# Generalized Integral Inequalities For Discontinuous Functions With One Or Two Independent Variables

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*Abstract:* In this paper, some new integral inequalities for discontinuous functions with one or two independent variables are established, which provide new bounds for unknown functions in certain integral equations. The established inequalities generalize the main results in [14,15,16,17].

*Key–Words:* Integral inequality; Discontinuous function; Integral equation; Bounded; Qualitative analysis

## 1 Introduction

In recent years many integral inequalities have been established, which provide handy tools for investigating the quantitative and qualitative properties of solutions to integral and differential equations, for example, see [1-20], and the references therein. In these investigations, most of the known integral inequalities are concerned of continuous functions [1-13], while few authors take research in integral inequalities for discontinuous functions [14-17]. Now let us first recall some known inequalities in [14-17].

In [14, Theorem 2.1, 2.2, 3.1], the author established the following three integral inequalities for discontinuous functions:

$$(a_1): \quad \varphi(t) \le n(t) + \int_{t_0}^{t} g(s)\varphi(\tau(s))ds + \sum_{t_0<t_i<t} \beta_i \varphi^m(t_i - 0), \ m > 0;$$

$$(a_2): \quad \varphi(t) \le \psi(t) + q(t)\int_{t_0}^{t} g(s)\varphi^m(\tau(s))ds + \sum_{t_0<t_i<t} \beta_i \varphi^m(t_i - 0), \ m > 0;$$

$$(a_3): \quad \varphi(t) \le n(t) + q(t)[\int_{t_0}^{t} f(s)\varphi(\sigma(s))ds + \int_{t_0}^{t} f(s)(\int_{t_0}^{\tau} g(t)\varphi(\tau(t))dt)ds] + \sum_{t_0<t_i<t} \beta_i \varphi^m(t_i - 0), \ m > 0$$

where $\varphi(t)$ is unknown nonnegative piecewise continuous function defined on $[t_0, \infty)$ with the first kind of discontinuities in the points $x_i$, $i = 1, 2, \cdots$.

Based on $(a_1) - (a_3)$, some new bounds are derived for the unknown function $\varphi(t)$ in [14].

Recently, in [15, Theorem 3, 5], the author established two more general integral inequalities for discontinuous functions as follows:

$$(b_1): \quad u(x) \le \varphi(x) + q(x)\int_{x_0}^{x} f(\tau)W(u(p(\tau)))d\tau + \sum_{t_0<t_i<t} \beta_i \varphi^m(t_i - 0), \ m > 0;$$

$$(b_2): \quad u(x) \le u(0) + q(x)[\int_{x_0}^{x} f(s)u(p(s))ds + \int_{x_0}^{x} f(s)(\int_{x_0}^{x} g(\tau)u(p(\tau))d\tau)ds] + \int_{x_0}^{x} h(s)W(u(\sigma(s)))ds + \sum_{t_0<t_i<t} \beta_i \varphi^m(t_i - 0), \ m > 0$$

where $u(x)$ is unknown function as $\varphi(x)$ in $(a_1) - (a_3)$, and $W \in (R_+, R_+)$, $W(\gamma\beta) \le W(\gamma)W(\beta)$, $W(0) = 0$, $W$ is nondecreasing.

In [16, Theorem 2.1-2.3], the author presented three inequalities for discontinuous functions with two independent variables:

$$(c_1): \quad \varphi(t,x) \le a(t,x) + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi(\xi,\eta)d\xi d\eta + \sum_{t_0<t_i<t} \gamma_i \varphi^m(t_i-0,x_i-0),\ m>0;$$

$$(c_2): \quad \varphi(t,x) \le a(t,x) + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi^m(\xi,\eta)d\xi d\eta + \sum_{t_0<t_i<t} \gamma_i \varphi^m(t_i-0,x_i-0),\ m>0;$$

$$(c_3): \quad \varphi(t,x) \le a(t,x) + g(t,x)\int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi^m(\xi,\eta)d\xi d\eta + \sum_{t_0<t_i<t} \gamma_i \varphi^m(t_i-0,x_i-0),\ m>0$$

where $\varphi(t,x)$ is unknown nonnegative continuous function with the exception in the points $(x_i,y_i)$, $i=1,2,\cdots$.

As one can see, $(c_1)-(c_3)$ are the direct generalization for $(a_1)-(a_3)$ from one independent variable to two independent variables.

More recently, in [17, Theorem 2.1-2.3], the author presented the following inequalities for discontinuous functions with two independent variables:

$$(d_1): \quad u(t,x) \le \varphi(t,x) + q(t,x)\int_{t_0}^{t}\int_{x_0}^{x} f(\tau,s)\omega(u(\tau,s))d\tau ds + \sum_{t_0<t_i<t} \beta_i u(t_i-0,x_i-0);$$

$$(d_2): \quad u(t,x) \le \varphi(t,x) + q(t,x)\int_{t_0}^{t}\int_{x_0}^{x} f(\tau,s)\omega(u(\tau,s))d\tau ds + \sum_{t_0<t_i<t} \beta_i u^m(t_i-0,x_i-0);$$

$$(d_3): \quad u^m(t,x) \le \varphi(t,x) + \frac{nq(t,x)}{m-n}\int_{t_0}^{t}\int_{x_0}^{x} f(\tau,s)u^n(\tau,s)\omega(u(\tau,s))d\tau ds + \sum_{t_0<t_i<t} \beta_i u(t_i-0,x_i-0)$$

where $u(t,x)$ is unknown function as $\varphi(t,x)$ in $(c_1)-(c_3)$, and $\omega$ is similar to $W$ in $(b_1)-(b_2)$.

One easily see $(d_1)-(d_3)$ are the generalization of $(c_1)-(c_3)$.

The inequalities $(a_1)-(a_3)$, $(b_1)-(b_2)$, $(c_1)-(c_3)$, $(d_1)-(d_3)$ have proven to be effective in deriving bounds for discontinuous solutions of certain integration equations and differential equations. More details about them can be referred to [14-17].

In this paper, motivated by the work above, we will establish more general integral inequalities for discontinuous functions with one or two independent variables. Also we will present some applications for them.

## 2 Main Results

In the rest of the paper we denote the set of real numbers as $R$, and $R_+=[0,\infty)$ is a subset of $R$. For two given sets $G$, $H$, we denote the set of maps from $G$ to $H$ by $(G,H)$.

**Theorem 1** *Suppose $u(x)$ is a nonnegative piecewise continuous function defined on $[x_0,\infty)$ with discontinuities of the first kind in the points $x_i$, $i=1,2,...$, and $0\le x_0<x_1<...<x_n<...$, $\lim_{n\to\infty} x_n=\infty$. $q$, $\varphi\in(R,R_+)$, and $q(x)\ge 1$, $\varphi(x)>0$, $\varphi(x)$ is nondecreasing. $\beta_i$ are constants. $\beta_i\ge 0$. $f\in(R_+,R_+)$. $\sigma\in\Im-$ class of continuous functions, that is, $\sigma\in(R,R)$, $\sigma(t)\le t$, $\lim_{|t|\to\infty}\sigma(t)=\infty$. $\omega\in(R_+,R_+)$, $\omega(0)=0$, $\omega(\alpha\beta)\le\omega(\alpha)\omega(\beta)$, and $\omega$ is nondecreasing. $\phi\in C(R_+,R_+)$ is strictly increasing. $\psi\in(R_+,R_+)$. Furthermore, assume $\sigma(x)>x_{i-1}$ for $x\in(x_{i-1},x_i]$, $i=1,2....$ If for $x\ge x_0$, $u(x)$ satisfies the following inequality*

$$\phi(u(x)) \le \varphi(x) + q(x)\int_{x_0}^{x} f(\tau)\omega(\phi(u(\sigma(\tau))))d\tau + \sum_{x_0<x_j<x} \beta_i \psi(\phi(u(x_j-0))), \tag{1}$$

*then*

$$u(x) \le \phi^{-1}\{q(x)\varphi(x)G_i^{-1}[\int_{x_i}^{x} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)}d\tau]\},\quad x\in(x_i,x_{i+1}],\ i=0,1,2,\cdots, \tag{2}$$

*where $G_i(s)=\int_{c_i}^{s}\frac{1}{\omega(s)}ds$, $i=0,1,2...$, $c_i$, $i=0,1,2...$ are constants, and $c_0=1$,*

$$c_i = G_{i-1}^{-1}[\int_{x_{i-1}}^{x_i} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)}d\tau] + \frac{1}{\varphi(x_i-0)}\beta_i\psi\{q(x_i-0)\varphi(x_i-0)\times G_{i-1}^{-1}[\int_{x_{i-1}}^{x_i} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)}d\tau]\}$$

*for $i=1,2,\cdots$.*

**Proof:** From (1), considering $\varphi(x)$ is nondecreasing, it follows

$$\begin{aligned}\frac{\phi(u(x))}{\varphi(x)} &\le 1+\frac{q(x)}{\varphi(x)}\int_{x_0}^{x} f(\tau)\omega(\phi(u(\sigma(\tau))))d\tau \\ &+\sum_{x_0<x_j<x}\frac{\beta_j\psi(\phi(u(x_j-0)))}{\varphi(x)} \\ &\le q(x)[1+\int_{x_0}^{x}\frac{f(\tau)\omega(\phi(u(\sigma(\tau))))}{\varphi(\tau)}d\tau \\ &+\sum_{x_0<x_j<x}\frac{\beta_j\psi(\phi(u(x_j-0)))}{\varphi(x_j-0)}]. \quad (3)\end{aligned}$$

Let $v(x)=\frac{\phi(u(x))}{\varphi(x)q(x)}$ and

$$\overline{v}_i(x)=c_i+\int_{x_i}^{x}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)}d\tau$$

$i=0,1,2....$ Under the assumption $\sigma(x)>x_{i-1},\ x\in(x_{i-1},x_i],\ i=1,2...$, considering $\sigma$ is continuous, we have in fact $\sigma(x)\ge x_0,\ for\ \forall x\ge x_0$. Since $u$ is nonnegative on $[x_0,\infty)$, so $u(\sigma(x))\ge 0,\ \forall x\in[x_0,\infty)$. Furthermore we have $v(\sigma(x))\ge 0,\ \forall x\in[x_0,\infty)$, and

$$\begin{aligned}v(x) &\le 1+\int_{x_0}^{x}\frac{f(\tau)\omega(\phi(u(\sigma(\tau))))}{\varphi(\tau)}d\tau \\ &+\sum_{x_0<x_j<x}\frac{\beta_j\psi(\phi(u(x_j-0)))}{\varphi(x_j-0)} \\ &= 1+\int_{x_0}^{x}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau))v(\sigma(\tau)))}{\varphi(\tau)}d\tau \\ &+\sum_{x_0<x_j<x}\frac{\beta_j\psi(q(x_j-0)\varphi(x_j-0)v(x_j-0))}{\varphi(x_j-0)} \\ &\le 1+\int_{x_0}^{x}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)}d\tau \\ &+\sum_{x_0<x_j<x}\frac{\beta_j\psi(q(x_j-0)\varphi(x_j-0)v(x_j-0))}{\varphi(x_j-0)}. \quad (4)\end{aligned}$$

*Case 1:* If $x\in(x_0,x_1]$, the from (4) and the definition of $\overline{v}_0(x)$ we obtain

$$v(x)\le\overline{v}_0(x),\ x\in(x_0,x_1]. \quad (5)$$

According to the assumption for $\sigma$ we have $x_0<\sigma(x)\le x$. Then $v(\sigma(x))\le\overline{v}_0(\sigma(x))\le\overline{v}_0(x)$, and

$$\begin{aligned}\overline{v}_0'(x) &= \frac{f(x)\omega(q(\sigma(x))\varphi(\sigma(x)))\omega(v(\sigma(x)))}{\varphi(x)} \\ &\le \frac{f(x)\omega(q(\sigma(x))\varphi(\sigma(x)))\omega(\overline{v}_0(x))}{\varphi(x)},\end{aligned}$$

that is,

$$\frac{\overline{v}_0'(x)}{\omega(\overline{v}_0(x))}\le\frac{f(x)\omega(q(\sigma(x))\varphi(\sigma(x)))}{\varphi(x)}. \quad (6)$$

An integration for (6) from $x_0$ to $x$ yields

$$G_0(\overline{v}_0(x))-G_0(\overline{v}_0(0))\le\int_{x_0}^{x}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)}d\tau.$$

Considering $\overline{v}_0(0)=1=c_0$, then $G_0(\overline{v}_0(0))=0$, and

$$v(x)\le\overline{v}_0(x)\le G_0^{-1}[\int_{x_0}^{x}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)}d\tau]. \quad (7)$$

Especially we have

$$\begin{aligned}&v(x_1-0)\le\overline{v}_0(x_1-0)\le\overline{v}_0(x_1) \\ &\le G_0^{-1}[\int_{x_0}^{x_1}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)}d\tau].\end{aligned}$$

*Case 2:* If $x\in(x_1,x_2]$, considering (7) holds for $\forall x\in(x_0,x_1]$, then from (4) we deduce

$$\begin{aligned}v(x) &\le 1+\int_{x_0}^{x}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)}d\tau \\ &+\frac{\beta_1\psi(q(x_1-0)\varphi(x_1-0)v(x_1-0))}{\varphi(x_1-0)} \\ &=1+\int_{x_0}^{x_1}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)}d\tau \\ &+\int_{x_1}^{x}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)}d\tau \\ &+\frac{\beta_1\psi(q(x_1-0)\varphi(x_1-0)v(x_1-0))}{\varphi(x_1-0)} \\ &=\overline{v}_0(x_1)+\int_{x_1}^{x}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)}d\tau \\ &+\frac{\beta_1\psi(q(x_1-0)\varphi(x_1-0)v(x_1-0))}{\varphi(x_1-0)} \\ &\le G_0^{-1}[\int_{x_0}^{x_1}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)}d\tau] \\ &+\frac{\beta_1}{\varphi(x_1-0)}\psi\{q(x_1-0)\varphi(x_1-0) \\ &G_0^{-1}[\int_{x_0}^{x_1}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)}d\tau]\} \\ &+\int_{x_1}^{x}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)}d\tau \\ &=c_1+\int_{x_1}^{x}\frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)}d\tau \\ &=\overline{v}_1(x). \quad (8)\end{aligned}$$

Then similar to the process of (5)-(7), we can deduce

$$v(x) \le \overline{v}_1(x) \le G_1^{-1}[\int_{x_1}^{x} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)} d\tau]. \tag{9}$$

Especially we have

$$v(x_2-0) \le \overline{v}_1(x_2-0) \le \overline{v}_1(x_2) \le$$
$$G_1^{-1}[\int_{x_1}^{x_2} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)} d\tau].$$

*Case 3:* Suppose

$$v(x) \le \overline{v}_{j-1}(x) \le$$
$$G_{j-1}^{-1}[\int_{x_{j-1}}^{x} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)} d\tau]$$

holds for $x \in (x_{j-1}, x_j]$, $j = 1, 2...i$. Then for $x \in (x_i, x_{i+1}]$, from (4) we obtain

$$v(x) \le 1 + \int_{x_0}^{x} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)} d\tau$$
$$+ \sum_{x_0<x_j<x} \frac{\beta_j \psi(q(x_j-0)\varphi(x_j-0)v(x_j-0))}{\varphi(x_j-0)}$$
$$= 1 + \int_{x_0}^{x_i} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)} d\tau$$
$$+ \int_{x_i}^{x} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)} d\tau$$
$$+ \sum_{x_0<x_j<x} \frac{\beta_j \psi(q(x_j-0)\varphi(x_j-0)v(x_j-0))}{\varphi(x_j-0)}$$
$$\le G_{i-1}^{-1}[\int_{x_{i-1}}^{x_i} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)} d\tau]$$
$$\frac{1}{\varphi(x_i-0)} \beta_i \psi\{q(x_i-0)\varphi(x_i-0)\times$$
$$G_{i-1}^{-1}[\int_{x_{i-1}}^{x_i} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)} d\tau]\}$$
$$+ \int_{x_i}^{x} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)} d\tau$$
$$= c_i + \int_{x_i}^{x} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))\omega(v(\sigma(\tau)))}{\varphi(\tau)} d\tau$$
$$= \overline{v}_i(x). \tag{10}$$

Then similar to (5)-(7) we get that

$$v(x) \le \overline{v}_i(x) \le G_i^{-1}[\int_{x_i}^{x} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)} d\tau]. \tag{11}$$

Considering $u(x) = \phi^{-1}\{q(x)\varphi(x)v(x)\}$, then it follows

$$u(x) \le \phi^{-1}\{q(x)\varphi(x)G_i^{-1}[\int_{x_i}^{x} \frac{f(\tau)\omega(q(\sigma(\tau))\varphi(\sigma(\tau)))}{\varphi(\tau)} d\tau]\},$$

which is the desired result. □

**Remark 2** *Theorem 1 generalize many known theorems in the literature. For example, if we take $\phi(u) = u$, $\psi(u) = u^m$, $m > 0$, $\omega(u) = u$, then Theorem 1 becomes [14, Theorem 2.1]. If we take $\phi(u) = u$, $\psi(u) = u^m$, $\omega(u) = u^m$, $m > 0$, then Theorem 1 becomes [14, Theorem 2.2]. If we take $\phi(u) = u$, $\psi(u) = u^m$, $m > 0$, then Theorem 1 becomes [15, Theorem 3].*

**Corollary 3** *Suppose $u(x)$, $\phi(x)$, $f(x)$, $q(x)$, $\omega(x)$, $\varphi(x)$, $\psi(x)$, $\beta_i, i = 1, 2, \cdots$ are the same as in Theorem 1. If for $x \ge x_0$,*

$$\phi(u(x)) \le \varphi(x) + q(x)\int_{x_0}^{x} f(\tau)\omega(\phi(u(\tau)))d\tau$$
$$+ \sum_{x_0<x_j<x} \beta_j \psi(\phi(u(x_j-0))),$$

*then*

$$u(x) \le \phi^{-1}\{q(x)\varphi(x)G_i^{-1}[\int_{x_i}^{x} \frac{f(\tau)\omega(q(\tau)\varphi(\tau))}{\varphi(\tau)} d\tau]\},$$
$$x \in (x_i, x_{i+1}],\ i = 0, 1, 2...,$$

*where*

$$G_i(s) = \int_{c_i}^{s} \frac{1}{\omega(s)} ds,\ i = 0, 1, 2...,$$
$$c_i = G_{i-1}^{-1}[\int_{x_{i-1}}^{x_i} \frac{f(\tau)\omega(q(\tau)\varphi(\tau))}{\varphi(\tau)} d\tau]$$
$$+ \frac{\beta_i \psi\{q(x_i-0)\varphi(x_i-0)G_{i-1}^{-1}[\int_{x_{i-1}}^{x_i} \frac{f(\tau)\omega(q(\tau)\varphi(\tau))}{\varphi(\tau)} d\tau]\}}{\varphi(x_i-0)}$$

*for $i = 1, 2...$, and $c_0 = 1$.*

Now we consider the integral inequality containing multiple integrals for discontinuous function with one independent variable.

**Theorem 4** *Suppose $u$, $\phi$, $\varphi$, $\omega$, $\psi$, $\beta_i, i = 1, 2...$ are the same as in Theorem 1, $\sigma_i$, $i = 1, 2, 3 \in \Im$ class of functions, $f$, $g$, $h_1$, $h_2 \in (R_+, R_+)$, $q_i(x) \in (R, R_+)$, $q_i(x) \ge 1, i = 1, 2, 3$. Furthermore, assume $min\{\sigma_i(x),\ i = 1, 2, 3\} >$*

$x_{j-1}$ *for* $x \in (x_{j-1}, x_j]$, $j = 1, 2, \cdots$. *If for* $x \geq x_0$, $u(x)$ *satisfies the following inequality*

$$\phi(u(x)) \leq \varphi(x) + q_1(x)\int_{x_0}^{x} f(\tau)\omega(\phi(u(\sigma_1(\tau))))ds$$
$$+q_2(x)\int_{x_0}^{x} g(\tau)\phi(u(\sigma_2(\tau))ds$$
$$+q_3(x)\int_{x_0}^{x} h_1(\tau)\int_{x_0}^{\tau} h_2(s)\phi(u(\sigma_3(s)))dsd\tau$$
$$+\sum_{x_0<x_j<x}\beta_j\psi(\phi(u(x_j-0))), \tag{12}$$

*then for* $x \in (x_i, x_{i+1}]$, $i = 0, 1, 2...$ *we have*

$$u(x) \leq \phi^{-1}\{q(x)\varphi(x)G_i^{-1}[\int_{x_0}^{x} \exp(-F_{i+1}(\tau))$$
$$\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}$$
$$\omega(\exp(F_{i+1}(\tau)))d\tau]\exp(F_{i+1}(x))\}. \tag{13}$$

*where*

$$F_{i+1}(x) = \int_{x_i}^{x}[\frac{g(\tau)\varphi(\sigma_2(\tau))q(\sigma_2(\tau))}{\varphi(\tau)}$$
$$+h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\varphi(\sigma_3(s))q(\sigma_3(s))}{\varphi(s)}ds]d\tau,\ i = 0, 1, 2...,$$

$$G_i(s) = \int_{c_i}^{s}\frac{1}{\omega(s)}ds,\ i = 0, 1, 2...,$$

$$c_0 = 1,\ c_i = b_i + \frac{\beta_i\psi[\varphi(x_i-0)q(x_i-0)b_i]}{\varphi(x_i-0)},\ i = 1, 2...,$$

$$b_i = G_{i-1}^{-1}[\int_{x_{i-1}}^{x_i}\exp(-F_i(\tau))\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}$$
$$\omega(\exp(F_i(\tau)))d\tau]\exp(F_i(x_i)),\ i = 1, 2...$$

$b_i$, $c_i$ *are all constants.*

**Proof:** Let $q(x) = max\{q_i(x), i = 1, 2, 3\}$, considering $\varphi$ is nondecreasing, from (12) we have

$$\frac{\phi(u(x))}{\varphi(x)} \leq q(x)[1 + \int_{x_0}^{x}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau$$
$$+\int_{x_0}^{x}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_0}^{x}h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau$$
$$+\sum_{x_0<x_j<x}\frac{\beta_j\psi(\phi(u(x_j-0)))}{\varphi(x_j-0)}]. \tag{14}$$

Let $v(x) = \frac{\phi(u(x))}{\varphi(x)q(x)}$, and

$$\overline{v}_i(x) = c_i + \int_{x_i}^{x}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau$$
$$+\int_{x_i}^{x}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_i}^{x}h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau,\ i = 0, 1...$$

Then for $x \geq x_0$, we have

$$v(x) \leq 1 + \int_{x_0}^{x}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau$$
$$+\int_{x_0}^{x}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_0}^{x}h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau$$
$$+\sum_{x_0<x_j<x}\frac{\beta_j\psi(\phi(u(x_j-0)))}{\varphi(x_j-0)}. \tag{15}$$

Under the assumption $min\{\sigma_i(x),\ i = 1, 2, 3\} > x_{j-1}$, $x \in (x_{j-1}, x_j]$, $j = 1, 2...$, considering $\sigma_i$ is continuous, we have in fact $min\{\sigma_i(x),\ i = 1, 2, 3\} \geq x_0$, $for\ \forall x \geq x_0$. Since $u$ is nonnegative on $[x_0, \infty)$, so $u(\sigma_i(x)) \geq 0$, $i = 1, 2.3$, $\forall x \in [x_0, \infty)$. Furthermore $v(\sigma_i(x)) \geq 0$, $i = 1, 2.3$, $\forall x \in [x_0, \infty)$.

*Case 1:* If $x \in (x_0, x_1]$, then from (15) it follows

$$v(x) \leq \overline{v}_0(x). \tag{16}$$

From the assumption for $\sigma_i$ we have $x_0 < \sigma_i(x) \leq x$, $i = 1, 2, 3$. Then $v(\sigma_i(x)) \leq \overline{v}_0(\sigma_i(x)) \leq \overline{v}_0(x)$, and

$$\overline{v}_0'(x) = \frac{f(x)\omega(\phi(u(\sigma_1(x))))}{\varphi(x)} + \frac{g(x)\phi(u(\sigma_2(x))}{\varphi(x)}$$
$$+h_1(x)\int_{x_0}^{x}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}ds$$
$$\leq \frac{f(x)\omega(\varphi(\sigma_1(x))q(\sigma_1(x)))}{\varphi(x)}\omega(v(\sigma_1(x)))$$
$$+\frac{g(x)\varphi(\sigma_2(x))q(\sigma_2(x))}{\varphi(x)}v(\sigma_2(x))$$
$$+h_1(x)\int_{x_0}^{x}\frac{h_2(s)\varphi(\sigma_3(s))q(\sigma_3(s))v(\sigma_3(s))}{\varphi(s)}ds$$
$$\leq \frac{f(x)\omega(\varphi(\sigma_1(x))q(\sigma_1(x)))}{\varphi(x)}\omega(\overline{v}_0(x))$$

$$+\frac{g(x)\varphi(\sigma_2(x))q(\sigma_2(x))}{\varphi(x)}\overline{v}_0(x)$$
$$+\overline{v}_0(x)h_1(x)\int_{x_0}^{x}\frac{h_2(s)\varphi(\sigma_3(s))q(\sigma_3(s))}{\varphi(s)}ds.$$

Furthermore,

$$\overline{v}_0'(x)-[\frac{g(x)\varphi(\sigma_2(x))q(\sigma_2(x))}{\varphi(x)}$$
$$+h_1(x)\int_{x_0}^{x}\frac{h_2(s)\varphi(\sigma_3(s))q(\sigma_3(s))}{\varphi(s)}ds]\overline{v}_0(x)$$
$$\le \frac{f(x)\omega(\varphi(\sigma_1(x))q(\sigma_1(x)))}{\varphi(x)}\omega(\overline{v}_0(x)),$$

that is,

$$\overline{v}_0'(x)-F_1'(x)\overline{v}_0(x)\le \frac{f(x)\omega(\varphi(\sigma_1(x))q(\sigma_1(x)))}{\varphi(x)}\omega(\overline{v}_0(x)). \tag{17}$$

Multiplying $\exp(-F_1(x))$ on both sides of (17), it follows

$$[\overline{v}_0(x)\exp(-F_1(x))]'$$
$$\le \exp(-F_1(x))\frac{f(x)\omega(\varphi(\sigma_1(x))q(\sigma_1(x)))}{\varphi(x)}\omega(\overline{v}_0(x)). \tag{18}$$

An integration for (18) from $x_0$ to $x$ yields

$$\overline{v}_0(x)\exp(-F_1(x))-1$$
$$\le \int_{x_0}^{x}[\exp(-F_1(\tau))\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}\omega(\overline{v}_0(\tau))]d\tau.$$

Let $c(x)=\{1+\int_{x_0}^{x}[\exp(-F_1(\tau))\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}\omega(\overline{v}_0(\tau))]d\tau\}$.

Then $\overline{v}_0(x)\le c(x)\exp(F_1(x))$.

Moreover,

$$c'(x)=\exp(-F_1(x))\frac{f(x)\omega(\varphi(\sigma_1(x))q(\sigma_1(x)))}{\varphi(x)}\omega(\overline{v}_0(x))$$
$$\le \exp(-F_1(x))\frac{f(x)\omega(\varphi(\sigma_1(x))q(\sigma_1(x)))}{\varphi(x)}$$
$$\omega(c(x))\omega(\exp(F_1(x))), \tag{19}$$

that is,

$$\frac{c'(x)}{\omega(c(x))}\le \exp(-F_1(x))\frac{f(x)\omega(\varphi(\sigma_1(x))q(\sigma_1(x)))}{\varphi(x)}$$
$$\omega(\exp(F_1(x))). \tag{20}$$

Integrating (20) from $x_0$ to $x$, it follows

$$G_0(c(x))-G_0(c(0))\le \int_{x_0}^{x}[\exp(-F_1(\tau))$$
$$\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}\omega(\exp(F_1(\tau)))]d\tau.$$

Considering $G_0(c(0))=0$, then

$$c(x)\le G_0^{-1}[\int_{x_0}^{x}\exp(-F_1(\tau))$$
$$\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}\omega(\exp(F_1(\tau)))d\tau],$$

and

$$v(x)\le \overline{v}_0(x)\le c(x)\exp(F_1(x))$$
$$\le G_0^{-1}[\int_{x_0}^{x}\exp(-F_1(\tau))\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}$$
$$\omega(\exp(F_1(\tau)))d\tau]\exp(F_1(x)). \tag{21}$$

Especially we have

$$\frac{\phi(u(x_1-0))}{\varphi(x_1-0)q(x_1-0)}=v(x_1-0)\le \overline{v}_0(x_1-0)\le \overline{v}_0(x_1)$$
$$\le G_0^{-1}[\int_{x_0}^{x_1}\exp(-F_1(\tau))\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}$$
$$\omega(\exp(F_1(\tau)))d\tau]\exp(F_1(x_1))=b_1.$$

*Case 2:* If $x\in(x_1,x_2]$, then from (15) we have

$$v(x)\le 1+\int_{x_0}^{x}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau$$
$$+\int_{x_0}^{x}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_0}^{x}h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau$$
$$+\frac{\beta_1\psi(\phi(u(x_1-0)))}{\varphi(x_1-0)}$$
$$=1+\int_{x_0}^{x_1}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau+\int_{x_0}^{x_1}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_0}^{x_1}h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau$$
$$+\frac{\beta_1\psi(\phi(u(x_1-0)))}{\varphi(x_1-0)}+\int_{x_1}^{x}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau$$
$$+\int_{x_1}^{x}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_1}^{x}h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau$$
$$=\overline{v}_0(x_1)+\frac{\beta_1\psi(\phi(u(x_1-0)))}{\varphi(x_1-0)}+\int_{x_1}^{x}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau$$
$$+\int_{x_1}^{x}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$

$$+\int_{x_1}^{x} h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau. \quad (22)$$

Considering (21) holds for $\forall x \in (x_0, x_1]$, and the definition of $b_1$, then it follows

$$v(x) \le b_1 + \frac{\beta_1\psi[\varphi(x_1-0)q(x_1-0)b_1]}{\varphi(x_1-0)}$$
$$+\int_{x_1}^{x}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau+\int_{x_1}^{x}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_1}^{x} h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau$$
$$= c_1+\int_{x_1}^{x}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau$$
$$+\int_{x_1}^{x}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_1}^{x} h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau$$
$$=\overline{v}_1(x). \quad (23)$$

Then similar to the process of (16)-(21), we can reach the estimate

$$v(x)\le \overline{v}_1(x)$$
$$\le G_1^{-1}[\int_{x_1}^{x}\exp(-F_2(\tau))\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}$$
$$\omega(\exp(F_2(\tau)))d\tau]\exp(F_2(x)). \quad (24)$$

Especially we have

$$\frac{\phi(u(x_2-0))}{\varphi(x_2-0)q(x_2-0)}=v(x_2-0)\le\overline{v}_1(x_2-0)\le\overline{v}_1(x_2)$$
$$\le G_1^{-1}[\int_{x_1}^{x_2}\exp(-F_2(\tau))\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}$$
$$\omega(\exp(F_2(\tau)))d\tau]\exp(F_2(x_2))=b_2.$$

*Case 3:* Suppose

$$v(x)\le\overline{v}_{j-1}(x)$$
$$\le G_{j-1}^{-1}[\int_{x_{j-1}}^{x}\exp(-F_j(\tau))\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}$$
$$\omega(\exp(F_j(\tau)))d\tau]\exp(F_j(x))$$

holds for $x \in (x_{j-1}, x_j]$, $j = 1, 2, ...i$, then for $x \in (x_i, x_{i+1}]$, from (15) we have

$$v(x)\le 1+\int_{x_0}^{x}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau$$
$$+\int_{x_0}^{x}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_0}^{x} h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau$$
$$+\sum_{x_0<x_j<x}\frac{\beta_j\psi(\phi(u(x_j-0)))}{\varphi(x_j-0)}$$
$$=1+\int_{x_0}^{x_i}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau+\int_{x_0}^{x_i}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_0}^{x_i} h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau$$
$$+\sum_{x_0<x_j<x}\frac{\beta_j\psi(\phi(u(x_j-0)))}{\varphi(x_j-0)}$$
$$+\int_{x_i}^{x}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau+\int_{x_i}^{x}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_i}^{x} h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau$$
$$\le b_i+\frac{\beta_i\psi[\varphi(x_i-0)q(x_i-0)b_i]}{\varphi(x_i-0)}$$
$$+\int_{x_i}^{x}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau+\int_{x_i}^{x}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_i}^{x} h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau$$
$$=c_i+\int_{x_i}^{x}\frac{f(\tau)\omega(\phi(u(\sigma_1(\tau))))}{\varphi(\tau)}d\tau+\int_{x_i}^{x}\frac{g(\tau)\phi(u(\sigma_2(\tau))}{\varphi(\tau)}d\tau$$
$$+\int_{x_i}^{x} h_1(\tau)\int_{x_0}^{\tau}\frac{h_2(s)\phi(u(\sigma_3(s)))}{\varphi(s)}dsd\tau$$
$$=\overline{v}_i(x). \quad (25)$$

Then similar to (16)-(21), we deduce that

$$v(x)\le\overline{v}_i(x)$$
$$\le G_i^{-1}[\int_{x_i}^{x}\exp(-F_{i+1}(\tau))\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}$$
$$\omega(\exp(F_{i+1}(\tau)))d\tau]\exp(F_{i+1}(x)). \quad (26)$$

From the analysis above, considering

$$u(x)=\phi^{-1}\{q(x)\varphi(x)v(x)\},$$

we have completed the proof. □

**Remark 5** *If $\varphi(x) \equiv u_0$ and $u_0$ is a constant, $\sigma_1(x) = \sigma_3(x)$, $f(x) = h_1(x)$, $q_2(x) \equiv 1$, $q_1(x) = q_3(x)$, $\phi(u) = u$, $\psi(u) = u^m$, $m > 0$, then Theorem 4 becomes [15, Theorem 5]. If $q_1(x) = q_2(x) = q_3(x)$, $f(x) = h_1(x)$, $g(x) \equiv 0$, $\phi(u) = u$, $\omega(u) = u$, $\psi(u) = u^m$, $m > 0$, then Theorem 4 becomes [14, Theorem 3.1].*

**Corollary 6** *Suppose $u$, $\phi$, $\varphi$, $\omega$, $\psi$, $\beta_i, i = 1, 2...$ are the same as in Theorem 1, $f$ , $g$, $\sigma_i$, $q_i$, $i = 1, 2$ are the same as in Theorem 4. If for $x \geq x_0$*

$$\phi(u(x)) \leq \varphi(x) + q_1(x)\int_{x_0}^{x} f(\tau)\omega(\phi(u(\sigma_1(\tau))))ds$$

$$+q_2(x)\int_{x_0}^{x} g(\tau)\phi(u(\sigma_2(\tau))ds + \sum_{x_0<x_j<x} \beta_j\psi(\phi(u(x_j-0))),$$

*then for $x \in (x_i, x_{i+1}]$, $i = 0, 1, 2...$ we have*

$$u(x) \leq \phi^{-1}\{q(x)\varphi(x)G_i^{-1}[\int_{x_0}^{x} \exp(-F_{i+1}(\tau))$$

$$\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}\omega\{\exp(F_{i+1}(\tau))d\tau]\exp(F_{i+1}(x))\},$$

*where*

$$F_{i+1}(x) = \exp\{\int_{x_i}^{x} \frac{g(\tau)\varphi(\sigma_2(\tau))q(\sigma_2(\tau))}{\varphi(\tau)}d\tau\},\ i = 0, 1, 2...,$$

$$G_i(s) = \int_{c_i}^{s} \frac{1}{\omega(s)}ds,\ i = 0, 1, 2...,$$

$$c_0 = 1,\ c_i = b_i + \frac{\beta_i\psi[\varphi(x_i-0)q(x_i-0)b_i]}{\varphi(x_i-0)},\ i = 1, 2...,$$

$$b_i = G_{i-1}^{-1}[\int_{x_{i-1}}^{x_i} \exp(-F_i(\tau))\frac{f(\tau)\omega(\varphi(\sigma_1(\tau))q(\sigma_1(\tau)))}{\varphi(\tau)}\times$$

$$\omega(\exp(F_i(\tau)))d\tau]\exp(F_i(x)),\ i = 1, 2...$$

If we take $\sigma_1(x) = \sigma_2(x) = x$ in Corollary 6, then we can obtain another corollary, which can be left to the readers.

In the following we study the integral inequality for discontinuous functions with two independent variables.

**Theorem 7** *Suppose $u(x, y)$ is a nonnegative continuous function on $\Omega = \bigcup_{i,j\geq 1} \Omega_{i,j}$, $\Omega_{i,j} = \{(x, y)|x_{i-1} < x \leq x_i,\ y_{j-1} < y \leq y_j\}$ with the exception in the points $(x_i, y_i)$, $i = 1, 2, ...,$ where there are finite jumps, and $x_0 < x_1 < ... < x_n < ...,$ $y_0 < y_1 < ... < y_n < ...,$ $\lim_{n\to\infty} x_n = \infty$, $\lim_{n\to\infty} y_n = \infty$. $\varphi(x, y)$ is a positive nondecreasing function, that is, for $\forall\ (p, q), (P, Q) \in \Omega$ and $p \leq P$, $q \leq Q$ it follows $\varphi(p, q) \leq \varphi(P, Q)$. Furthermore, suppose $q(x, y) \geq 1$, $f(x, y) \geq 0$ and $f(x, y) = 0$ for $(x, y) \in \Omega_{i,j}$, $i \neq j$. $\omega$, $\phi$, $\psi$, $\beta_i$ are the same as in Theorem 1. If for $x > x_0$, $y > y_0$ $u(x, y)$ satisfies the following inequality*

$$\phi(u(x, y)) \leq \varphi(x, y)$$

$$+q(x, y)\int_{y_0}^{y}\int_{x_0}^{x} f(s, t)\omega(\phi(u(x(s, t))))dsdt$$

$$+\sum_{x_0<x_j<x, y_0<y_j<y} \beta_j\psi(\phi(u(x_j-0, y_j-0))) \quad (27)$$

*then*

$$u(x, y) \leq \phi^{-1}\{q(x, y)\varphi(x, y)$$

$$G_i^{-1}[\int_{y_i}^{y}\int_{x_i}^{x} \frac{f(s, t)\omega(\varphi(s, t)q(s, t))}{\varphi(s, t)}dsdt]\},$$

$$\forall (x, y) \in \Omega_{i,i},\ i = 1, 2..., \quad (28)$$

*where*

$$G_i(s) = \int_{c_i}^{s} \frac{1}{\omega(s)}ds,\ i = 0, 1, 2...,$$

$$c_i = G_{i-1}^{-1}[\int_{y_{i-1}}^{y_i}\int_{x_{i-1}}^{x_i} \frac{f(s, t)\omega(\varphi(s, t)q(s, t))}{\varphi(s, t)}dsdt] +$$

$$\frac{\beta_i}{\varphi(x_i-0, y_i-0)}\psi\{\varphi(x_i-0, y_i-0)q(x_i-0, y_i-0)$$

$$G_{i-1}^{-1}[\int_{y_{i-1}}^{y_i}\int_{x_{i-1}}^{x_i} \frac{f(s, t)\omega(\varphi(s, t)q(s, t))}{\varphi(s, t)}dsdt]\}$$

*for $i = 1, 2...,$ and $c_0 = 1$.*

**Proof:** Let $v(x, y) = \frac{\phi(u(x,y))}{\varphi(x,y)q(x,y)}$, and

$$\overline{v}_i(x, y) = c_i + \int_{y_i}^{y}\int_{x_i}^{x} \frac{f(s, t)\omega(\varphi(s, t)q(s, t))\omega(v(s, t))}{\varphi(s, t)}dsdt,$$

$$i = 0, 1, 2, \cdots$$

Considering $\varphi$ is nondecreasing, for $x > x_0$, $y > y_0$ we have

$$v(x, y) \leq 1 + \int_{y_0}^{y}\int_{x_0}^{x} \frac{f(s, t)\omega(\phi(u(s, t)))}{\varphi(s, t)}dsdt$$

$$+\sum_{x_0<x_j<x, y_0<y_j<y} \frac{\beta_j\psi(\phi(u(x_j-0, y_j-0)))}{\varphi(x_j-0, y_j-0)}$$

$$= 1 + \int_{y_0}^{y}\int_{x_0}^{x} \frac{f(s, t)\omega[\varphi(s, t)q(s, t)v(s, t)]}{\varphi(s, t)}dsdt$$

$$+\sum_{x_0<x_j<x, y_0<y_j<y} \{\frac{\beta_j}{\varphi(x_j-0, y_j-0)}$$

$$\psi[\varphi(x_j-0, y_j-0)q(x_j-0, y_j-0)v(x_j-0, y_j-0)]\}$$

$$\leq 1 + \int_{y_0}^{y}\int_{x_0}^{x} \frac{f(s, t)\omega(\varphi(s, t)q(s, t))\omega(v(s, t))}{\varphi(s, t)}dsdt$$

$$+\sum_{x_0<x_j<x, y_0<y_j<y} \{\frac{\beta_j}{\varphi(x_j-0, y_j-0)}$$

$$\psi[\varphi(x_j-0, y_j-0)q(x_j-0, y_j-0)v(x_j-0, y_j-0)]\} \quad (29)$$

*Case 1:* If $(x,y)\in\Omega_{1,1}$, then from (29) we have

$$v(x,y)\le\overline{v}_0(x,y). \tag{30}$$

Given a fixed $X$ such that $x_0<X\le x_1$ and $x\in(x_0,X]$, then $v(x,y)\le\overline{v}_0(x,y)\le\overline{v}_0(X,y)$, and

$$[\overline{v}_0(X,y)]'_y=\int_{x_0}^{X}\frac{f(s,y)\omega(\varphi(s,y)q(s,y))\omega(v(s,y))}{\varphi(s,y)}ds$$
$$\le\omega(\overline{v}_0(X,y))\int_{x_0}^{X}\frac{f(s,y)\omega(\varphi(s,y)q(s,y))}{\varphi(s,y)}ds,$$

that is,

$$\frac{[\overline{v}_0(X,y)]'_y}{\omega(\overline{v}_0(X,y))}\le\int_{x_0}^{X}\frac{f(s,y)\omega(\varphi(s,y)q(s,y))}{\varphi(s,y)}ds. \tag{31}$$

Considering $\overline{v}_0(X,y_0)=1,\ G_0(\overline{v}_0(X,y_0))=0$, an integration for (31) from $y_0$ to $y$ yields

$$G_0(\overline{v}_0(X,y))\le\int_{y_0}^{y}\int_{x_0}^{X}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))}{\varphi(s,t)}dsdt.$$

Then

$$v(x,y)\le\overline{v}_0(X,y)$$
$$\le G_0^{-1}[\int_{y_0}^{y}\int_{x_0}^{X}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))}{\varphi(s,t)}dsdt]. \tag{32}$$

Take $x=X$ and considering $X\in(x_0,x_1]$ is arbitrary, it follows

$$v(x,y)\le\overline{v}_0(x,y)$$
$$\le G_0^{-1}[\int_{y_0}^{y}\int_{x_0}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))}{\varphi(s,t)}dsdt],(x,y)\in\Omega_{1,1}. \tag{33}$$

Especially we have

$$v(x_1-0,y_1-0)\le\overline{v}_0(x_1-0,y_1-0)\le\overline{v}_0(x_1,y_1)$$
$$\le G_0^{-1}[\int_{y_0}^{y_1}\int_{x_0}^{x_1}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))}{\varphi(s,t)}dsdt].$$

*Case 2:* If $(x,y)\in\Omega_{2,2}$, then from (29) we have

$$v(x,y)\le 1+\int_{y_0}^{y}\int_{x_0}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))\omega(v(s,t))}{\varphi(s,t)}dsdt$$
$$+\frac{\beta_1\psi[\varphi(x_1-0,y_1-0)q(x_1-0,y_1-0)v(x_1-0,y_1-0)]}{\varphi(x_1-0,y_1-0)}$$
$$=1+\int_{y_0}^{y_1}\int_{x_0}^{x_1}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))\omega(v(s,t))}{\varphi(s,t)}dsdt$$
$$+\int_{y_1}^{y}\int_{x_1}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))\omega(v(s,t))}{\varphi(s,t)}dsdt$$
$$+\frac{\beta_1\psi[\varphi(x_1-0,y_1-0)q(x_1-0,y_1-0)v(x_1-0,y_1-0)]}{\varphi(x_1-0,y_1-0)}$$
$$=\overline{v}_0(x_1,y_1)+\int_{y_1}^{y}\int_{x_1}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))\omega(v(s,t))}{\varphi(s,t)}dsdt$$
$$+\frac{\beta_1\psi[\varphi(x_1-0,y_1-0)q(x_1-0,y_1-0)v(x_1-0,y_1-0)]}{\varphi(x_1-0,y_1-0)}$$
$$\le G_0^{-1}[\int_{y_0}^{y_1}\int_{x_0}^{x_1}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))}{\varphi(s,t)}dsdt]$$
$$+\int_{y_1}^{y}\int_{x_1}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))\omega(v(s,t))}{\varphi(s,t)}dsdt$$
$$+\frac{\beta_1}{\varphi(x_1-0,y_1-0)}\psi\{\varphi(x_1-0,y_1-0)q(x_1-0,y_1-0)$$
$$G_0^{-1}[\int_{y_0}^{y_1}\int_{x_0}^{x_1}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))}{\varphi(s,t)}dsdt]\}$$
$$=c_1+\int_{y_1}^{y}\int_{x_1}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))\omega(v(s,t))}{\varphi(s,t)}dsdt$$
$$=\overline{v}_1(x,y) \tag{34}$$

Following in the same manner as the process of (30)-(33) we can deduce

$$v(x,y)\le\overline{v}_1(x,y)$$
$$\le G_1^{-1}[\int_{y_1}^{y}\int_{x_1}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))}{\varphi(s,t)}dsdt],\ (x,y)\in\Omega_{2,2}. \tag{35}$$

Especially we have

$$v(x_2-0,y_2-0)\le\overline{v}_1(x_2-0,y_2-0)\le\overline{v}_1(x_2,y_2)$$
$$\le G_1^{-1}[\int_{y_1}^{y_2}\int_{x_1}^{x_2}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))}{\varphi(s,t)}dsdt].$$

*Case 3:* Suppose

$$v(x,y)\le G_{j-1}^{-1}[\int_{y_{j-1}}^{y}\int_{x_{j-1}}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))}{\varphi(s,t)}dsdt]$$

holds for $(x,y)\in\Omega_{jj},\ j=1,2,...i$, then for $(x,y)\in\Omega_{i+1,i+1}$, from (29) we have

$$v(x,y)\le 1+\int_{y_0}^{y}\int_{x_0}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))\omega(v(s,t))}{\varphi(s,t)}dsdt$$
$$+\sum_{x_0<x_j<x,y_0<y_j<y}\{\frac{\beta_j}{\varphi(x_j-0,y_j-0)}$$
$$\psi[\varphi(x_j-0,y_j-0)q(x_j-0,y_j-0)v(x_j-0,y_j-0)]\}$$
$$=1+\int_{y_0}^{y_i}\int_{x_0}^{x_i}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))\omega(v(s,t))}{\varphi(s,t)}dsdt$$
$$+\int_{y_i}^{y}\int_{x_i}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))\omega(v(s,t))}{\varphi(s,t)}dsdt$$

$$+\sum_{x_0<x_j<x,y_0<y_j<y}\{\frac{\beta_j}{\varphi(x_j-0,y_j-0)}$$

$$\psi[\varphi(x_j-0,y_j-0)q(x_j-0,y_j-0)v(x_j-0,y_j-0)]\}$$

$$\leq G_{i-1}^{-1}[\int_{y_{i-1}}^{y_i}\int_{x_{i-1}}^{x_i}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))}{\varphi(s,t)}dsdt]$$

$$+\int_{y_i}^{y}\int_{x_i}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))\omega(v(s,t))}{\varphi(s,t)}dsdt$$

$$+\frac{\beta_i}{\varphi(x_i-0,y_i-0)}\psi\{\varphi(x_i-0,y_i-0)q(x_i-0,y_i-0)$$

$$G_{i-1}^{-1}[\int_{y_{i-1}}^{y_i}\int_{x_{i-1}}^{x_i}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))}{\varphi(s,t)}dsdt]\}$$

$$=c_i+\int_{y_i}^{y}\int_{x_i}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))\omega(v(s,t))}{\varphi(s,t)}dsdt$$

$$=\overline{v}_i(x,y) \quad (36)$$

Similar to Case 2 we can reach the estimate

$$v(x,y)\leq \overline{v}_i(x,y)$$

$$\leq G_i^{-1}[\int_{y_i}^{y}\int_{x_i}^{x}\frac{f(s,t)\omega(\varphi(s,t)q(s,t))}{\varphi(s,t)}dsdt],$$

$$(x,y)\in\Omega_{i+1,i+1}. \quad (37)$$

Considering $u(x,y)=\phi^{-1}\{q(x,y)\varphi(x,y)v(x,y)\}$, then we have completed the proof. □

**Remark 8** *Theorem 7 generalize many known results. For example, if we take* $q(x,y)\equiv 1,\ \phi(u)=u,\ \omega(u)=u,\ \psi(u)=u^m,\ m>0$, *then Theorem 7 becomes [16, Theorem 2.1]. If we take* $\phi(u)=u,\ \omega(u)=\psi(u)=u^m,\ m>0$, *then Theorem 2.3 reduces to [16, Theorem 2.3]. If we take* $q(x,y)\equiv 1,\ \phi(u)=u,\ \omega(u)=\psi(u)=u^m,\ m>0$, *then Theorem 7 reduces to [16, Theorem 2.2]. If we take* $\phi(u)=u,\ \psi(u)=u$, *then Theorem 2.3 reduces to [17, Theorem 2.1]. If we take* $\phi(u)=u,\ \psi(u)=u^m,\ m>0$, *then Theorem 7 reduces to [17, Theorem 2.2]. If we take* $\phi(u)=u^m,\ \omega(u)=\widetilde{\omega}(u)u^n,\ \psi(u)=u,\ m>n>0$, *then Theorem 7 becomes [17, Theorem 2.3].*

**Remark 9** *Theorem 7 can easily be generalized to the situation with delay items and four iterated integrals, and the process of proof is almost the same as in Theorem 7.*

## 3 Some Applications

In this section, we will present two examples so as to illustrate the validity of the above results in making estimate for the bounds of the solutions of certain integral equations.

**Example 1**: Consider the following integral equation:

$$ln(1+u(x))=C+\int_{x_0}^{x}F(s,u(s))ds+\sum_{x_0<x_j<x}u(x_j-0) \quad (38)$$

with the initial condition $u(x_0)=e^C-1>0$, where $u(x)$ is a nonnegative piecewise continuous function defined on $[x_0,\infty)$ with discontinuities of the first kind in $x_i,\ i=1,2...$, and $0\leq x_0<x_1<...<x_n<...,\ \lim_{n\to\infty}x_n=\infty$. Assume $0\leq F(x,u)\leq f(x)(ln(1+u))^m,\ m>0$, where $f\in(R_+,R_+)$.

If we let $\phi(u)=ln(u+1),\ \psi(u)=e^u-1,\ \omega(u)=u^m,\ \varphi(x)\equiv C,\ q(x)\equiv 1,\ \beta_i\equiv 1$, then from (38) we have:

$$\phi(u(x))=ln(1+u(x))$$

$$\leq C+\int_{x_0}^{x}f(s)(ln(1+u(s)))^m ds+\sum_{x_0<x_j<x}u(x_j-0)$$

$$=C+\int_{x_0}^{x}f(s)\omega(\phi(u(s)))ds+\sum_{x_0<x_j<x}\psi(\phi(u(x_j-0))).$$

So according to Corollary 2.1 we can give the bound of $u(x)$ as

$$u(x)\leq\phi^{-1}\{CG_i^{-1}[\int_{x_i}^{x}f(\tau)C^{m-1}d\tau]\},\ x\in(x_i,x_{i+1}]$$

where

$$G_i(s)=\int_{c_i}^{s}\frac{1}{\omega(s)}ds,\ i=0,1,2...,$$

$$c_i=G_{i-1}^{-1}[\int_{x_{i-1}}^{x_i}f(\tau)C^{m-1}d\tau]$$

$$+\frac{\psi\{CG_{i-1}^{-1}[\int_{x_{i-1}}^{x_i}f(\tau)C^{m-1}d\tau]\}}{C}$$

for $i=1,2...$ and $c_0=1$.

**Example 2**: Consider the following integral equation with two independent variables:

$$e^{u(x,y)}=C+\int_{y_0}^{y}\int_{x_0}^{x}F(s,t,u(s,t))dsdt$$

$$+\sum_{x_0<x_j<x,y_0<y_j<y}e^{nu(x_j-0,y_j-0)} \quad (39)$$

with the initial condition $u(x_0, y_0) = lnC$, where $u(x, y)$ is a nonnegative continuous function defined on $\Omega = \bigcup_{i,j\geq 1} \Omega_{i,j}$, $\Omega_{i,j} = \{(x, y)|x_{i-1} < x \leq x_i,\ y_{j-1} < y \leq y_j\}$ with the exception in the points $(x_i, y_i)$, $i = 1, 2, ...$, and $0 \leq x_0 < x_1 < ... < x_n < ...$, $0 \leq y_0 < y_1 < ... < y_n < ...$, $\lim_{n\to\infty} x_n = \infty$, $\lim_{n\to\infty} y_n = \infty$. Furthermore, assume $0 \leq F(x, y, u) \leq f(x, y)e^{mu}$, where $m, n$ are positive numbers, and $f(x, y) \equiv 0$, $\forall (x, y) \in \Omega_{i,j}$, $i \neq j$.

If we take $\phi(u) = e^u$, $\omega(u) = u^m$, $\psi(u) = u^n, q(x, y) \equiv 1$, $\varphi(x, y) \equiv C$, $\beta_i \equiv 1$, then according to Theorem 2.3 we can obtain the bound of $u(x, y)$ as

$$u(x, y) \leq \phi^{-1}\{CG_i^{-1}[\int_{y_i}^{y}\int_{x_i}^{x} f(s, t)C^{m-1}dsdt]\},$$

$$(x, y) \in \Omega_{i,i},\ i = 1, 2...$$

where

$$G_i(s) = \int_{c_i}^{s} \frac{1}{\omega(s)}ds,\ i = 0, 1, 2...,$$

$$c_i = G_{i-1}^{-1}[\int_{y_{i-1}}^{y_i}\int_{x_{i-1}}^{x_i} f(s, t)C^{m-1}dsdt]$$

$$+\frac{\beta_i\psi\{CG_{i-1}^{-1}[\int_{y_{i-1}}^{y_i}\int_{x_{i-1}}^{x_i} f(s, t)C^{m-1}dsdt]\}}{C}$$

for $i = 1, 2...$ and $c_0 = 1$.

**Remark 10** *We note that the methods in [1-17] are not available here to make estimate for the bound of the solutions of the presented two integral equations.*

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