Gronwall-Bellman Type Inequalities On Time Scales And Their Applications

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Abstract: In this work, we investigate some new Gronwall-Bellman type dynamic inequalities on time scales in two independent variables, which provide a handy tool in deriving explicit bounds on unknown functions in certain dynamic equations on time scales. The established results generalize the main results on integral inequalities for continuous functions in [1] and their corresponding discrete analysis in [2].

Key–Words: Dynamic inequality; Gronwall-Bellman inequality; Time scales; Dynamic equation; Bounded

1 Introduction

During the past decades, with the development of the theory of differential and integral equations, a lot of integral and difference inequalities, for example, [3-12] and the references therein, have been discovered, which play an important role in the research of boundedness, global existence, stability of solutions of differential and integral equations as well as difference equations. In these inequalities, Gronwall-Bellman type inequalities and their generations have been paid much attention by many authors (for example, see [3-8]). On the other hand, Hilger [13] initiated the theory of time scales, and one of the purposes of the theory of time scales is to unify continuous and discrete analysis. Since then, many integral inequalities on time scales have been established(for example, see [14-20] and the references therein), which on one hand provide a handy tool in the study of qualitative as well as quantitative properties of solutions of certain dynamic equations on time scales, on the other hand unify continuous and discrete analysis to some extent. But to our knowledge, Gronwall-Bellman type inequalities in two independent on time scales containing integration on infinite intervals have been paid little attention in the literature so far.

In this paper, we will establish some new Gronwall-Bellman type dynamic inequalities in two independent variables on time scales containing integration on infinite intervals, which generalize the main results in [1] and [2], and present some applications for them.

First we will give some preliminaries on calculus of time scales and some universal symbols used in this paper. Throughout the paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$. $\mathbb{Z}$ denotes the set of integers. For two given sets G, H, we denote the set of maps from G to H by (G, H). A time scale is an arbitrary nonempty closed subset of the real numbers. In this paper, $\mathbb{T}$ denotes an arbitrary time scale. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma \in (\mathbb{T}, \mathbb{T})$ and $\rho \in (\mathbb{T}, \mathbb{T})$ such that $\sigma(t) = inf\{s \in \mathbb{T}, s > t\}$, $\rho(t) = sup\{s \in \mathbb{T}, s < t\}$.

Definition 1 *The graininess* $\mu(t) \in (\mathbb{T}, \mathbb{R}_+)$ *is defined by* $\mu(t) = \sigma(t) - t$.

Remark 2 *Obviously,* $\mu(t) = 0$ *if* $\mathbb{T} = \mathbb{R}$ *while* $\mu(t) = 1$ *if* $\mathbb{T} = \mathbb{Z}$.

Definition 3 *A point* $t \in \mathbb{T}$ *is said to be left-dense if* $\rho(t) = t$ *and* $t \neq \inf \mathbb{T}$*, right-dense if* $\sigma(t) = t$ *and* $t \neq \sup \mathbb{T}$*, left-scattered if* $\rho(t) < t$ *and right-scattered if* $\sigma(t) > t$.

Definition 4 *The set* $\mathbb{T}^\kappa$ *is defined to be* $\mathbb{T}$ *if* $\mathbb{T}$ *does not have a left-scattered maximum, otherwise it is* $\mathbb{T}$ *without the left-scattered maximum.*

Definition 5 *A function $f(t) \in (\mathbb{T}, \mathbb{R})$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1+\mu(t)f(t) \neq 0$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f | f \in \mathfrak{R},\ 1+\mu(t)f(t) > 0,\ \forall t \in \mathbb{T}\}$.*

Definition 6 *For some $t \in \mathbb{T}^\kappa$, and a function $f \in (\mathbb{T}, \mathbb{R})$, the delta derivative of f at t is denoted by $f^\Delta(t)$ (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighborhood $\mathfrak{U}$ of t satisfying*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in \mathfrak{U}$.

Similarly, for some $x \in \mathbb{T}^\kappa$, and a function $f(x, y) \in (\mathbb{T} \times \mathbb{T}, \mathbb{R})$, the partial delta derivative of $f(x, y)$ with respect to x is denoted by $(f(x, y))^\Delta_x$, and satisfies

$$\begin{aligned} & |f(\sigma(x), y) - f(s, y) - (f(x, y))^\Delta_x(\sigma(x) - s)| \\ \leq \ & \varepsilon|\sigma(x) - s|,\ for\ \forall\varepsilon > 0, \end{aligned}$$

where $s \in \mathfrak{U}$, and $\mathfrak{U}$ is a neighborhood of x. The function $f(x, y)$ is called partial delta $differential$ with respect to x on $\mathbb{T}^\kappa$.

Remark 7 *If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t)$ becomes the usual derivative $f'(t)$, while $f^\Delta(t) = f(t+1) - f(t)$ if $\mathbb{T} = \mathbb{Z}$, which represents the forward difference.*

Definition 8 *If $F^\Delta(t) = f(t),\ t \in \mathbb{T}^\kappa$, then F is called an antiderivative of f, and the Cauchy integral of f is defined by*

$\int_a^b f(t)\Delta t = F(b) - F(a)$,
where $a,\ b \in \mathbb{T}$.

Similarly, for $a,\ b \in \mathbb{T}$ and a function $f(x, y) : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$, the Cauchy partial integral of $f(x, y)$ with respect to x is defined by
$\int_a^b f(x, y)\Delta x = F(b, y) - F(a, y)$,
where $(F(x, y))^\Delta_x = f(x, y),\ x \in \mathbb{T}^\kappa$.

The following theorem includes some important properties for *Cauchy partial integral* on time scales.

Theorem 9 *If $a,\ b,\ c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f(x, y),\ g(x, y) \in C_{rd}(\mathbb{T} \times \mathbb{T}, \mathbb{R})$, then*

(i) $\int_a^b [f(x, y) + g(x, y)]\Delta x = \int_a^b f(x, y)\Delta x + \int_a^b g(x, y)\Delta x$,

(ii) $\int_a^b (\alpha f)(x, y)\Delta t = \alpha \int_a^b f(x, y)\Delta x$,

(iii) $\int_a^b f(x, y)\Delta x = -\int_b^a f(x, y)\Delta x$,

(iv) $\int_a^b f(x, y)\Delta x = \int_a^c f(x, y)\Delta x + \int_c^b f(x, y)\Delta x$,

(v) $\int_a^a f(x, y)\Delta x = 0$,

(vi) if $f(x, y) \geq 0$ for all $a \leq x \leq b$, then $\int_a^b f(x, y)\Delta x \geq 0$.

Remark 10 *If $b = \infty$, then all the conclusions of Theorem 1.1 still hold.*

Definition 11 *The cylinder transformation ξ_h is defined by*

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h},\ if\ h \neq 0\ (for\ z \neq -\frac{1}{h}), \\ z, \qquad\qquad if\ h = 0, \end{cases}$$

where Log is the principal logarithm function.

Definition 12 *For $p(x, y) \in \mathfrak{R}$ with respect to x, the exponential function is defined by*

$$e_p(x, s) = exp(\int_s^x \xi_{\mu(\tau)}(p(\tau, y))\Delta\tau)$$

for $s, x \in \mathbb{T}$.

Remark 13 *If $\mathbb{T} = \mathbb{R}$, then for $x \in \mathbb{R}$ the following formula holds*

$$e_p(x, s) = exp(\int_s^x p(\tau, y)d\tau)$$

$$for \quad s \in \mathbb{T}.$$

If $\mathbb{T} = \mathbb{Z}$, then for $t \in \mathbb{Z}$,

$$e_p(x, s) = \prod_{\tau=s}^{x-1} [1 + p(\tau, y)]$$

$$for \quad s \in \mathbb{T} \quad and \quad s < x.$$

The following two theorems include some known properties on the $exponential function$.

Theorem 14 *If $p(x,y) \in \mathfrak{R}$ with respect to x, then the following conclusions hold*

(i) $e_p(x,x) \equiv 1$, and $e_0(x,s) \equiv 1$,

(ii) $e_p(\sigma(x),s) = (1+\mu(x)p(x,y))e_p(x,s)$,

(iii) If $p \in \mathfrak{R}^+$ with respect to x, then $e_p(x,s) > 0$ for $\forall s,\ x \in \mathbb{T}$,

(iv) If $p \in \mathfrak{R}^+$ with respect to x, then $\ominus p \in \mathfrak{R}^+$,

(v) $e_p(x,s) = \frac{1}{e_p(s,x)} = e_{\ominus p}(s,x)$, where $(\ominus p)(x,y) = -\frac{p(x,y)}{1+\mu(x)p(x,y)}$.

Theorem 15 *If $p(x,y) \in \mathfrak{R}$ with respect to x, $x_0 \in \mathbb{T}$ is a fixed number, then the exponential function $e_p(x,x_0)$ is the unique solution of the following initial value problem*

$$\begin{cases} (z(x,y))^{\Delta}_x = p(x,y)z(x,y), \\ z(x_0,y) = 1. \end{cases}$$

Remark 16 *Theorem 9 14 and 15 are extensions of [16, Theorem 2.2] and [21, Theorem 5.2, 5.1] respectively.*

Remark 17 *For more details about time scales, we advise the reader to refer to [22].*

2 Main Results

For convenience of notation, in the rest of this paper we always assume that $\mathbb{T}_0 = [x_0,\infty)\bigcap\mathbb{T}$, $\widetilde{\mathbb{T}}_0 = [y_0,\infty)\bigcap\mathbb{T}$, where $x_0,\ y_0 \in \mathbb{T}$, and furthermore assume $\mathbb{T}_0 \subseteq \mathbb{T}^\kappa$, $\widetilde{\mathbb{T}}_0 \subseteq \mathbb{T}^\kappa$.

We will give some lemmas for further use.

Lemma 18 *Suppose $Y \in \widetilde{\mathbb{T}}_0$ is an arbitrarily fixed number, and $u(x,Y) \in C_{rd}$, $m(x,Y) \in \mathfrak{R}_+$ with respect to x, $m(x,Y) \geq 0$, then*

$$u(x,Y) \leq a(x,Y) + \int_{x_0}^{x} m(s,Y)u(s,Y)\Delta s, x \in \mathbb{T}_0$$

implies

$$u(x,Y) \leq a(x,Y) + \int_{x_0}^{x} e_{m(.,Y)}(x,\sigma(s))a(s,Y)$$
$$m(s,Y)\Delta s,\ x \in \mathbb{T}_0,$$

where $e_{m(.,Y)}(x,x_0)$ is the unique solution of the following problem

$$\begin{cases} (z(x,Y))^{\Delta}_x = m(x,Y)z(x,Y), \\ z(x_0,Y) = 1. \end{cases}$$

The proof for Lemma 18 is similar to [21, Theorem 5.6], and we omit it here.

Lemma 19 *Under the conditions of Lemma 18, and furthermore assume $a(x,y)$ is nondecreasing in x for every fixed y, then we have*

$$u(x,Y) \leq a(x,Y)e_{m(.,Y)}(x,x_0).$$

Proof: Since $a(x,y)$ is nondecreasing in x for every fixed y, then from Lemma 18 we have

$$u(x,Y) \leq a(x,Y) + \int_{x_0}^{x} e_{m(.,Y)}(x,\sigma(s))a(s,Y)$$
$$m(s,Y)\Delta s$$
$$\leq a(x,Y)[1 + \int_{x_0}^{x} e_{m(.,Y)}(x,\sigma(s))m(s,Y)\Delta s].$$

On the other hand, from [22, Theorem 2.39 and 2.36 (i)],

$$1+\int_{x_0}^{x} e_{m(.,Y)}(x,\sigma(s))m(s,Y)\Delta s = e_{m(.,Y)}(x,x_0).$$

Then Collecting the above information we can obtain the desired inequality.

Lemma 20 *[23] Assume that $a \geq 0, p \geq q \geq 0$, and $p \neq 0$, then for any $K > 0$*

$$a^{\frac{q}{p}} \leq \tfrac{q}{p}K^{\frac{q-p}{p}}a + \tfrac{p-q}{p}K^{\frac{q}{p}}.$$

Theorem 21 *Suppose $\sup\limits_{y\in\widetilde{\mathbb{T}}_0} y = \infty$, $u,\ f,\ g,\ h,\ a,\ b \in C_{rd}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, R_+)$, p is a constant with $p \geq 1$. If for $(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality*

$$u^p(x,y) \leq a(x,y) + b(x,y)\int_{x_0}^{x}\int_{y}^{\infty}[f(s,t)u(s,t)$$
$$+g(s,t)+\int_{x_0}^{s}\int_{t}^{\infty} h(\xi,\eta)u(\xi,\eta)\Delta\eta\Delta\xi]\Delta t\Delta s, \quad (1)$$

then

$$u(x,y) \leq \{a(x,y) + b(x,y)[B_1(x,y)$$
$$+\int_{x_0}^{x} e_{B_2(.,y)}(x,\sigma(s))B_2(s,y)B_1(s,y)\Delta s]\}^{\frac{1}{p}},$$
$$X(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \quad (2)$$

where

$$B_1(x,y) = \int_{x_0}^{x}\int_{y}^{\infty}[f(s,t)(\frac{1}{p}K^{\frac{1-p}{p}}a(s,t)+\frac{p-1}{p}K^{\frac{1}{p}})$$

$$+g(s,t)+\int_{x_0}^{s}\int_{t}^{\infty}h(\xi,\eta)(\frac{1}{p}K^{\frac{1-p}{p}}a(\xi,\eta)$$
$$+\frac{p-1}{p}K^{\frac{1}{p}})\Delta\eta\Delta\xi]\Delta t\Delta s, \quad (3)$$

$$B_2(x,y)=\int_{y}^{\infty}[f(x,t)+\int_{x_0}^{x}\int_{t}^{\infty}h(\xi,\eta)\Delta\eta\Delta\xi]$$
$$\frac{1}{p}K^{\frac{1-p}{p}}b(x,t)\Delta t. \quad (4)$$

Proof : Let

$$v(x,y)=\int_{x_0}^{x}\int_{y}^{\infty}[f(s,t)u(s,t)+g(s,t)$$
$$+\int_{x_0}^{s}\int_{t}^{\infty}h(\xi,\eta)u(\xi,\eta)\Delta\eta\Delta\xi]\Delta t\Delta s. \quad (5)$$

Then

$$u(x,y)\le(a(x,y)+b(x,y)v(x,y))^{\frac{1}{p}},\ (x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0 \quad (6)$$

Fix $Y\in\widetilde{\mathbb{T}}_0$, and let $y\in[Y,\infty)\bigcap\mathbb{T},\ x\in\mathbb{T}_0$. Then

$$v(x,Y)=\int_{x_0}^{x}\int_{Y}^{\infty}[f(s,t)u(s,t)+g(s,t)$$
$$+\int_{x_0}^{s}\int_{t}^{\infty}h(\xi,\eta)u(\xi,\eta)\Delta\eta\Delta\xi]\Delta t\Delta s$$
$$\le\int_{x_0}^{x}\int_{Y}^{\infty}[f(s,t)(a(s,t)+b(s,t)v(s,t))^{\frac{1}{p}}+g(s,t)$$
$$+\int_{x_0}^{s}\int_{t}^{\infty}h(\xi,\eta)(a(\xi,\eta)+b(\xi,\eta)v(\xi,\eta))^{\frac{1}{p}}\Delta\eta\Delta\xi]\Delta t\Delta s. \quad (7)$$

From Lemma 20, for $\forall K>0$ we have

$$(a(x,y)+b(x,y)v(x,y))^{\frac{1}{p}}$$
$$\le\frac{1}{p}K^{\frac{1-p}{p}}(a(x,y)+b(x,y)v(x,y))+\frac{p-1}{p}K^{\frac{1}{p}},$$
$$(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0. \quad (8)$$

So combining (7) and (8), considering $v(x,y)$ is decreasing in y, it follows

$$v(x,Y)\le\int_{x_0}^{x}\int_{Y}^{\infty}[f(s,t)(\frac{1}{p}K^{\frac{1-p}{p}}(a(s,t)$$
$$+b(s,t)v(s,t))+\frac{p-1}{p}K^{\frac{1}{p}})+g(s,t)$$
$$+\int_{x_0}^{s}\int_{t}^{\infty}h(\xi,\eta)(\frac{1}{p}K^{\frac{1-p}{p}}(a(\xi,\eta)$$
$$+b(\xi,\eta)v(\xi,\eta))+\frac{p-1}{p}K^{\frac{1}{p}})\Delta\eta\Delta\xi]\Delta t\Delta s$$
$$=\int_{x_0}^{x}\int_{Y}^{\infty}[f(s,t)(\frac{1}{p}K^{\frac{1-p}{p}}a(s,t)+\frac{p-1}{p}K^{\frac{1}{p}})+g(s,t)$$
$$+\int_{x_0}^{s}\int_{t}^{\infty}h(\xi,\eta)(\frac{1}{p}K^{\frac{1-p}{p}}a(\xi,\eta)+\frac{p-1}{p}K^{\frac{1}{p}})\Delta\eta\Delta\xi]\Delta t\Delta s$$
$$+\int_{x_0}^{x}\int_{Y}^{\infty}[f(s,t)\frac{1}{p}K^{\frac{1-p}{p}}b(s,t)v(s,t)$$
$$+\int_{x_0}^{s}\int_{t}^{\infty}h(\xi,\eta)\frac{1}{p}K^{\frac{1-p}{p}}b(\xi,\eta)v(\xi,\eta)\Delta\eta\Delta\xi]\Delta t\Delta s$$
$$\le B_1(x,Y)+\int_{x_0}^{x}B_2(s,Y)v(s,Y)\Delta s, \quad (9)$$

where $B_1(x,y),\ B_2(x,y)$ are defined in (3) and (4) respectively.

From Lemma 18, it follows

$$v(x,Y)\le B_1(x,Y)$$
$$+\int_{x_0}^{x}e_{B_2(.,Y)}(x,\sigma(s))B_2(s,Y)B_1(s,Y)\Delta s,\ x\in\mathbb{T}_0. \quad (10)$$

Since $Y\in\widetilde{\mathbb{T}}_0$ is arbitrary, then in fact (10) holds for $\forall(x,y)\in(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0)$, that is,

$$v(x,y)\le B_1(x,y)+\int_{x_0}^{x}e_{B_2(.,y)}(x,\sigma(s))B_2(s,y)$$
$$B_1(s,y)\Delta s,(x,y)\in(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0). \quad (11)$$

Then combining (6) and (11) we obtain the desired inequality.

Theorem 22 *Under the conditions of Theorem 2.1, if for $(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies (1), then*
then

$$u(x,y)\le\{a(x,y)+b(x,y)B_1(x,y)e_{B_2(.,y)}(x,x_0)\}^{\frac{1}{p}},$$
$$(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0, \quad (12)$$

where $B_1(x,y),B_2(x,y)$ are defined the same as in Theorem 21.

Proof: Considering $B_1(x,y)$ is nondecreasing in x, from (2) we have

$$u(x,y)\le\{a(x,y)+b(x,y)[B_1(x,y)$$
$$+\int_{x_0}^{x}e_{B_2(.,y)}(x,\sigma(s))B_2(s,y)B_1(s,y)\Delta s]\}^{\frac{1}{p}}$$
$$\le\{a(x,y)+b(x,y)B_1(x,y)[1+$$

$$\int_{x_0}^{x} e_{B_2(.,y)}(x,\sigma(s))B_2(s,y)\Delta s]\}^{\frac{1}{p}},\quad (x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0. \tag{13}$$

On the other hand, according to [12, Theorem 2.39 and 2.36 (i)] we have

$$\int_{x_0}^{x} e_{B_2(x,y)}(x,\sigma(s))B_2(s,y)\Delta s = e_{B_2(x,y)}(x,x_0)-1. \tag{14}$$

Combining (13) and (14) we obtain the desired inequality.

Corollary 23 *Under the conditions of Theorem 22, if for $(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality*

$$u(x,y)\le a(x,y)+\int_{x_0}^{x}\int_{y}^{\infty}[f(s,t)u(s,t)+g(s,t)$$
$$+\int_{x_0}^{s}\int_{t}^{\infty}h(\xi,\eta)u(\xi,\eta)\Delta\eta\Delta\xi]\Delta t\Delta s, \tag{15}$$

then

$$u(x,y)\le a(x,y)+B_1(x,y)e_{B_2(.,y)}(x,x_0),$$
$$(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0, \tag{16}$$

where

$$B_1(x,y)=\int_{x_0}^{x}\int_{y}^{\infty}[f(s,t)a(s,t)+g(s,t)$$
$$+\int_{x_0}^{s}\int_{t}^{\infty}h(\xi,\eta)a(\xi,\eta)\Delta\eta\Delta\xi]\Delta t\Delta s, \tag{17}$$

$$B_2(x,y)=\int_{y}^{\infty}[f(x,t)+\int_{x_0}^{x}\int_{t}^{\infty}h(\xi,\eta)\Delta\eta\Delta\xi]\Delta t. \tag{18}$$

Theorem 24 *Suppose $\sup_{y\in\widetilde{\mathbb{T}}_0} y=\infty$, u, f, g, p are the same as in Theorem 21, a, h_1, $h_2\in C_{rd}(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0,R_+)$, and $a(x,y)$ is nondecreasing in x for every fixed y. If for $(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality*

$$u^p(x,y)\le a(x,y)+\int_{x_0}^{x}[f(s,y)u^p(s,y)\Delta s$$
$$+\int_{x_0}^{x}\int_{y}^{\infty}[g(s,t)u(s,t)+h_1(s,t)$$
$$+\int_{x_0}^{s}\int_{t}^{\infty}h_2(\xi,\eta)u(\xi,\eta)\Delta\eta\Delta\xi]\Delta t\Delta s, \tag{19}$$

then

$$u(x,y)\le\{[a(x,y)+\widetilde{B}_1(x,y)e_{\widetilde{B}_2(.,y)}(x,x_0)]$$
$$e_{f(.,y)}(x,x_0)\}^{\frac{1}{p}},\quad (x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0, \tag{20}$$

where

$$\widetilde{f}(x,y)=g(x,y)(e_{f(.,y)}(x,x_0))^{\frac{1}{p}}\frac{1}{p}K^{\frac{1-p}{p}}$$
$$+\int_{x_0}^{x}\int_{y}^{\infty}h_2(\xi,\eta)(e_{f(.,\eta)}(\xi,x_0))^{\frac{1}{p}}\frac{1}{p}K^{\frac{1-p}{p}}\Delta\eta\Delta\xi, \tag{21}$$

$$\widetilde{g}(x,y)=g(x,y)(e_{f(.,y)}(x,x_0))^{\frac{1}{p}}[\frac{1}{p}K^{\frac{1-p}{p}}a(x,y)+$$
$$\frac{p-1}{p}K^{\frac{1}{p}}]+h_1(x,y)$$
$$+\int_{x_0}^{x}\int_{y}^{\infty}h_2(\xi,\eta)(e_{f(.,\eta)}(\xi,x_0))^{\frac{1}{p}}[\frac{1}{p}K^{\frac{1-p}{p}}a(\xi,\eta)$$
$$+\frac{p-1}{p}K^{\frac{1}{p}}\Delta\eta\Delta\xi, \tag{22}$$

$$\widetilde{B}_1(x,y)=\int_{x_0}^{x}\int_{y}^{\infty}\widetilde{g}(s,t)\Delta t\Delta s, \tag{23}$$

$$\widetilde{B}_2(x,y)=\int_{y}^{\infty}\widetilde{f}(x,t)\Delta t. \tag{24}$$

Proof: Let

$$v(x,y)=a(x,y)+\int_{x_0}^{x}\int_{y}^{\infty}[g(s,t)u(s,t)+h_1(s,t)$$
$$+\int_{x_0}^{s}\int_{t}^{\infty}h_2(\xi,\eta)u(\xi,\eta)\Delta\eta\Delta\xi]\Delta t\Delta s. \tag{25}$$

Then

$$u^p(x,y)\le v(x,y)+\int_{x_0}^{x}[f(s,y)u^p(s,y)\Delta s,$$
$$(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0. \tag{26}$$

Fix $Y\in\widetilde{\mathbb{T}}_0$, and let $y\in[Y,\infty)\bigcap\widetilde{\mathbb{T}}$. Then

$$u^p(x,Y)\le v(x,Y)+\int_{x_0}^{x}[f(s,Y)u^p(s,Y)\Delta s,\ x\in\mathbb{T}_0. \tag{27}$$

Considering $v(x,Y)$ is nondecreasing in x, by Lemma 19 we obtain

$$u^p(x,Y)\le v(x,Y)e_{f(.,Y)}(x,x_0),\ x\in\mathbb{T}_0. \tag{28}$$

Since Y is selected from $\widetilde{\mathbb{T}}_0$ arbitrarily, then in fact (28) holds for $\forall y\in\widetilde{\mathbb{T}}_0$, that is

$$u^p(x,y)\le v(x,y)e_{f(.,y)}(x,x_0),\ (x,y)\in(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0). \tag{29}$$

Let

$$c(x,y)=\int_{x_0}^{x}\int_{y}^{\infty}[g(s,t)u(s,t)+h_1(s,t)$$

$$+\int_{x_0}^{s}\int_{t}^{\infty}h_2(\xi,\eta)u(\xi,\eta)\Delta\eta\Delta\xi]\Delta t\Delta s. \quad (30)$$

Then

$$v(x,y)=a(x,y)+c(x,y), \quad (31)$$

and

$$u^p(x,y)\le(a(x,y)+c(x,y))e_{f(.,y)}(x,x_0),$$

$$(x,y)\in(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0). \quad (32)$$

Combining (30) and (32) we have

$$c(x,y)=\int_{x_0}^{x}\int_{y}^{\infty}[g(s,t)u(s,t)+h(s,t)]\Delta t\Delta s$$

$$\le\int_{x_0}^{x}\int_{y}^{\infty}\{g(s,t)[(a(s,t)+c(s,t))e_{f(.,t)}(s,x_0)]^{\frac{1}{p}}$$

$$+h_1(s,t)+\int_{x_0}^{s}\int_{t}^{\infty}h_2(\xi,\eta)(a(\xi,\eta)$$

$$+c(\xi,\eta))e_{f(.,\eta)}(\xi,x_0)]^{\frac{1}{p}}\Delta\eta\Delta\xi\}\Delta t\Delta s. \quad (33)$$

On the other hand, From Lemma 2.3 the following inequality holds

$$(a(x,y)+c(x,y))^{\frac{1}{p}}$$

$$\le\frac{1}{p}K^{\frac{1-p}{p}}(a(x,y)+c(x,y))+\frac{p-1}{p}K^{\frac{1}{p}},$$

$$(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0. \quad (34)$$

So from (33) and (34) it follows

$$c(x,y)\le\int_{x_0}^{x}\int_{y}^{\infty}\{g(s,t)(e_{f(.,t)}(s,x_0))^{\frac{1}{p}}$$

$$[\frac{1}{p}K^{\frac{1-p}{p}}(a(s,t)+c(s,t))+\frac{p-1}{p}K^{\frac{1}{p}}]+h_1(s,t)$$

$$+\int_{x_0}^{s}\int_{t}^{\infty}h_2(\xi,\eta)(e_{f(.,\eta)}(\xi,x_0))^{\frac{1}{p}}$$

$$[\frac{1}{p}K^{\frac{1-p}{p}}(a(\xi,\eta)+c(\xi,\eta))+\frac{p-1}{p}K^{\frac{1}{p}}]\Delta\eta\Delta\xi\}\Delta t\Delta s$$

$$=\int_{x_0}^{x}\int_{y}^{\infty}[\widetilde{f}(s,t)c(s,t)+\widetilde{g}(s,t)]\Delta t\Delta s, \quad (35)$$

where $\widetilde{f}(x,y)$, $\widetilde{g}(x,y)$ are defined in (21) and (22) respectively.

According to Corollary 23 we obtain

$$c(x,y)\le\widetilde{B}_1(x,y)e_{\widetilde{B}_2(.,y)}(x,x_0), \quad (36)$$

where $\widetilde{B}_1(x,y)$, $\widetilde{B}_2(x,y)$ are defined in (23) and (24) respectively.

Combining (32) and (36) we get the desired inequality (20).

Corollary 25 *Under the conditions of Theorem 24, if for $(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality*

$$u(x,y)\le a(x,y)+\int_{x_0}^{x}[f(s,y)u(s,y)\Delta s$$

$$+\int_{x_0}^{x}\int_{y}^{\infty}[g(s,t)u(s,t)+h_1(s,t)$$

$$+\int_{x_0}^{s}\int_{t}^{\infty}h_2(\xi,\eta)u(\xi,\eta)\Delta\eta\Delta\xi]\Delta t\Delta s, \quad (37)$$

then

$$u(x,y)\le[a(x,y)+\widetilde{B}_1(x,y)e_{\widetilde{B}_2(x,y)}(x,x_0)]e_{f(x,y)}(x,x_0),$$

$$(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0. \quad (38)$$

where

$$\widetilde{f}(x,y)=g(x,y)e_{f(.,y)}(x,x_0)$$

$$+\int_{x_0}^{x}\int_{y}^{\infty}h_2(\xi,\eta)e_{f(.,\eta)}(\xi,x_0)\Delta\eta\Delta\xi,$$

$$\widetilde{g}(x,y)=g(x,y)e_{f(.,y)}(x,x_0)a(x,y)+h_1(x,y)$$

$$+\int_{x_0}^{x}\int_{y}^{\infty}h_2(\xi,\eta)e_{f(.,\eta)}(\xi,x_0)a(\xi,\eta)\Delta\eta\Delta\xi,$$

$$\widetilde{B}_1(x,y)=\int_{x_0}^{x}\int_{y}^{\infty}\widetilde{g}(s,t)\Delta t\Delta s,$$

$$\widetilde{B}_2(x,y)=\int_{y}^{\infty}\widetilde{f}(x,t)\Delta t.$$

Theorem 26 *Suppose $\sup_{y\in\widetilde{\mathbb{T}}_0}y=\infty$, u, a, f, p are the same as in Theorem 24, $L\in C(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0\times R_+,R_+)$, and $0\le L(s,t,x)-L(s,t,y)\le M(s,t,y)(x-y)$ for $x\ge y\ge0$, where $M\in C(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0\times R_+,R_+)$. If for $(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality*

$$u^p(x,y)\le a(x,y)+\int_{x_0}^{x}f(s,y)u^p(s,y)\Delta s$$

$$+\int_{x_0}^{x}\int_{y}^{\infty}L(s,t,u(s,t))\Delta t\Delta s, \quad (39)$$

then

$$u(x,y) \le \{[a(x,y) + \widehat{B}_1(x,y)e_{\widehat{B}_2(x,y)}(x,x_0)]$$

$$e_{f(x,y)}(x,x_0)\}^{\frac{1}{p}},\ (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0. \qquad (40)$$

where

$$\widehat{f}(x,y) = M(x,y,(e_{f(x,y)}(x,x_0))^{\frac{1}{p}}(\frac{1}{p}K^{\frac{1-p}{p}}a(x,y)$$

$$+\frac{p-1}{p}K^{\frac{1}{p}}))(e_{f(x,y)}(x,x_0))^{\frac{1}{p}}\frac{1}{p}K^{\frac{1-p}{p}}, \qquad (41)$$

$$\widehat{g}(x,y) = L(x,y,(e_{f(x,y)}(x,x_0))^{\frac{1}{p}}(\frac{1}{p}K^{\frac{1-p}{p}}a(x,y)$$

$$+\frac{p-1}{p}K^{\frac{1}{p}})), \qquad (42)$$

$$\widehat{B}_1(x,y) = \int_{x_0}^{x}\int_{y}^{\infty}\widehat{g}(s,t)\Delta t\Delta s, \qquad (43)$$

$$\widehat{B}_2(x,y) = \int_{y}^{\infty}\widehat{f}(x,t)\Delta t. \qquad (44)$$

Proof: Let

$$v(x,y) = a(x,y) + \int_{x_0}^{x}\int_{y}^{\infty}L(s,t,u(s,t))\Delta t\Delta s. \qquad (45)$$

Then

$$u^p(x,y) \le v(x,y) + \int_{x_0}^{x}f(s,y)u^p(s,y)\Delta s,$$

$$(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0. \qquad (46)$$

Fix a $Y \in \widetilde{\mathbb{T}}_0$, and let $y \in [Y,\infty)\bigcap\widetilde{\mathbb{T}}$, then

$$u^p(x,Y) \le v(x,Y) + \int_{x_0}^{x}f(s,Y)u^p(s,Y)\Delta s,\ x \in \mathbb{T}_0. \qquad (47)$$

Considering $v(x,Y)$ is nondecreasing in x, then by Lemma 19 we obtain

$$u^p(x,Y) \le v(x,Y)e_{f(x,Y)}(x,x_0),\ x \in \mathbb{T}_0. \qquad (48)$$

Since Y is selected from $\widetilde{\mathbb{T}}_0$ arbitrarily, then in fact (48) holds for $\forall y \in \widetilde{\mathbb{T}}_0$, that is,

$$u^p(x,y) \le v(x,y)e_{f(x,y)}(x,x_0),\ (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0. \qquad (49)$$

Let

$$c(x,y) = \int_{x_0}^{x}\int_{y}^{\infty}L(s,t,u(s,t))\Delta t\Delta s. \qquad (50)$$

Then we have

$$v(x,y) = a(x,y) + c(x,y), \qquad (51)$$

and

$$u^p(x,y) \le (a(x,y) + c(x,y))e_{f(x,y)}(x,x_0),$$

$$(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0. \qquad (52)$$

Combining (34), (50) and (52) we have

$$c(x,y) \le \int_{x_0}^{x}\int_{y}^{\infty}L(s,t,((a(s,t)$$

$$+c(s,t))e_{f(s,t)}(s,x_0))^{\frac{1}{p}})\Delta t\Delta s$$

$$\le \int_{x_0}^{x}\int_{y}^{\infty}L(s,t,(e_{f(s,t)}(s,x_0))^{\frac{1}{p}}(\frac{1}{p}K^{\frac{1-p}{p}}(a(s,t)$$

$$+c(s,t)) + \frac{p-1}{p}K^{\frac{1}{p}}))\Delta t\Delta s$$

$$= \int_{x_0}^{x}\int_{y}^{\infty}[L(s,t,(e_{f(s,t)}(s,x_0))^{\frac{1}{p}}(\frac{1}{p}K^{\frac{1-p}{p}}(a(s,t)$$

$$+c(s,t)) + \frac{p-1}{p}K^{\frac{1}{p}}))$$

$$-L(s,t,(e_{f(s,t)}(s,x_0))^{\frac{1}{p}}(\frac{1}{p}K^{\frac{1-p}{p}}a(s,t)+\frac{p-1}{p}K^{\frac{1}{p}}))$$

$$+L(s,t,(e_{f(s,t)}(s,x_0))^{\frac{1}{p}}(\frac{1}{p}K^{\frac{1-p}{p}}a(s,t)+\frac{p-1}{p}K^{\frac{1}{p}}))]\Delta t\Delta s$$

$$\le \int_{x_0}^{x}\int_{y}^{\infty}[M(s,t,(e_{f(s,t)}(s,x_0))^{\frac{1}{p}}(\frac{1}{p}K^{\frac{1-p}{p}}a(s,t)$$

$$+\frac{p-1}{p}K^{\frac{1}{p}}))(e_{f(s,t)}(s,x_0))^{\frac{1}{p}}\frac{1}{p}K^{\frac{1-p}{p}}c(s,t)$$

$$+L(s,t,(e_{f(s,t)}(s,x_0))^{\frac{1}{p}}(\frac{1}{p}K^{\frac{1-p}{p}}a(s,t)+\frac{p-1}{p}K^{\frac{1}{p}}))]\Delta t\Delta s$$

$$= \int_{x_0}^{x}\int_{y}^{\infty}[\widehat{f}(s,t)c(s,t) + \widehat{g}(s,t)]\Delta t\Delta s, \qquad (53)$$

where $\widehat{f}(x,y)$, $\widehat{g}(x,y)$ are defined in (41) and (42) respectively.

By use of Corollary 2.3 we obtain

$$c(x,y) \le \widehat{B}_1(x,y)e_{\widehat{B}_2(x,y)}(x,x_0),\ (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \qquad (54)$$

where $\widehat{B}_1(x,y)$, $\widehat{B}_2(x,y)$ are defined in (43) and (44) respectively.

Combining (52) and (54) we obtain the desired inequality (40).

3 Some Applications

In this section, we will present some applications for the established results above, and new explicit bounds of solutions of certain dynamic equations will be derived.

Example 27 *Consider the following dynamic equation*

$$u^p(x,y) = a(x,y) + \int_{x_0}^{x}\int_{y}^{\infty} F(s,t,u(s,t))\Delta t\Delta s,$$

$$(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \qquad (55)$$

where $u,\ a \in C_{rd}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, R)$, *and* p *is a constant with* $p \geq 1$.

Theorem 28 *If* $u(x,y)$ *is a solution of (55), and* $|F(s,t,u)| \leq f(s,t)|u| + g(s,t)$, *where* $f,\ g \in C_{rd}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \mathbb{R}_+)$, *then we have*

$$|u(x,y)| \leq \{|a(x,y)| + [B_1(x,y) + \int_{x_0}^{x} e_{B_2(x,y)}(x,\sigma(s))$$

$$B_2(s,y)B_1(s,y)\Delta s]\}^{\frac{1}{p}}, \quad (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \quad (56)$$

where

$$B_1(x,y) = \int_{x_0}^{x}\int_{y}^{\infty} [f(s,t)(\frac{1}{p}K^{\frac{1-p}{p}}|a(s,t)|$$

$$+\frac{p-1}{p}K^{\frac{1}{p}}) + g(s,t)]\Delta t\Delta s,$$

$$B_2(x,y) = \int_{y}^{\infty} f(x,t)\frac{1}{p}K^{\frac{1-p}{p}}\Delta t.$$

Proof: From (55) we have

$$|u(x,y)|^p \leq |a(x,y)| + \int_{x_0}^{x}\int_{y}^{\infty} |F(s,t,u(s,t))|\Delta t\Delta s$$

$$\leq |a(x,y)| + \int_{x_0}^{x}\int_{y}^{\infty} [f(s,t)|u(s,t)| + g(s,t)]\Delta t\Delta s.$$

Then a suitable application of Theorem 2.1 yields the desired inequality (56).

Example 29 *Consider the following dynamic equation*

$$u^2(x,y) = \varphi(y) + \int_{x_0}^{x} F_1(s,y)\Delta s$$

$$+\int_{x_0}^{x}\int_{y}^{\infty} F_2(s,t,u(s,t))\Delta t\Delta s, \qquad (57)$$

where $u \in C_{rd}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, R)$, $(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0$.

Theorem 30 *If* $u(x,y)$ *is a solution of (57), and* $|F_1(x,y)| \leq f(x,y)u^2(x,y)$, $|F_2(x,y,u)| \leq g(x,y)|u| + h(x,y)$, *where* $f,\ g,\ h \in C_{rd}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, R_+)$, *then the following estimate holds*

$$|u(x,y)| \leq$$

$$\sqrt{[|\varphi(y)| + (\widetilde{B}_1(x,y)e_{\widetilde{B}_2(x,y)}(x,x_0))]e_{f(x,y)}(x,x_0)},$$

$$(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \qquad (58)$$

where

$$\widetilde{f}(x,y) = g(x,y)\frac{\sqrt{e_{f(x,y)}(x,x_0)}}{2\sqrt{K}},$$

$$\widetilde{g}(x,y) = g(x,y)(e_{f(x,y)}(x,x_0))^{\frac{1}{p}}[\frac{|\varphi(y)|}{2\sqrt{K}} + \frac{\sqrt{K}}{2}] + h(x,y),$$

$$\widetilde{B}_1(x,y) = \int_{x_0}^{x}\int_{y}^{\infty} \widetilde{g}(s,t)\Delta t\Delta s,$$

$$B_2(x,y) = \int_{y}^{\infty} \widetilde{f}(x,t)\Delta t.$$

Proof: from (57) we have

$$|u^2(x,y)| \leq |\varphi(y)| + \int_{x_0}^{x} |F_1(s,y)|\Delta s$$

$$+\int_{x_0}^{x}\int_{y}^{\infty} |F_2(s,t,u(s,t))|\Delta t\Delta s$$

$$\leq |\varphi(y)| + \int_{x_0}^{x} f(s,y)|u(s,y)|^2\Delta s$$

$$+\int_{x_0}^{x}\int_{y}^{\infty} [g(s,t)|u(s,t)| + h(s,t)]\Delta t\Delta s,$$

and then a suitable application of Theorem 24 yields (58).

Remark 31 *From these applications, one can see some explicit bounds for unknown functions to certain dynamic equations on time scales are established by the present theorems.*

Remark 32 *If we take* $\mathbb{T} = \mathbb{R},\ x_0 = 0,\ y_0 = 0$, *then our Theorem 22,24 and 26 reduce to [1, Theorem 1, 3, 5(α)] respectively.*

Remark 33 *If we take* $\mathbb{T} = \mathbb{Z},\ x_0 = 0,\ y_0 = 0$, *then our Theorem 22,24 and 26 reduce to [2, Theorem 1, 3, 5] respectively with slight difference.*

4 Conclusions

In this paper, we established some new Gronwall-Bellman Type dynamic inequalities in two independent variables on time scales containing integration on infinite intervals. As one can see from the presented examples, the established results provide a handy tool in the study of boundedness of solutions of certain dynamic equations on time scales. Furthermore, the established results generalize some known inequalities for continuous functions and their corresponding discrete analysis in the literature.

References:

[1] F. W. Meng, W. N. Li, On some new integral inequalities and their applications, *Appl. Math. Comput.* 148 (2004), pp. 381-392

[2] F. W. Meng, W. N. Li, On some newnonlinear discrete inequalities and their applications, *J. Comput. Appl. Math.* 158 (2003). pp.407-417.

[3] O. Lipovan, Integral inequalities for retarded Volterra equations, *J. Math. Anal. Appl.* 322 (2006), pp.349-358.

[4] B. G. Pachpatte, On Some New Inequalities Related to a Certain Inequality Arising in the Theory of Differential Equations, *J. Math. Anal. Appl.* 251 (2000), pp.736-751.

[5] Rui A.C. Ferreira, Delfim F.M. Torres, Generalized retarded integral inequalities, *Applied Mathematics Letters* 22 (2009), pp.876-881.

[6] H. X. Zhang, F. W. Meng, Integral inequalities in two independent variables for retarded Volterra equations, *Appl. Math. Comput.* 199 (2008), pp.90-98.

[7] W. S. Cheung, J. L. Ren, Discrete nonlinear inequalities and applications to boundary value problems, *J. Math. Anal. Appl.* 319 (2006), pp.708-724.

[8] Y. H. Kim, Gronwall, Bellman and Pachpatte type integral inequalities with applications, *Nonlinear Analysis* 71 (2009), e2641-e2656.

[9] W. N. Li, M. A. Han, F.W. Meng, Some new delay integral inequalities and their applications, *J. Comput. Appl. Math.* 180 (2005), pp.191-200.

[10] Q. H. Ma, E.H.Yang, Some new Gronwall-Bellman-Bihari type integral inequalities with delay, *Period. Math. Hungar.* 44 (2002), No.2, pp.225-238.

[11] Z. L. Yuan, X.W. Yuan, F.W. Meng, Some new delay integral inequalities and their applications, *Applied Mathematics and Computation* 208 (2009), pp.231-237.

[12] B. G. Pachpatte, Explicit bounds on certain integral inequalities, *J. Math. Anal. Appl.* 267 (2002), pp.48-61.

[13] S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, *Results Math.* 18 (1990), pp.18-56.

[14] W. N. Li, Some Pachpatte type inequalities on time scales, *Comput. Math. Appl.* 57 (2009), pp.275-282.

[15] X. L. Cheng, Improvement of some Ostrowski-Grüss type inequalities, *Comput. Math. Appl.* 42 (2001), pp.109-114.

[16] W. N. Li, Some new dynamic inequalities on time scales, *J. Math. Anal. Appl.* 319 (2006), pp.802-814.

[17] Q. A. Ngô, Some mean value theorems for integrals on time scales, *Appl. Math. Comput.* 213 (2009), pp.322-328.

[18] W. j. Liu, Q. A. Ngô, Some Iyengar-type inequalities on time scales for functions whose second derivatives are bounded, *Appl. Math. Comput.* 216 (2010), pp.3244-3251.

[19] Y. Xing, M. Han, and G. Zheng, Initial value problem for first-order integro-differential equation of Volterra type on time scales, *Nonlinear Analysis: Theory, Methods Applications*, 60(2005), No.3, pp.429-442.

[20] Q. H. Ma, J. Pečarić, The bounds on the solutions of certain two-dimensional delay dynamic systems on time scales, (Article In Press), *Comput. Math. Appl.*

[21] R. Agarwal, M. Bohner, A. Peterson, Inequalities on time scales: a survey, *Math. Inequal. Appl.* 4 (2001), No. 4, pp.535-557.

[22] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[23] F. C. Jiang, F. W. Meng, Explicit bounds on some new nonlinear integral inequality with delay, *J. Comput. Appl. Math.* 205 (2007), pp.479-486.