Traveling Wave Solutions For Some Nonlinear Evolution Equations By The First Integral Method

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Abstract: In this paper, based on the known first integral method, we try to seek the traveling wave solutions of several nonlinear evolution equations. As a result, some exact travelig wave solutions and solitary solutions for Whitham-Broer-Kaup (WBK) equations, Gardner equation, Boussinesq-Burgers equations, nonlinear schrodinger equation and mKDV equation are established successfully.

Key–Words: First integral method; Traveling wave solution; WBK equations, Gardner equation; Boussinesq-Burgers equations; nonlinear schrodinger equation; mKDV equation; Exact solution; Solitary solution

1 Introduction

Recently searching for exact traveling wave solutions of nonlinear equations has gained more and more popularity, and many effective methods have been presented so far. Some of these approaches are the homogeneous balance method [1], the extended hyperbolic functions method [2], the tanh-method [3-6], the inverse scattering transform [7], the Backlund transform [8], the Exp-function method [9], the generalized Riccati equation method [10,11], the F-expansion method [12,13], and so on.

In [14], Feng presented the first integral method originally. The main steps of the first integral method are summarized as follows:
Step 1. Consider a general NLEE in the form

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, ...) = 0 \qquad (1)$$

Using a wave variable $\xi = x - ct$, we can rewrite Eq. (1) in the following nonlinear ODE

$$Q(u, u', u'', u''', ...) = 0, \qquad (2)$$

where $u = u(\xi)$.
Step 2. Suppose the solution of Eq. (2) can be written as $u(x,t) = X(\xi)$, and furthermore, we introduce a new independent variable $Y = Y(\xi)$ such that $Y(\xi) = \frac{dX}{d\xi}$.
Step 3. Under the conditions of Step 2, Eq. (2) can be converted into a system of nonlinear ordinary differential equations as follows.

$$\begin{cases} \frac{dX}{d\xi} = Y(\xi), \\ \frac{dY}{d\xi} = F(X(\xi), Y(\xi)), \end{cases} \qquad (3)$$

If we can find the integrals to Eq. (3), then the general solutions to Eq. (3) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the so-called Division Theorem to obtain one first integral to Eq. (3) which reduces Eq. (2) to a first order integrable ordinary differential equation. An exact solution to Eq. (1) is then obtained by solving this equation.

Division theorem [14]: Suppose that $P(w, z)$, $Q(w, z)$ are polynomials in $C(w, z)$ and $P(w, z)$ is irreducible in the complex domain $C(w, z)$. If $Q(w, z)$ vanishes at all zero points of $P(w, z)$, then there exists a polynomial $G(w, z)$ in $C(w, z)$ such that

$$Q(w, z) = P(w, z)G(w, z). \qquad (4)$$

Because of its simpleness and validity, the first integral method is soon applied to other problems (see [15-17]).

In this paper, we test the power of the first integral method by applying it to several nonlinear equations, and construct exact traveling wave solutions for them. In Section 2, WBK equations, Gardner equation, Boussinesq-Burgers equations, nonlinear schrodinger(NLS^+) equation and mKDV equation are considered, and some solitary wave solutions are obtained respectively by use of the first integral method. Some conclusions are presented at the end of the paper.

2 Applications of the first integral method to some nonlinear equations

2.1 Whitham-Broer-Kaup Equations

In this subsection, we will consider the Whitham-Broer-Kaup (WBK) equations [18-20]:

$$u_t + uu_x + v_x + \beta u_{xx} = 0. \tag{5}$$

$$v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} = 0. \tag{6}$$

where $\alpha,\ \beta$ are constants.

Suppose that

$$\xi = k(x - ct).$$

Then (5) and (6) are converted to ODEs

$$-cu' + uu' + v' + \beta ku'' = 0. \tag{7}$$

$$-cv' + (uv)' + \alpha k^2 u''' - \beta kv'' = 0. \tag{8}$$

Integrating (7) and (8) once, and considering the zero constants for integration we have:

$$-cu + \frac{1}{2}u^2 + v + \beta ku' = 0. \tag{9}$$

$$-cv + uv + \alpha k^2 u'' - \beta kv' = 0. \tag{10}$$

From (9), it follows

$$v = cu - \frac{1}{2}u^2 - \beta ku'. \tag{11}$$

Substituting (11) into (10), we have

$$(\beta + \alpha)k^2 u'' + \frac{3}{2}cu^2 - c^2 u - \frac{1}{2}u^3 = 0. \tag{12}$$

Let

$$X(\xi) = u(\xi), Y(\xi) = u_\xi(\xi).$$

Then we have

$$\frac{dX}{d\xi} = Y(\xi), \quad \frac{dY}{d\xi} = \frac{c^2X + \frac{1}{2}X^3 - \frac{3}{2}cX^2}{(\beta+\alpha)k^2}. \tag{13}$$

Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (12), and $R(X, Y)$ is an irreducible polynomial in the complex domain $C(X, Y)$ such that

$$R(X(\xi), Y(\xi)) = \sum_{i=0}^{m} s_i(X)Y^i = 0, \tag{14}$$

where $s_i(X), i = 0, 1...m$ are polynomials of X, and $s_m(X) \neq 0$. Eq. (14) is called the first integral to (13).

Due to the Division Theorem, there exists a polynomial $h_1(X) + h_2(X)Y$ in the complex domain $C(X, Y)$ such that

$$\frac{dR}{d\xi} = \frac{\partial R}{\partial X}\frac{dX}{d\xi} + \frac{\partial R}{\partial Y}\frac{dY}{d\xi}$$

$$= [h_1(X) + h_2(X)Y]\sum_{i=0}^{m} s_i(X)Y^i. \tag{15}$$

In this example, for the sake of convenience we take $m = 1$. Then by equating the coefficients of Y^i on both sides of Eq. (15), we have

$$\begin{cases} \frac{ds_1(X)}{dX} = h_2(X)s_1(X) \\ \frac{ds_0(X)}{dX} = h_1(X)s_1(X) + h_2(X)s_0(X) \\ s_1(X)\frac{c^2X+\frac{1}{2}X^3-\frac{3}{2}cX^2}{(\beta+\alpha)k^2} = h_1(X)s_0(X). \end{cases} \tag{16}$$

Then it follows

$$\begin{cases} deg(s_1(X)) = 0 \\ h_2(X) = 0 \\ deg(h_1(X)) = 1 \\ deg(s_0(X)) = 2 \end{cases} \tag{17}$$

For simplicity, let $s_1(X) = 1$, and let $h_1(x) = a_1X + a_0$, a_1 and a_0 are constants to be determined later. Then from (16) we can obtain $s_0(X) = \frac{a_1}{2}X^2 + a_0X + d$, where d is the integral constant. Also from (16), we have

$$\frac{c^2X + \frac{1}{2}X^3 - \frac{3}{2}cX^2}{(\beta+\alpha)k^2} = (\frac{a_1}{2}X^2 + a_0X + d)(a_1X + a_0).$$

Equating all the coefficients of X^i on both sides, yields a serials of nonlinear algebraic equations as follows:

$$\begin{cases} \frac{a_1^2}{2} = \frac{1}{2(\beta+\alpha)k^2} \\ \frac{3a_1a_0}{2} = \frac{-3c}{2(\beta+\alpha)k^2} \\ a_0^2 + da_1 = \frac{c^2}{(\beta+\alpha)k^2} \\ da_0 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} a_0 = \frac{-c}{\sqrt{(\beta+\alpha)k^2}} \\ a_1 = \frac{1}{\sqrt{(\beta+\alpha)k^2}} \\ d = 0 \end{cases} \quad or \quad \begin{cases} a_0 = \frac{c}{\sqrt{(\beta+\alpha)k^2}} \\ a_1 = -\frac{1}{\sqrt{(\beta+\alpha)k^2}} \\ d = 0 \end{cases} \tag{18}$$

Case I:

If $a_0 = \frac{-c}{\sqrt{(\beta+\alpha)k^2}}, a_1 = \frac{1}{\sqrt{(\beta+\alpha)k^2}}, d = 0$, then from (13) and (14) we have

$$\frac{dX}{d\xi} = Y = -s_0(X)$$

$$= -\frac{1}{2\sqrt{(\beta+\alpha)k^2}}X^2 + \frac{c}{\sqrt{(\beta+\alpha)k^2}}X. \quad (19)$$

Solving (19), we have

$$X = \coth[\frac{c(\xi + c_0)}{2\sqrt{(\beta+\alpha)k^2}}].c + c \quad (20)$$

and

$$X = \tanh[\frac{c(\xi + c_0)}{2\sqrt{(\beta+\alpha)k^2}}].c + c \quad (21)$$

Then combining with (11) we can obtain the solitary wave solutions of WBK equations as denoted in (22)-(23):

$$u(x,t) = \coth[\frac{c(k(x-ct)+c_0)}{2\sqrt{(\beta+\alpha)k^2}}].c + c$$

$$v(x,t) = \frac{c^2}{2}\{1 - \coth^2[\frac{c(k(x-ct)+c_0)}{2\sqrt{(\beta+\alpha)k^2}}]$$

$$+\frac{\beta k}{\sqrt{(\beta+\alpha)k^2}}\text{csch}^2[\frac{c(k(x-ct)+c_0)}{2\sqrt{(\beta+\alpha)k^2}}]\} \quad (22)$$

$$u(x,t) = \tanh[\frac{c(k(x-ct)+c_0)}{2\sqrt{(\beta+\alpha)k^2}}].c + c$$

$$v(x,t) = \frac{c^2}{2}\{1 - \tanh^2[\frac{c(k(x-ct)+c_0)}{2\sqrt{(\beta+\alpha)k^2}}]$$

$$-\frac{\beta k}{\sqrt{(\beta+\alpha)k^2}}\text{sech}^2[\frac{c(k(x-ct)+c_0)}{2\sqrt{(\beta+\alpha)k^2}}]\} \quad (23)$$

where c_0 is an arbitrary constant.

Case II:

If $a_0 = \frac{c}{\sqrt{(\beta+\alpha)k^2}}, a_1 = -\frac{1}{\sqrt{(\beta+\alpha)k^2}}, d = 0$, then from (13) and (14) we have

$$\frac{dX}{d\xi} = Y = -s_0(X)$$

$$= \frac{1}{\sqrt{(\beta+\alpha)k^2}}X^2 - \frac{c}{\sqrt{(\beta+\alpha)k^2}}X \quad (24)$$

Solving (24), then we can obtain the solitary wave solutions of WBK equations as denoted in (25)-(26) in the same manner:

$$u(x,t) = -\coth[\frac{c(k(x-ct)+c_0)}{2\sqrt{(\beta+\alpha)k^2}}].c + c$$

$$v(x,t) = \frac{c^2}{2}\{1 - \coth^2[\frac{c(k(x-ct)+c_0)}{2\sqrt{(\beta+\alpha)k^2}}]$$

$$-\frac{\beta k}{\sqrt{(\beta+\alpha)k^2}}\text{csch}^2[\frac{c(k(x-ct)+c_0)}{2\sqrt{(\beta+\alpha)k^2}}]\} \quad (25)$$

$$u(x,t) = -\tanh[\frac{c(k(x-ct)+c_0)}{2\sqrt{(\beta+\alpha)k^2}}].c + c$$

$$v(x,t) = \frac{c^2}{2}\{1 - \tanh^2[\frac{c(k(x-ct)+c_0)}{2\sqrt{(\beta+\alpha)k^2}}]$$

$$+\frac{\beta k}{\sqrt{(\beta+\alpha)k^2}}\text{sech}^2[\frac{c(k(x-ct)+c_0)}{2\sqrt{(\beta+\alpha)k^2}}]\} \quad (26)$$

where c_0 is an arbitrary constant.

Remark 1 *Considering the difference of the form of constants, our results (23) and (26) are consistent with the results derived by tanh method in Ref. [18], and (22), (25) can be compared with the results by homotopy perturbation method in Ref. [19].*

Remark 2 *For the case of $m \geq 2$, the discussions becomes more complicated and involves the irregular singular point theory, and the elliptic integrals of the second kind and the hyper-elliptic integrals. Some solutions in the functional form cannot be expressed explicitly. We should note here that one does not need to consider the cases $m \geq 5$ due to the fundamental fact that an algebraic equation with the degree greater than or equal to 5 is generally not solvable.*

2.2 Gardner Equation

In this subsection we will consider the Gardner equation [21] as follows:

$$u_t + 2auu_x - 3bu^2u_x + u_{xxx} = 0, \ a > 0, \ b > 0. \quad (27)$$

Let $\xi = k(x-ct)$, $u(x,t) = u(\xi)$. Then Eq. (27) can be reduced to an ODE:

$$-cu' + 2auu' - 3bu^2u' + k^2u''' = 0. \quad (28)$$

Integrating it with respect to ξ once, considering the zero constant for the integration, yields:

$$-cu + au^2 - bu^3 + k^2u'' = 0. \quad (29)$$

Let

$$X(\xi) = u(\xi), Y(\xi) = u_\xi(\xi).$$

Then

$$\frac{dX}{d\xi} = Y(\xi), \quad \frac{dY}{d\xi} = \frac{cX - aX^2 + bX^3}{k^2}. \quad (30)$$

Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (30), and $\eta(X,Y)$ is an irreducible polynomial in the complex domain $C(X,Y)$ such that

$$\eta(X(\xi), Y(\xi)) = \sum_{i=0}^{m} s_i(X)Y^i = 0, \quad (31)$$

where $s_i(X), i = 0, 1...m$ are polynomials of X, and $s_m(X) \neq 0$. Eq. (31) is called the first integral to (30).

According to the Division Theorem, there exists a polynomial $h_1(X)+h_2(X)Y$ in the complex domain $C(X,Y)$ such that

$$\frac{d\eta}{d\xi}=\frac{\partial\eta}{\partial X}\frac{dX}{d\xi}+\frac{\partial\eta}{\partial Y}\frac{dY}{d\xi}$$

$$=[h_1(X)+h_2(X)Y]\sum_{i=0}^{m}s_i(X)Y^i. \qquad (32)$$

In this example we take $m=1$. Then by collecting all the terms with the same power of Y together and equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows

$$\begin{cases}\frac{ds_1(X)}{dX}=h_2(X)s_1(X)\\ \frac{ds_0(X)}{dX}=h_1(X)s_1(X)+h_2(X)s_0(X)\\ s_1(X)(\frac{cX-aX^2+bX^3}{k^2})=h_1(X)s_0(X)\end{cases} \qquad (33)$$

Solving (33), we have

$$\begin{cases}deg(s_1(X))=0\\ h_2(X)=0\\ deg(h_1(X))=1\\ deg(s_0(X))=2\end{cases} \qquad (34)$$

For simplicity, let $s_1(X)=1,\ h_1(x)=b_1X+b_0$, b_1 and b_0 are constants to be determined later. Then from (33) it follows $s_0(X)=\frac{b_1}{2}X^2+b_0X+d$, where d is the integral constant. Also we obtain

$$\frac{cX-aX^2+bX^3}{k^2}=(\frac{b_1}{2}X^2+b_0X+d)(b_1X+b_0)$$

Equating all the coefficients of X^i on both sides, we obtain a serials of nonlinear algebraic equations as follows:

$$\begin{cases}\frac{b_1^2}{2}=\frac{b}{k^2}\\ \frac{3b_1b_0}{2}=-\frac{a}{k^2}\\ b_0^2+db_1=\frac{c}{k^2}\\ db_0=0\end{cases}\Rightarrow\begin{cases}b_0=\pm\frac{a}{3k}\sqrt{\frac{2}{b}}\\ b_1=\mp\frac{\sqrt{2b}}{k}\\ d=0\\ c=\frac{2a^2}{9b}\end{cases} \qquad (35)$$

From (30) and (31) we have

$$\frac{dX}{d\xi}=Y=-s_0(X)=\pm\frac{\sqrt{2b}}{k}X^2\mp\frac{a}{3k}\sqrt{\frac{2}{b}}X$$

Solving it, we can obtain the solitary wave solutions of Gardner equation as follows:

$$X(\xi)=\frac{a}{3b}[1\pm\tanh\frac{a}{3k\sqrt{2b}}(\xi+c_0)] \qquad (36)$$

$$X(\xi)=\frac{a}{3b}[1\pm\coth\frac{a}{3k\sqrt{2b}}(\xi+c_0)] \qquad (37)$$

where c_0 is an arbitrary constant. Then

$$u(x,t)=\frac{a}{3b}[1\pm\tanh\frac{a}{3k\sqrt{2b}}(k(x-\frac{2a^2}{9b}t)+c_0)] \qquad (38)$$

$$u(x,t)=\frac{a}{3b}[1\pm\coth\frac{a}{3k\sqrt{2b}}(k(x-\frac{2a^2}{9b}t)+c_0)] \qquad (39)$$

Remark 3 *If we take $c_0=0,\ k=1$ in (38) and (39), then the results are consistent with the results derived by the tanh method in Ref. [21].*

2.3 Boussinesq-Burgers Equations

In this subsection, we will consider the Boussinesq-Burgers equations [22-24]:

$$u_t+2uu_x-\frac{1}{2}v_x=0. \qquad (40)$$

$$v_t+2(uv)_x-\frac{1}{2}u_{xxx}=0. \qquad (41)$$

Suppose that

$$\xi=k(x-ct)$$

Then (40) and (41) are converted to ODEs

$$-cu'+2uu'-\frac{1}{2}v'=0. \qquad (42)$$

$$-cv'+2(uv)'-\frac{1}{2}k^2u'''=0. \qquad (43)$$

Integrating (42) and (43) once, and considering the zero constants for integration we have:

$$-cu+u^2-\frac{1}{2}v=0. \qquad (44)$$

$$-cv+2uv-\frac{1}{2}k^2u''=0. \qquad (45)$$

From (44), it follows

$$v=2(u^2-cu). \qquad (46)$$

Substituting (46) into (45), we have

$$k^2u''=8u^3-12cu^2+4c^2u. \qquad (47)$$

Let

$$X(\xi)=u(\xi),Y(\xi)=u_\xi(\xi).$$

Then we have

$$\frac{dX}{d\xi}=Y(\xi),\frac{dY}{d\xi}=\frac{8X^3-12cX^2+4c^2X}{k^2}. \qquad (48)$$

Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (47), and $R(X,Y)$ is an irreducible polynomial in the complex domain $C(X,Y)$ such that

$$R(X(\xi),Y(\xi))=\sum_{i=0}^{m}s_i(X)Y^i=0. \tag{49}$$

where $s_i(X), i=0,1...m$ are polynomials of X, and $s_m(X)\neq 0$. Eq. (49) is called the first integral to (48).

Due to the Division Theorem, there exists a polynomial $h_1(X)+h_2(X)Y$ in the complex domain $C(X,Y)$ such that

$$\frac{dR}{d\xi}=\frac{\partial R}{\partial X}\frac{dX}{d\xi}+\frac{\partial R}{\partial Y}\frac{dY}{d\xi}$$

$$=[h_1(X)+h_2(X)Y]\sum_{i=0}^{m}s_i(X)Y^i. \tag{50}$$

Take $m=1$. Then by equating the coefficients of Y^i on both sides of Eq. (50), we have

$$\begin{cases}\frac{ds_1(X)}{dX}=h_2(X)s_1(X)\\ \frac{ds_0(X)}{dX}=h_1(X)s_1(X)+h_2(X)s_0(X)\\ s_1(X)\frac{8X^3-12cX^2+4c^2X}{k^2}=h_1(X)s_0(X).\end{cases} \tag{51}$$

Then it follows

$$\begin{cases}deg(s_1(X))=0\\ h_2(X)=0\\ deg(h_1(X))=1\\ deg(s_0(X))=2\end{cases} \tag{52}$$

For simplicity, let $s_1(X)=1$, and let $h_1(x)=r_1X+r_0$, r_1 and r_0 are constants to be determined later. Then from (51) we can obtain $s_0(X)=\frac{r_1}{2}X^2+r_0X+d$, where d is the integral constant. Also from (51), we have

$$\frac{8X^3-12cX^2+4c^2X}{k^2}=(\frac{r_1}{2}X^2+r_0X+d)(r_1X+r_0).$$

Equating all the coefficients of X^i on both sides, yields a serials of nonlinear algebraic equations as follows:

$$\begin{cases}\frac{r_1^2}{2}=\frac{8}{k^2}\\ \frac{3r_1r_0}{2}=-\frac{12c}{k^2}\\ r_0^2+dr_1=\frac{4c^2}{k^2}\\ dr_0=0\end{cases}\Rightarrow\begin{cases}r_0=\pm\frac{2c}{k}\\ r_1=\mp\frac{4}{k}\\ d=0\end{cases} \tag{53}$$

From (48) and (49) we have

$$\frac{dX}{d\xi}=Y=-s_0(X)=\pm\frac{2}{k}X^2\mp\frac{2}{k}X.$$

Solving it, we have

$$X(\xi)=\frac{c}{2}[1\pm\tanh\frac{c}{k}(\xi+c_0)], \tag{54}$$

$$X(\xi)=\frac{c}{2}[1\pm\coth\frac{c}{k}(\xi+c_0)], \tag{55}$$

where c_0 is an arbitrary constant. Then combining with (46), we can obtain the solitary wave solutions of Boussinesq-Burgers equations as follows:

$$u(x,t)=\frac{c}{2}\{1\pm\tanh[\frac{c}{k}(k(x-ct)+c_0)]\},$$

$$v(x,t)=-\frac{c^2}{4}\mathrm{sech}^2[\frac{c}{k}(k(x-ct)+c_0)]. \tag{56}$$

$$u(x,t)=\frac{c}{2}\{1\pm\coth[\frac{c}{k}(k(x-ct)+c_0)]\},$$

$$v(x,t)=\frac{c^2}{4}\mathrm{csch}^2[\frac{c}{k}(k(x-ct)+c_0)]. \tag{57}$$

Remark 4 *Our results (56), (57) are analogous to some of the results in Ref. [22-24].*

2.4 NLS^+ Equation

In this subsection, we will consider the NLS^+ equation [25]:

$$i\Phi_t-\Phi_{xx}+2(|\Phi|^2-\rho^2)\Phi=0, \tag{58}$$

where Φ is a complex wave function and ρ is a constant.

Since $\Phi=\Phi(x,t)$ in Eq. (58) is a complex function, we suppose that

$$\Phi=u(\xi)e^{i(x+t)},\ \xi=x+2t. \tag{59}$$

By using (59), Eq. (58) is converted to an ODE as follows

$$-2\rho^2u+2u^3-u''=0. \tag{60}$$

Let

$$X(\xi)=u(\xi),Y(\xi)=u_\xi(\xi). \tag{61}$$

Then we have

$$\frac{dX}{d\xi}=Y(\xi),\frac{dY}{d\xi}=2X^3(\xi)-2\rho^2X(\xi). \tag{62}$$

Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (62), and $R(X,Y)$ is an irreducible polynomial in the complex domain $C(X,Y)$ such that

$$R(X(\xi),Y(\xi))=\sum_{i=0}^{m}s_i(X)Y^i=0, \tag{63}$$

where $s_i(X), i = 0, 1...m$ are polynomials of X, and $s_m(X) \neq 0$. Eq. (63) is called the first integral to (62).

Due to the Division Theorem, there exists a polynomial $h_1(X) + h_2(X)Y$ in the complex domain $C(X, Y)$ such that

$$\frac{dR}{d\xi} = \frac{\partial R}{\partial X}\frac{dX}{d\xi} + \frac{\partial R}{\partial Y}\frac{dY}{d\xi}$$

$$= [h_1(X) + h_2(X)Y]\sum_{i=0}^{m} s_i(X)Y^i. \quad (64)$$

Case 1: Take $m = 1,\ \rho \neq 0$. By equating the coefficients of Y^i on both sides of Eq. (64) we obtain

$$\begin{cases} \frac{ds_1(X)}{dX} = h_2(X)s_1(X) \\ \frac{ds_0(X)}{dX} = h_1(X)s_1(X) + h_2(X)s_0(X) \\ s_1(X)(2X^3 - 2\rho^2 X) = h_1(X)s_0(X) \end{cases} \quad (65)$$

which implies

$$\begin{cases} deg(s_1(X)) = 0 \\ h_2(X) = 0 \\ deg(h_1(X)) = 1 \\ deg(s_0(X)) = 2 \end{cases} \quad (66)$$

For simplicity, let $s_1(X) = 1$, and let $h_1(X) = a_1X + a_0$, where a_1 and a_0 are constants to be determined later. Then from (65) we can obtain $s_0(X) = \frac{a_1}{2}X^2 + a_0X + d$, where d is the integral constant. Also from (65), we have

$$2X^3 - 2\rho^2 X = (\frac{a_1}{2}X^2 + a_0X + d)(a_1X + a_0).$$

Equating all the coefficients of X^i on both sides of the equation above, yields

$$\begin{cases} \frac{a_1^2}{2} = 2 \\ \frac{3a_1a_0}{2} = 0 \\ a_0^2 + da_1 = -2\rho^2 \\ da_0 = 0 \end{cases} \Rightarrow \begin{cases} a_0 = 0 \\ a_1 = 2 \\ d = -\rho^2 \end{cases} \ or \ \begin{cases} a_0 = 0 \\ a_1 = -2 \\ d = \rho^2 \end{cases} \quad (67)$$

We will construct the traveling wave solutions of NLS^+ equation in two families.

Family I:

If $a_0 = 0, a_1 = 2, d = -\rho^2$, then from (62)-(63) we obtain

$$\frac{dX}{d\xi} = Y = -s_0(X) = -X^2 + \rho^2.$$

Considering $u(\xi) = X$, solving the equation above we get that $u(\xi) = X = \rho\frac{1+c_0e^{-2\rho\xi}}{1-c_0e^{-2\rho\xi}}$, where c_0 is an arbitrary constant. So combining with (59) we can demonstrate the exact complex traveling wave solution of NLS^+ equation as follows

$$\phi_1(x,t) = \rho\frac{1 + c_0e^{-2\rho(x+2t)}}{1 - c_0e^{-2\rho(x+2t)}}e^{i(x+t)}. \quad (68)$$

Especially, if we take $c_0 = 1$, we obtain the complex solitary wave solution as follows

$$\phi_2(x,t) = \rho e^{i(x+t)}\coth[\rho(x + 2t)].$$

If we take $c_0 = -1$, we obtain another solitary wave solution

$$\phi_3(x,t) = \rho e^{i(x+t)}\tanh[\rho(x + 2t)].$$

Family II:

If $a_0 = 0, a_1 = -2, d = \rho^2$, then from (62)-(63) we have

$$\frac{dX}{d\xi} = Y = -s_0(X) = X^2 - \rho^2$$

It follows $u(\xi) = X = \rho\frac{1+c_1e^{2\rho\xi}}{1-c_1e^{2\rho\xi}}$, where c_1 is an arbitrary constant. Then the exact complex traveling wave solution of NLS^+ equation can be denoted by

$$\phi_4(x,t) = \rho\frac{1 + c_1e^{2\rho(x+2t)}}{1 - c_1e^{2\rho(x+2t)}}e^{i(x+t)}. \quad (69)$$

Especially, if we take $c_1 = 1$, we obtain the complex solitary wave solution as follows

$$\phi_5(x,t) = -\rho e^{i(x+t)}\coth[\rho(x + 2t)].$$

If we take $c_1 = -1$, we obtain another solitary wave solution

$$\phi_6(x,t) = -\rho e^{i(x+t)}\tanh[\rho(x + 2t)].$$

Case 2: Take $m = 2, \rho = 0$. Equating the coefficients of Y^i on both sides of Eq. (64) we obtain

$$\begin{cases} \frac{ds_2(X)}{dX} = h_2(X)s_2(X) \\ \frac{ds_1(X)}{dX} = h_1(X)s_2(X) + h_2(X)s_1(X) \\ \frac{ds_0(X)}{dX} + 2(2X^3 - 2\rho^2 X) = h_1(X)s_1(X) + h_2(X)s_0(X) \\ s_1(X)(2X^3 - 2\rho^2 X) = h_1(X)s_0(X) \end{cases} \quad (70)$$

which implies

$$\begin{cases} deg(s_2(X)) = 0 \\ h_2(X) = 0 \\ deg(h_1(X)) = 1 \\ deg(s_1(X)) = 2 \\ deg(s_0(X)) = 3 \end{cases} \quad (71)$$

For simplicity, let $s_2(X) = 1$, and let $h_1(X) = a_1X + a_0$, where a_1 and a_0 are constants to be determined later. Then from (70) we can obtain

$$s_1(X) = \frac{a_1}{2}X^2 + a_0X + d,\ s_0(X) = (\frac{a_1^2}{8}-1)X^4+\frac{1}{2}a_0a_1X^3+\frac{da_1+a_0^2}{2}X^2+da_0X+c, \quad (72)$$

where $d,\ c$ are the integral constants. Also from (70), we have

$$2X^3(\frac{a_1}{2}X^2 + a_0X + d) = [(\frac{a_1^2}{8}-1)X^4+\frac{1}{2}a_0a_1X^3+\frac{da_1+a_0^2}{2}X^2+da_0X+c]\times (a_1X + a_0). \quad (73)$$

Equating all the coefficients of X^i on both sides of the equation above, yields

$$\begin{cases} a_1 = a_1(\frac{a_1^2}{8}-1) \\ 2a_0 = a_0(\frac{a_1^2}{8}-1) + \frac{a_0a_1^2}{2} \\ 2d = \frac{a_0^2a_1}{2} + \frac{da_1^2+a_1a_0^2}{2} \\ 0 = \frac{a_0}{2}(da_1 + a_0^2) + da_0a_1 \\ 0 = da_0^2 + ca_1 \\ 0 = ca_0 \end{cases} \Rightarrow \begin{cases} a_0 = 0 \\ a_1 = \pm 2\sqrt{2} \\ d = 0 \\ c = 0 \end{cases} \quad (74)$$

So the traveling wave solutions of NLS^+ equation can be deduced as follows.
Family III:
If $a_0 = 0,\ a_1 = \pm 2\sqrt{2},\ d = 0,\ c = 0$, then we obtain

$$s_1(X) = \sqrt{2}X^2,\ s_0(X) = X^4. \quad (75)$$

So from (62)-(63) we have

$$Y^2 + \sqrt{2}X^2Y + X^4 = 0, \quad (76)$$

which implies

$$\frac{dX}{d\xi} = Y = (-\frac{1}{2}\sqrt{2} \pm \frac{1}{2}i\sqrt{2})X^2. \quad (77)$$

Solving (77), yields

$$u(\xi) = X = \frac{1}{\frac{1}{2}\sqrt{2} \mp \frac{1}{2}i\sqrt{2} + c_0}, \quad (78)$$

where c_0 is an arbitrary constant. Then furthermore combining with (59) we obtain the complex solutions of NLS^+ equation as follows

$$\phi_7(x,t) = \frac{1}{\frac{1}{2}\sqrt{2} \mp \frac{1}{2}i\sqrt{2} + c_0}e^{i(x+t)} \quad (79)$$

Remark 5 *The traveling wave solutions of NLS^+ equation denoted by $\phi_1(x,t),\ \phi_4(x,t),\ \phi_7(x,t)$ are not given in [25], and are new complex solutions to our best knowledge.*

2.5 mKDV Equation

In this subsection we will consider the mKDV equation [25-26] as follows:

$$u_t - u^2u_x + \beta u_{xxx} = 0, \beta > 0. \quad (80)$$

Let $\xi = x - ct,\ u(x,t) = u(\xi)$. Then Eq. (80) can be reduced to an ODE as follows:

$$-cu' - u^2u' + \beta u''' = 0. \quad (81)$$

Integrating Eq. (81) with respect to ξ once yields:

$$p - cu - \frac{u^3}{3} + \beta u'' = 0,$$

where p is an integral constant.

Let $X(\xi) = u(\xi), Y(\xi) = u_\xi(\xi)$. Then we have

$$\frac{dX}{d\xi} = Y(\xi), \frac{dY}{d\xi} = \frac{cX + \frac{X^3}{3} - p}{\beta}. \quad (82)$$

Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (82), and $\eta(X,Y)$ is an irreducible polynomial in the complex domain $C(X,Y)$ such that

$$\eta(X(\xi), Y(\xi)) = \sum_{i=0}^{m} s_i(X)Y^i = 0, \quad (83)$$

where $s_i(X), i = 0, 1...m$ are polynomials of X, and $s_m(X) \neq 0$. Eq. (83) is called the first integral to (82).

According to the Division Theorem, there exists a polynomial $h_1(X) + h_2(X)Y$ in the complex domain $C(X,Y)$ such that

$$\frac{d\eta}{d\xi} = \frac{\partial\eta}{\partial X}\frac{dX}{d\xi} + \frac{\partial\eta}{\partial Y}\frac{dY}{d\xi} = [h_1(X) + h_2(X)Y]\sum_{i=0}^{m} s_i(X)Y^i. \quad (84)$$

In this example, for the sake of convenience we take $m = 1$. Then then by equating the coefficients of Y^i on both sides of Eq. (84) we obtain a set of algebraic equations as follows

$$\begin{cases} \frac{ds_1(X)}{dX} = h_2(X)s_1(X) \\ \frac{ds_0(X)}{dX} = h_1(X)s_1(X) + h_2(X)s_0(X) \\ \frac{s_1(X)}{\beta}(cX + \frac{X^3}{3} - p) = h_1(X)s_0(X) \end{cases} \quad (85)$$

Solving (85), yields

$$\begin{cases} deg(s_1(X)) = 0 \\ h_2(X) = 0 \\ deg(h_1(X)) = 1 \\ deg(s_0(X)) = 2 \end{cases} \quad (86)$$

For simplicity, let $s_1(X) = 1,\ h_1(x) = b_1X + b_0$, where b_1 and b_0 are constants to be determined later. Then from (85) it follows $s_0(X) = \frac{b_1}{2}X^2 + b_0X + b$, where b is the integral constant. Also we obtain

$$\frac{cX + \frac{X^3}{3} - p}{\beta} = (\frac{b_1}{2}X^2 + b_0X + b)(b_1X + b_0).$$

Equating all the coefficients of X^i on both sides of the equation above, yields

$$\left\{\begin{array}{c} \frac{b_1^2}{2} = \frac{1}{3\beta} \\ \frac{3b_1b_0}{2} = 0 \\ b_0^2 + bb_1 = \frac{c}{\beta} \\ bb_0 = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{c} b_0 = 0 \\ b_1 = \sqrt{\frac{2}{3\beta}} \\ b = c\sqrt{\frac{3}{2\beta}} \end{array}\right. \ or\ \left\{\begin{array}{c} b_0 = 0 \\ b_1 = -\sqrt{\frac{2}{3\beta}} \\ b = -c\sqrt{\frac{3}{2\beta}} \end{array}\right. \tag{87}$$

We will construct the traveling wave solutions for mKDV equation in two families.
$Family\ I$:

$$b_0 = 0, p = 0, b_1 = \sqrt{\frac{2}{3\beta}}, b = c\sqrt{\frac{3}{2\beta}}.$$

From (82)-(83) we have

$$\frac{dX}{d\xi} = Y = -s_0(X) = -\frac{1}{\sqrt{6\beta}}X^2 - c\sqrt{\frac{3}{2\beta}}$$

When $c > 0$: It follows

$$X = \sqrt{3c}\ \tan[\sqrt{3c}(-\frac{1}{\sqrt{6\beta}}\xi + c_0)],$$

where c_0 is an arbitrary constant. So we can demonstrate the exact traveling wave solution of mKDV equation as

$$u_{11}(x,t) = \sqrt{3c}\tan[\sqrt{3c}(-\frac{1}{\sqrt{6\beta}}(x - ct) + c_0)] \tag{88}$$

When $c < 0$: It can be solved that

$$X = \frac{\sqrt{-3c}(1 + c_0e^{-\sqrt{\frac{2c}{\beta}}\xi})}{1 - c_0e^{-\sqrt{\frac{2c}{\beta}}\xi}},$$

where c_0 is an arbitrary constant. So the exact traveling wave solution of mKDV equation can be denoted by

$$u_{12}(x,t) = \frac{\sqrt{-3c}(1 + c_0e^{-\sqrt{-\frac{2c}{\beta}}(x-ct)})}{1 - c_0e^{-\sqrt{-\frac{2c}{\beta}}(x-ct)}} \tag{89}$$

Especially, if we take $c_0 = 1$, then we obtain the solitary solution

$$u_{13}(x,t) = \sqrt{-3c}\coth[\sqrt{-\frac{c}{2\beta}}(x - ct)].$$

If we take $c_0 = -1$, we obtain another solitary solution

$$u_{14}(x,t) = \sqrt{-2c}\tanh[\sqrt{-\frac{c}{2\beta}}(x - ct)].$$

$Family\ II$:

$$b_0 = 0, p = 0, b_1 = -\sqrt{\frac{2}{3\beta}}, b = -c\sqrt{\frac{3}{2\beta}}$$

From (82)-(83) we have

$$\frac{dX}{d\xi} = Y = -s_0(X) = \frac{1}{\sqrt{6\beta}}X^2 + c\sqrt{\frac{3}{2\beta}}$$

When $c > 0$: It follows

$$X = \sqrt{3c}\ \tan[\sqrt{3c}(\frac{1}{\sqrt{6\beta}}\xi + c_1)]$$

where c_1 is an arbitrary constant. So we can demonstrate the exact traveling wave solution of mKDV equation as

$$u_{21}(x,t) = \sqrt{3c}\ \tan[\sqrt{3c}(\frac{1}{\sqrt{6\beta}}(x-ct)+c_1)] \tag{90}$$

When $c < 0$: We deduce $X = \frac{\sqrt{-3c}(1+c_1e^{\sqrt{-\frac{2c}{\beta}}\xi})}{1-c_1e^{\sqrt{-\frac{2c}{\beta}}\xi}}$, where c_1 is an arbitrary constant. Then the exact traveling wave solution of mKDV equation can be demonstrated as

$$u_{22}(x,t) = \frac{\sqrt{-3c}(1 + c_1e^{\sqrt{-\frac{2c}{\beta}}(x-ct)})}{1 - c_1e^{\sqrt{-\frac{2c}{\beta}}(x-ct)}} \tag{91}$$

Especially, if we take $c_1 = 1$, we obtain another solitary solution

$$u_{23}(x,t) = -\sqrt{-3c}\coth[\sqrt{-\frac{c}{2\beta}}(x - ct)].$$

However, if we take $c_1 = -1$, we obtain another solitary solution

$$u_{24}(x,t) = -\sqrt{-3c}\tanh[\sqrt{-\frac{c}{2\beta}}(x - ct)].$$

Remark 6 *If we take* $c_0 = -1,\ c_1 = -1,\ \sqrt{\frac{-2c}{\beta}} = \lambda,\ \beta = \delta$ *in* $u_{14}(x,t)$ *and* $u_{24}(x,t)$*, then we can obtain the solitary wave solutions* $u(x,t) = \pm\frac{\lambda}{2}\sqrt{6\delta}\tanh[\frac{1}{2}\lambda(x + \frac{1}{2}\beta\lambda^2 t)]$*, which are consistent with the results derived by the* (G'/G) *method in [26] by M.Wang.*

Remark 7 *The solutions* $u_{11}(x,t),\ u_{13}(x,t),\ u_{21}(x,t),\ u_{23}(x,t)$ *are not given in [25-26], and have not been reported so far in the literature to our best knowledge.*

Remark 8 *All of the solutions presented in Section 2 have been checked with Maple11 by putting them back into the original equations.*

3 Conclusions

In this paper, the solitary wave solutions of the W-BK equations, Gardner equation, Boussinesq-Burgers equations, nonlinear schrodinger equation and mKD-V equation are successfully constructed by use of the first integral method. The main point of the first integral method is to find the first integral for the reduced first order integrable ordinary differential equation from the original equation. Also it is the difficult part of the method to find the first integral.

As it is concise and effective, the first integral method is one of the most effective methods to establish the exact solutions of the nonlinear equations.

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