valued for $r > 0$.*

Proof: The proof follows from the definition of the generalized resolvent.

Definition 10 *Let X be a real Hilbert space. A map $M : X \to 2^X$ is said to be maximal monotone if*

(i) M is monotone, i.e.,

$$\langle u^*-v^*, u-v\rangle \geq 0\, \forall\, (u, u^*), (v, v^*) \in graph(M),$$

(ii) $(I + \rho M) = X$ for $\rho > 0$.

Furthermore, the classical resolvent J^M_ρ is defined by

$$R^M_\rho(u) = (I + \rho M)^{-1}(u).$$

Next, we include the following examples on the relative monotonicity.

Example 11 *Let $X = (-\infty, +\infty)$, $A(x) = -\frac{1}{2}x$ and $M(x) = -x$ for all $x \in X$. Then M is relatively monotone, but not monotone.*

Example 12 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a maximal monotone mapping. Suppose that $M_\rho = \rho^{-1}(I - R^M_\rho)$ denotes the Yosida approximation of M, where R^M_ρ is the classical resolvent defined by*

$$R^M_\rho(u) = (I + \rho M)^{-1}(u).$$

Then, for any $u, v \in X$, we have

$$\langle M_\rho(u) - M_\rho(v), R^M_\rho(u) - R^M_\rho(v)\rangle \geq 0.$$

Therefore, M_ρ is relatively monotone with respect to R^M_ρ.

Lemma 13 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a set-valued mapping, and let $A : X \to X$ be a strongly monotone mapping with a constant $r > 0$. Then the following implications hold:*

(a) M is relatively $(m)-$relaxed monotone iff its resolvent $R^M_{\rho,m,A}$ is $(r)-$cocoercive.

(b) M is relatively maximal relaxed monotone iff its resolvent $R^M_{\rho,m,A}$ is $(r)-$cocoercive and $dom(R^M_{\rho,m,A}) = X$.

Proof: To prove (a), we use the definition of the scaling, addition and inversion operations as follows:

$$(x, y) \in M \Leftrightarrow (x + \rho y, x) \in (I + \rho M)^{-1}.$$

As a result, $\forall (x, y), (x', y') \in M$, we begin with (for $r - \rho m > 0$)

$$\begin{aligned} & M\, is\, relatively\, (m) - relaxed\, monotone \\ \Leftrightarrow\; & \langle A(x') - A(x), y' - y\rangle \geq -m\|x' - x\|^2 \\ \Leftrightarrow\; & \langle A(x') - A(x), \rho y' - \rho y\rangle \geq -\rho m\|x' - x\|^2 \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow && \langle A(x') - A(x), x' - x + \rho y' - \rho y\rangle \\
&\geq && \langle A(x') - A(x), x' - x\rangle - \rho m\|x' - x\|^2 \\
&\Leftrightarrow && \langle x' + \rho y' - (x + \rho y), A(x') - A(x)\rangle \\
&\geq && (r - \rho m)\|x' - x\|^2 \\
&\Leftrightarrow && (I + \rho M)^{-1}\, is \\
& && (r - \rho m) - cocoercive\, relative\, to\, A.
\end{aligned}$$

Based on Lemma 13, we present the Generalized Representation Lemma as follows:

Lemma 14 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a relatively maximal $(m)-$relaxed monotone set-valued mapping for $m > 0$, and let $A : X \to X$ be a strongly monotone mapping with a constant $r > 0$. Then every element $z \in X$ can be represented exactly one way as $x + \rho y = z$ for $\rho > 0$ and $y \in M(x)$.*

Proof: The follows from Lemma 13.

3 New Generalization to Linear Convergence

This section deals with the significant generalizations to Rockafellar's theorem [16, Theorem 1], to the result of Eckstein and Bertsekas [14, Theorem 3], and to the result of Agarwal and Verma [2, Theorem 3.2] in light of the new framework of the relative maximal $(m)-$relaxed monotonicity. We will give some lemmas for further use.

Lemma 15 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a relatively maximal $(m)-$relaxed monotone set-valued mapping for $m > 0$, and let $A : X \to X$ be a strongly monotone mapping with a constant $r > 0$. Furthermore, suppose that $(I - A)oR^M_{\rho,m,A}$ is $(\lambda)-$cocoercive with a constant $\lambda > 0$. Then the generalized resolvent $R^M_{\rho,m,A} : X \to X$, defined by*

$$R^M_{\rho,m,A}(u) = (I + \rho M)^{-1}(u)\, for\, r - \rho m > 0,$$

satisfies the inequality

$$\langle R^M_{\rho,m,A}(u) - R^M_{\rho,m,A}(v), u - v\rangle$$

$$\geq (r - \rho m)\|R^M_{\rho,m,A}(u) - R^M_{\rho,m,A}(v)\|^2$$

where $r - \rho m > 0$.

Proof: Using the definition of the generalized resolvent in conjunction with other hypotheses, we have

$$\langle u - v, A(R^M_{\rho,m,A}(u)) - A(R^M_{\rho,m,A}(v))\rangle$$

$$\geq \langle A(R^M_{\rho,m,A}(u)) - A(R^M_{\rho,m,A}(v))$$

$$, R^M_{\rho,m,A}(u) - R^M_{\rho,m,A}(v)\rangle$$

$$-\rho m\|R^M_{\rho,m,A}(u) - R^M_{\rho,m,A}(v)\|^2$$

$$\geq (r - \rho m)\|R^M_{\rho,m,A}(u) - R^M_{\rho,m,A}(v)\|^2.$$

Thus, we have

$$\langle u - v, A(R^M_{\rho,m,A}(u)) - A(R^M_{\rho,m,A}(v))\rangle$$

$$\geq (r - \rho m)\|R^M_{\rho,m,A}(u) - R^M_{\rho,m,A}(v)\|^2,$$

where $r - \rho m > 0$. Based on the above inequality, we derive (for $(r - \rho m) > 0$)

$$\langle u - v, R^M_{\rho,m,A}(u) - R^M_{\rho,m,A}(v)\rangle$$

$$\geq (r - \rho m)\|R^M_{\rho,m,A}(u) - R^M_{\rho,m,A}(v)\|^2$$

$$+\lambda\|(I - A)oR^M_{\rho,m,A}(u) - (I - A)oR^M_{\rho,m,A}(v)\|^2$$

$$\geq (r - \rho m)\|R^M_{\rho,m,A}(u) - R^M_{\rho,m,A}(v)\|^2.$$

Lemma 16 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a relatively maximal monotone set-valued mapping for $m > 0$, and let $A : X \to X$ be a strongly monotone mapping with a constant $r > 0$. Furthermore, suppose that $(I - A)oR^M_{\rho,A}$ is $(\lambda)-$cocoercive with a constant $\lambda > 0$. Then the generalized resolvent $R^M_{\rho,A} : X \to X$, defined by*

$$R^M_{\rho,A}(u) = (I + \rho M)^{-1}(u)\, for\, \rho > 0,$$

satisfies the inequality

$$\langle R^M_{\rho,A}(u) - R^M_{\rho,A}(v), u - v\rangle$$

$$\geq r\|R^M_{\rho,A}(u) - R^M_{\rho,A}(v)\|^2.$$

Theorem 17 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a relatively maximal $(m)-$relaxed monotone set-valued mapping for $m > 0$, and let $A : X \to X$ be a strongly monotone mapping with a constant $r > 0$. Furthermore, suppose that $(I - A)oR^M_{\rho,m,A}$ is $(\lambda)-$cocoercive with a constant $\lambda > 0$. Then the following statements are mutually equivalent:*

(i) An element $u \in X$ is a solution to (1).

(ii) For an $u \in X$, we have

$$u = R^M_{\rho,m,A}(u),$$

where $R^M_{\rho,m,A}(u) = (I + \rho M)^{-1}(u)$ for $r - \rho m > 0$.

Proof: The proof follows applying the definition of the generalized resolvent corresponding to M.

Theorem 18 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a relatively maximal $(m)-$relaxed monotone set-valued mapping for $m > 0$, and let $A : X \to X$ be a strongly monotone mapping (with a constant $r > 0$) and weakly continuous. Furthermore, suppose that $(I-A)oR^M_{\rho_k,m,A}$ is $(\lambda)-$cocoercive with a constant $\lambda > 1$. For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the proximal point algorithm*

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k y^k \,\forall k \geq 0 \tag{5}$$

such that

$$\|y^k - J^M_{\rho_k,m,A}(x^k)\| \leq \epsilon_k, \tag{6}$$

where $\{\alpha_k\}$, $\{\epsilon_k\}$, $\{\rho_k\} \subseteq [0,\infty)$ are scalar sequences with $e_1 = \Sigma_{k=0}^{\infty}\epsilon_k < \infty$, $a_1 = \inf \alpha_k > 0$, $a_2 = \sup \alpha_k < 2$, $\rho = \inf \rho_k$. Suppose that the sequence $\{x^k\}$ is bounded in the sense that there exists at least one solution to (1). Then, for $J_k = I - R^M_{\rho_k,m,A}$, we have

$$\begin{aligned} & \langle J_k(u) - J_k(v), u - v\rangle \\ \geq\ & \|J_k(u) - J_k(v)\|^2 \\ +\ & (r - \rho_k m - 1)\|R^M_{\rho_k,m,A}(u) - R^M_{\rho_k,m,A}(v)\|^2, \end{aligned}$$

where $r - \rho_k m > 1$ and the sequence $\{x^k\}$ converges weakly to a solution of (1) for

$$e_1 = \Sigma_{k=0}^{\infty}\epsilon_k < \infty, a_1 = \inf \alpha_k > 0,$$

$a_2 = \sup \alpha_k < 2, \rho = \inf \rho_k$.

Proof: We start off the proof as follows: using Lemma 15, for $u, v \in X$, we have

$$\begin{aligned} & \langle J_k(u) - J_k(v), u - v\rangle \\ =\ & \langle J_k(u) - J_k(v), J_k(u) - J_k(v) \\ +\ & R^M_{\rho_k,m,A}(u) - R^M_{\rho_k,m,A}(v)\rangle \\ =\ & \|J_k(u) - J_k(v)\|^2 \\ +\ & \langle R^M_{\rho_k,m,A}(u) - R^M_{\rho_k,m,A}(v), u - v\rangle \\ -\ & \|R^M_{\rho_k,m,A}(u) - R^M_{\rho_k,m,A}(v)\|^2 \\ \geq\ & \|J_k(u) - J_k(v)\|^2 + (r - \rho_k m) \\ .\ & \|R^M_{\rho_k,m,A}(u) - R^M_{\rho_k,m,A}(v)\|^2 \\ -\ & \|R^M_{\rho_k,m,A}(u) - R^M_{\rho_k,m,A}(v)\|^2 \\ =\ & \|J_k(u) - J_k(v)\|^2 + (r - \rho_k m - 1) \\ .\ & \|R^M_{\rho_k,m,A}(u) - R^M_{\rho_k,m,A}(v)\|^2, \end{aligned}$$

where $r - \rho_k m > 1$.

Suppose that x^* is a zero of M. Note that any zero of M is a fixed point of $R^M_{\rho_k,m,A}$, so by Theorem 17, $J_k(x^*) = 0$ for all k. We define for all k

$$\begin{aligned} z^{k+1} &= (1-\alpha_k)x^k + \alpha_k R^M_{\rho_k,m,A}(x^k) \\ &= (I - \alpha_k J_k)(x^k). \end{aligned}$$

Next, we estimate using the above formulation

$$\begin{aligned} & \|z^{k+1} - x^*\|^2 \\ =\ & \|x^k - x^* - \alpha_k(J_k(x^k) - J_k(x^*))\|^2 \\ =\ & \|x^k - x^*\|^2 - 2\alpha_k\langle x^k - x^*, J_k(x^k) - J_k(x^*)\rangle \\ +\ & \alpha_k^2\|J_k(x^k) - J_k(x^*)\|^2 \\ \leq\ & \|x^k - x^*\|^2 - 2\alpha_k[\|J_k(x^k) - J_k(x^*)\|^2 \\ +\ & (r - \rho_k m - 1)\|R^M_{\rho_k,m,A}(u) - R^M_{\rho_k,m,A}(v)\|^2] \\ +\ & \alpha_k^2\|J_k(x^k) - J_k(x^*)\|^2 \\ \leq\ & \|x^k - x^*\|^2 - 2\alpha_k[\|J_k(x^k) - J_k(x^*)\|^2 \\ +\ & \alpha_k^2\|J_k(x^k) - J_k(x^*)\|^2 \\ =\ & \|x^k - x^*\|^2 - \alpha_k(2 - \alpha_k)\|J_k(x^k) - J_k(x^*)\|^2 \\ \leq\ & \|x^k - x^*\|^2 - a_1(2 - a_2)\|J_k(x^k) - J_k(x^*)\|^2. \end{aligned}$$

Since $a_1(2 - a_2) > 0$, it follows that

$$\|z^{k+1} - x^*\| \leq \|x^k - x^*\|,$$

while

$$\|x^{k+1} - z^{k+1}\| \leq \alpha_k\epsilon_k.$$

It follows from the above arguments that

$$\begin{aligned} & \|x^{k+1} - x^*\| = \|x^{k+1} - z^{k+1} + z^{k+1} - x^*\| \\ \leq\ & \alpha_k\epsilon_k + \|x^k - x^*\|. \end{aligned}$$

Combining the inequality for all k, we have

$$\begin{aligned} \|x^{k+1} - x^*\| &\leq \|x^0 - x^*\| + \Sigma_{i=0}^{k}\alpha_i\epsilon_i \,\forall\, i \\ &\leq \|x^0 - x^*\| + 2e_1. \end{aligned} \tag{7}$$

Hence, the sequence $\{x^k\}$ is bounded.

Now we turn our attention to establish the weak convergence. Since

$$\begin{aligned} & \|x^{k+1} - x^*\|^2 = \|z^{k+1} - x^* + x^{k+1} - z^{k+1}\|^2 \\ =\ & \|z^{k+1} - x^*\|^2 + \|x^{k+1} - z^{k+1}\|^2 \\ +\ & 2\langle z^{k+1} - x^*, x^{k+1} - z^{k+1}\rangle \\ \leq\ & \|z^{k+1} - x^*\|^2 + \|x^{k+1} - z^{k+1}\|^2 \\ +\ & 2\|z^{k+1} - x^*\|\|x^{k+1} - z^{k+1}\| \\ \leq\ & \|x^k - x^*\|^2 - a_1(2 - a_2)\|J_k(x^k) - J_k(x^*)\|^2 \\ +\ & 2\alpha_k\epsilon_k[\|x^0 - x^*\| + 2e_1] \\ +\ & \alpha_k^2\epsilon_k^2 \end{aligned}$$

and the summability of the sequence $\{\epsilon_k\}$ implies the summability of the sequence $\{\epsilon_k^2\}$ (and hence, $e_2 = \Sigma_{k=0}^{\infty}\epsilon_k^2 < \infty$), we have (for all k)

$$\begin{aligned} & \|x^{k+1} - x^*\|^2 \\ \leq & \|x^0 - x^*\|^2 - a_1(2 - a_2)\Sigma_{k=0}^{k}\|J_k(x^k) - J_k(x^*)\|^2 \\ + & 4e_1[\|x^0 - x^*\| + 2e_1] + 4e_2. \end{aligned}$$

Now, as $k \to \infty$, we have

$$\Sigma_{i=0}^{k}\|J_k(x^k) - J_k(x^*)\|^2 < \infty$$

and hence,

$$\lim_{k\to\infty} J_k(x^k) = 0.$$

At this stage of the proof, by Lemma 14, Generalized Representation Property, there exists a unique element $(u^k, v^k) \in M$ such that $u^k + \rho_k v^k = x^k$ for all k. Since $u^k = R^M_{\rho_k,m,A}(x^k)$ and

$$x^k - u^k = x^k - R^M_{\rho_k,m,A}(x^k) \to 0,$$

this, in turn, implies

$$v^k = \rho_k^{-1}(x^k - u^k) \to 0,$$

where ρ_k is bounded away from zero. On the other hand, since the sequence $\{x^k\}$ is bounded, it must have a weak cluster point, say x'. Let $\{x^{k(j)}\}$ be a subsequence of $\{x^k\}$ such that $x^{k(j)} \to x'$ weakly.

Let us consider a point $(u, v) \in M$. Then in conjunction with the relative maximal $(m)-$relaxed monotonicity of M with a positive constant m and the weak continuity of A, we have

$$\langle A(u) - A(u^k), v - v^k\rangle \geq -m\|u - u^k\|^2.$$

It follows that

$$\langle A(u) - A(x^{k(j)}), v - v^k\rangle \geq -m\|u - x^{k(j)}\|^2,$$

or

$$\langle A(u) - A(x'), v - 0\rangle \geq -m\|u - x')\|^2.$$

Therefore, $0 \in M(x')$. Thus, x' is a solution to (1).

Finally, all we need is to show the uniqueness of the weak cluster point $\{x^k\}$. Assume that x^* is a zero of M. Then using

$$\|x^k - x^*\| \leq \|x^0 - x^*\| + 2e_1 \,\forall\, k,$$

we have

$$a^* = \lim_{k\to\infty} \inf \|x^k - x^*\|,$$

which is finite and nonnegative, and it follows

$$\|x^k - x^*\| \to a^*.$$

Now consider any two weak cluster points x_1^* and x_2^* of $\{x^k\}$. Then it follows in light of the above argument that both are zeros of M, and hence

$$a_1^* = \liminf_{k\to\infty} \|x^k - x_1^*\| \text{ and } a_2^* = \liminf_{k\to\infty} \|x^k - x_2^*\|$$

and both exist and are finite. If we express

$$\begin{aligned} & \|x^k - x_2^*\|^2 = \|x^k - x_1^*\|^2 \\ + & 2\langle x^k - x_1^*, x_1^* - x_2^*\rangle \\ + & \|x_1^* - x_2^*\|^2, \end{aligned}$$

then it follows that

$$\begin{aligned} & \lim_{k\to\infty} \langle x^k - x_1^*, x_1^* - x_2^*\rangle \\ = & \frac{1}{2}[a_2^2 - a_1^2 - \|x_1^* - x_2^*\|^2]. \end{aligned}$$

Since x_1^* is a limit point of $\{x^k\}$, the left hand side limit tends to zero. Therefore,

$$a_1^2 = a_2^2 - \|x_1^* - x_2^*\|^2.$$

Similarly, we obtain

$$a_2^2 = a_1^2 - \|x_1^* - x_2^*\|^2.$$

Thus, we conclude that $x_1^* = x_2^*$. □

Next, we observe that Theorem 18, reduces to the following result on the relative maximal monotonicity ([2], Theorem 3.2), which happens to generalize a significant result [14, Theorem 3] as well in literature.

Theorem 19 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a relatively maximal monotone set-valued mapping, and let $A : X \to X$ be a strongly monotone mapping (with a constant $r > 0$) and weakly continuous. We further suppose that $(I - A)oR^M_{\rho_k,A}$ is $(\lambda)-$cocoercive with a constant $\lambda > 1$. For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the proximal point algorithm*

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \,\forall k \geq 0 \qquad (8)$$

such that

$$\|y^k - R^M_{\rho_k,A}(x^k)\| \leq \epsilon_k, \qquad (9)$$

where $\{\alpha_k\}$, $\{\epsilon_k\}$, $\{\rho_k\} \subseteq [0.\infty)$ are scalar sequences with $e_1 = \Sigma_{k=0}^{\infty}\epsilon_k < \infty$, $a_1 = \inf \alpha_k > 0$,

$a_2 = \sup \alpha_k < 2$, $\rho = \inf \rho_k$. *Suppose that the sequence $\{x^k\}$ is bounded in the sense that there exists at least one solution to (1). Then, for $J_k = I - R^M_{\rho_k,A}$, we have*

$$\begin{aligned} & \langle J_k(u) - J_k(v), u - v\rangle \\ \geq\ & \|J_k(u) - J_k(v)\|^2 \\ +\ & (r-1)\|R^M_{\rho_k,A}(u) - R^M_{\rho_k,A}(v)\|^2, \end{aligned}$$

where $r > 1$ and the sequence $\{x^k\}$ converges weakly to a solution of (1) for

$$e_1 = \Sigma_{k=0}^{\infty}\epsilon_k < \infty, a_1 = \inf \alpha_k > 0,$$

$$a_2 = \sup \alpha_k < 2, \rho = \inf \rho_k.$$

4 Some Observations

Under the assumptions of Theorem 18, we observe the following implications:

(a) The resolvent $R^M_{\rho_k,m,A}$ is $(r - \rho_k m)-$cocoercive, i.e.,

$$\langle R^M_{\rho_k,m,A}(u) - R^M_{\rho_k,m,A}(v), u - v\rangle$$
$$\geq (r - \rho_k m)\|R^M_{\rho_k,m,A}(u) - R^M_{\rho_k,m,A}(v)\|^2$$

where $r - \rho_k m > 0$.

(b) Any relatively $(m)-$relaxed monotone mapping M is relatively maximal $(m)-$relaxed monotone if $R(I + \rho M) = X$ for $\rho > 0$.

(c) For $J_k = I - R^M_{\rho_k,m,A}$, we have

$$\langle J_k(u) - J_k(v), u - v\rangle \geq \|J_k(u) - J_k(v)\|^2,$$

for $r - \rho_k m - 1 > 0$.

(d) A mapping M is relatively maximal relaxed monotone if $R^M_{\rho_k,m,A}$ is cocoercive and $D(R^M_{\rho_k,m,A}) = X$.

We further remark that our results hold for relatively maximal $(m)-$relaxed monotone and relatively maximal monotone mappings, while these results do not hold for maximal $(m)-$relaxed monotone mappings since the relative $(m)-$relaxed monotonicity may not imply the $(m)-$relaxed monotonicity, especially the constant term $r - \rho_k m > 1$ in Theorem 18 changes to $1 - \rho_k m < 1$ instead when $A = I$. On the other hand, Theorem 19 on the relative maximal monotonicity seems to be compatible with the of the maximal monotonicity in the sense that it reduces to the following result of Eckstein and Bertsekas [14] for $A = I$.

Theorem 20 *Let X be a real Hilbert space. Let $M : X \to 2^X$ be a maximal monotone set-valued mapping. For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the proximal point algorithm*

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \quad \forall k \geq 0 \tag{10}$$

such that

$$\|y^k - J^M_{\rho_k}(x^k)\| \leq \epsilon_k, \tag{11}$$

where $\{\alpha_k\}$, $\{\epsilon_k\}$, $\{\rho_k\} \subseteq [0, \infty)$ are scalar sequences with $e_1 = \Sigma_{k=0}^{\infty}\epsilon_k < \infty$, $a_1 = \inf \alpha_k > 0$, $a_2 = \sup \alpha_k < 2$, $\rho = \inf \rho_k$. Suppose that the sequence $\{x^k\}$ is bounded in the sense that there exists at least one solution to (1). Then, for $J_k = I - R^M_{\rho_k}$, we have

$$\begin{aligned} & \langle J_k(u) - J_k(v), u - v\rangle \\ \geq\ & \|J_k(u) - J_k(v)\|^2, \end{aligned}$$

where the sequence $\{x^k\}$ converges weakly to a solution of (1) for

$$e_1 = \Sigma_{k=0}^{\infty}\epsilon_k < \infty, a_1 = \inf \alpha_k > 0,$$

$$a_2 = \sup \alpha_k < 2, \rho = \inf \rho_k.$$

5 Douglas–Rachford Splitting Methods

This section is devoted to the Douglas-Rachford splitting method by choosing relatively maximal $(m)-$relaxed monotone mappings S and T with corresponding resolvents (for $\rho > 0$)

$$R^S_{\rho,m,A} = (I + \rho S)^{-1}\, and\, R^T_{\rho,m,A} = (I + \rho T)^{-1}$$

for S and T, respectively. Eckstein and Bertsekas [14] studied the Douglas-Rachford splitting of maximal monotone mapping M in a Hilbert space setting. We generalize this splitting to the case of the map M without the monotonicity assumptions. Let us consider two relatively maximal $(m)-$relaxed monotone mappings S and T such that $M = S + T$ with generalized resolvents $R^S_{\rho,m,A}$ and $R^T_{\rho,m,A}$ corresponding to S and T, respectively. Suppose that the following recursion relating to the sequence $\{z^k\}$ holds:

$$\begin{aligned} z^{k+1} = & R^S_{\rho,m,A}((2R^T_{\rho,m,A} - I)(z^k)) \\ & +(I - R^T_{\rho,m,A})(z^k). \end{aligned} \tag{12}$$

Based on recurrence relation (12), suppose that(x^k, b^k) be a unique element of T such that $x^k + \rho b^k = z^k$ for all $k \geq 0$. Then we have

$$(I - R^T_{\rho,m,A})(z^k) = \rho\, b^k,$$

$$(2R^T_{\rho,m,A} - I)(z^k) = x^k - \rho\, b^k.$$

Similarly, for $(y^k, a^k) \in S$, it follows that

$$R^S_{\rho,m,A}(y^k + \rho\, a^k) = y^k.$$

Based on the above identities, one may write z^{k+1} in terms of z^k alternatively as follows:

I. Find the unique element in $(y^{k+1}, a^{k+1}) \in S$ such that

$$y^{k+1} + \rho\, a^{k+1} = x^k - \rho\, b^k.$$

II. Find the unique element in $(x^{k+1}, b^{k+1}) \in T$ such that

$$x^{k+1} + \rho\, b^{k+1} = y^{k+1} + \rho\, b^k.$$

Next, we consider the mapping

$$G^{S,T}_{\rho,A} = R^S_{\rho,m,A} o (2R^T_{\rho,m,A} - I) + (I - R^T_{\rho,m,A}).$$

Based on the above definition, it can be written as

$$G^{S,T}_{\rho,A} = \{(u + \rho b, v + \rho b) | (u, b) \in T, (v, a) \in S\},$$

where $v + \rho a = u - \rho b$.

Next, we define another mapping in terms of $G^{S,T}_{\rho,A}$ by

$$\begin{aligned} & W^{S,T}_{\rho,A} = (G^{S,T}_{\rho,A})^{-1} - I \\ = & \{(v + \rho b, u - v) | (u, b) \in T, (v, a) \in S\}, \end{aligned}$$

where $v + \rho a = u - \rho b$.

Theorem 21 *Let $W^{S,T}_{\rho,A}$ represent the splitting mapping for S and T for $\rho > 0$. Then following implications hold:*

(i) If S and T are relatively $(m)-$relaxed monotone, and if S and T are $(\lambda)-$strongly monotone with respect to $(I - A)$, then $W^{S,T}_{\rho,A}$ is relatively $(\rho m)-$relaxed monotone.

(ii) If S and T are relatively maximal $(m)-$relaxed monotone, and if S and T are $(\lambda)-$strongly with respect to $(I - A)$, then $W^{S,T}_{\rho,A}$ is relatively maximal $(\rho m)-$relaxed monotone.

Proof: To prove (i), let $u, b, v, a, u', b', v', a' \in X$ be such that

$$(u, b), (u', b') \in T,\ (v, a), (v', a') \in S,$$

$$v + \rho\, a = u - \rho\, b \text{ and } v' + \rho\, a' = u' - \rho\, b'.$$

Then we have

$$a = \frac{1}{\rho}(u - v) - b \text{ and } a' = \frac{1}{\rho}(u' - v') - b',$$

and using the definition of $W^{S,T}_{\rho,A}$, we proceed with

$$\begin{aligned}
& \langle (v' + \rho b') - (v + \rho b), u' - v' - (u - v)\rangle \\
= & \rho\langle (v' + \rho b') - (v + \rho b), \\
& \rho^{-1}(u' - v') - b' - \rho^{-1}(u - v) + b\rangle \\
+ & \rho\langle (v' + \rho b') - (v + \rho b), b' - b\rangle \\
= & \rho\langle (v' - v), \rho^{-1}(u' - v') - b' - \rho^{-1}(u - v) + b\rangle \\
+ & \rho^2\langle b' - b, \rho^{-1}(u' - v') - b' - \rho^{-1}(u - v) + b\rangle \\
+ & \rho\langle (v' - v), b' - b\rangle + \rho^2\langle b' - b, b' - b\rangle \\
= & \rho\langle v' - v, a' - a\rangle - \rho^2\langle b' - b, b' - b\rangle \\
+ & \rho\langle b' - b, u' - u\rangle - \rho\langle b' - b, v' - v\rangle \\
+ & \rho\langle (v' - v), b' - b\rangle + \rho^2\langle b' - b, b' - b\rangle \\
= & \rho\langle v' - v, a' - a\rangle + \rho\langle b' - b, u' - u\rangle \\
= & \rho\langle A(v') - A(v) + (v' - v) - (A(v') - A(v)), \\
& a' - a\rangle \\
+ & \rho\langle b' - b, \\
& A(u') - A(u) + (u' - u) - (A(u') - A(u))\rangle \\
= & \rho\langle A(v') - A(v) + (I - A)(v') - (I - A)(v), \\
& a' - a\rangle \\
+ & \rho\langle b' - b, \\
& A(u') - A(u) + (I - A)(u') - (I - A)(u)\rangle \\
\geq & -\rho(m - \lambda)[\|v' - v\|^2 + \|u' - u\|^2] \\
\geq & -\rho m[\|v' - v\|^2 + \|u' - u\|^2]
\end{aligned}$$

for $m - \lambda > 0$. Hence, $W^{S,T}_{\rho,A}$ is relatively $(\rho m)-$relaxed monotone.

To prove (ii) in light of (i), it remains just to show that $W^{S,T}_{\rho,A}$ is maximal, while S and T are relatively maximal relaxed monotone already. It only remains to show that $(I + W^{S,T}_{\rho,A})^{-1} = G^{S,T}_{\rho,A}$ has the full domain. Based on Lemma 13, $(I + W^{S,T}_{\rho,A})^{-1}$ has the full domain since resolvents $R^S_{\rho,m,A}$ and $R^T_{\rho,m,A}$ are defined everywhere.

The next theorem reflects the relationship between zeros of $W^{S,T}_{\rho,A}$ and $M = S + T$.

Theorem 22 *Let S and T be the setvalued mappings on X with a positive constant ρ. Suppose that $zer(W^{S,T}_{\rho,A})$ denotes the zeros of $W^{S,T}_{\rho,A}$ and Z^*_ρ is defined by*

$$\begin{aligned} & Z^*_\rho = \{u + \rho b | b \in T(u), -b \in S(u)\} \\ \subseteq & \{u + \rho b | zer(S + T), b \in T(u)\}. \end{aligned}$$

Then $zer(W^{S,T}_{\rho,A}) = Z^*_\rho$.

Proof: First, we show, $zer(W^{S,T}_{\rho,A}) \subseteq Z^*_\rho$. Then, for $z \in zer(W^{S,T}_{\rho,A})$, there are some $u, b, v, a \in X$, $v + \rho b = z$ such that $u - v = 0$, $(u, b) \in T$ and $(v, a) \in S$. This implies

$$u - v = 0 \Leftrightarrow u = v \Leftrightarrow \rho a = -\rho b \Leftrightarrow a = -b.$$

On the other hand, it follows that $u + \rho b = z$, $(u, b) \in T$ and $(u, -b) \in S$, so $z \in Z^*_\rho$. Similarly, one can show the converse inclusion. Hence, we conclude $z \in zer(S + T)$. □

We observe based on Theorem 22 that for any given zero z of $W^{S,T}_{\rho,A}$, $R^T_{\rho,m,A}$ is a zero of $A + B$, and as a result, one may apply the generalized proximal point algorithm for $W^{S,T}_{\rho,A}$ to find a zero of $S + T$, and apply the resolvent $R^T_{\rho,m,A}$ to the result. This is equivalent to applying the Douglas-Rachford splitting method to the relatively maximal relaxed monotone map $W^{S,T}_{\rho,A}$ with stepsize $\rho_k = 1$ and exact evaluation of resolvents.

6 Conclusions

We observe that results obtained in Section3 can be applied to generalize the first-order evolution inclusions/evolution equations of the form [21] and [7, Theorem 4.1]

$$u'(t) + M\,u(t) - \omega\,u(t) \ni b(t),\ u(0) = u_0 \quad (13)$$

for almost all $t \in$(0,T), where T is fixed, $0 < t < \infty$, $b \in W^1_2(0,\mathrm{T}; X)$, $\omega \in \Re$, $M : X \to 2^X$ is a setvalued mapping, and $u_0 \in D(M)$.

References:

[1] R. P. Agarwal and R. U. Verma, General system of $(A, \eta)-$maximal relaxed monotone variational inclusion problems based on generalized hybrid algorithms, *Communications in Nonlinear Science and Numerical Simulation* 15 (2010) 238–251.

[2] R. P. Agarwal and R. U. Verma, Relatively maximal monotone mappings and applications to general inclusions, *Applicable Analysis* (2011) 1–16, iFirst.

[3] O. A. Boikanyo and G. Morosanu, Modified Rockafellar's algorithms, *Mathematical Sciences Research Journal* (2009), in press.

[4] M. B. Solodov and B. F. Svaiter, Forcing strong convergence of proximal point iterations in a Hilbert space, *Mathematical Programming Ser.* A 87 (2000) 189–202.

[5] A. Kaplan and R. Tichatschke, Proximal point methods and nonconvex optimization, *Journal of Global Optimization* 13 (1998) 389–406.

[6] Y. P. Fang and N. J. Huang, $H-$monotone operators and system of variational inclusions, *Communications on Applied Nonlinear Analysis* 11 (1) (2004) 95–101.

[7] R. U. Verma, First-order nonlinear evolution equations and Yosida approximations based on the relative maximal monotonicity, *International Journal of Evolution Equations* 5 (2) (2010) 135–153.

[8] R. U. Verma, Hybrid super-relaxed proximal point algorithms and general nonlinear variational inclusion problems, *International Journal of Pure and Applied Mathematics* 60 (2010) 461–479.

[9] R. U. Verma, General over-relaxed $A-$proximal point algorithms and applications to nonlinear variational inclusions, *Advances in Nonlinear Variational Inequalities* 13 (2) (2010) 103–114.

[10] R. U. Verma, Generalized maximal monotonicities, super-relaxed proximal point algorithms with applications in Hilbert spaces, *AJSE Mathematics* 35 (2010) 1–8.

[11] R. U. Verma, On general over-relaxed proximal point algorithm and applications, *Optimization* (2010) 1–6, iFirst.

[12] R. U. Verma, New approach based on relatively inexact proximal point algorithms to linear convergence analysis and applications, *Journal of Advanced Modeling and Optimization* 13 (2) (2011) 233–243.

[13] R. U. Verma, Generalized relatively relaxed proximal point algorithms and nonlinear variational inclusion problems, *International Journal of Pure and Applied Mathematics* 64 (4) (2010) 517–529.

[14] J. Eckstein and D. P. Bertsekas, On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators, *Mathematical Programming* 55 (1992) 293–318.

[15] A. Moudafi and M. Thera, Finding a zero of the sum of two maximal monotone operators, *Journal of Optimization Theory and Applications* 94 (1997) 425–448.

[16] R. T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM Journal of Control and Optimization* 14 (1976) 877–898.

[17] R. T. Rockafellar, Augmented Lagrangians and applications of the proximal point algorithm in convex programming, *Mathematics of Operations Research* 1 (1976) 97–116.

[18] R. T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pacific Journal of Mathematics* 33 (1970) 209–216.

[19] R. T. Rockafellar and R. J-B. Wets, *Variational Analysis*, Springer, Berlin, 2004.

[20] H. K. Xu, Iterative algorithms for nonlinear operators, *Journal of the London Mathematical Society* 66 (2) (2002) 240–256.

[21] E. Zeidler, *Nonlinear Functional Analysis and its Applications* III, Springer-Verlag, New York, 1985.

[22] P. Tossings, The perturbed proximal point algorithm and some of its applications, *Applied Mathematics and Optimization* 29 (1994) 125–159.