The Asymptotic Behavior of a Doubly Nonlinear Parabolic Equation with a Absorption Term Related to the Gradient

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Abstract: By comparing the solution $u(x,t)$ of the doubly degenerate parabolic equation

$$u_t = \text{div}(\mid Du^m \mid^{p-2} Du^m) - u^{q_1} \mid \nabla u^m \mid^{p_1}$$

with the Barenblatt type solution of the equation

$$u_t = \text{div}(\mid Du^m \mid^{p-2} Du^m),$$

the large time asymptotic behavior of $u(x,t)$ are got. Here the exponents m, p, p_1 and q_1 satisfy $p > p_1$, $p > 1, m > 1, q_1 + p_1 m > m(p-1) > 1$.

Key–Words: Doubly degenerate parabolic equation, weak solution, very singular solution, asymptotic behavior

1 Introduction

We will consider the large time asymptotic behavior of weak solutions of doubly degenerate parabolic equations of the following type

$$u_t = \text{div}(\mid Du^m \mid^{p-2} Du^m) - u^{q_1} \mid Du^m \mid^{p_1}, \quad (1)$$

$$u(x,0) = u_0(x). \quad (2)$$

Here, D is the gradient operator, the variables $(x,t) \in S = R^N \times (0,\infty)$, the exponent constants $p > p_1$, $q_1 + p_1 m > m(p-1) > 1, p > 1, m > 1, N \geq 1$, and $u_0(x) \in L^1(R^N)$. Equation (1.1) has been suggested as a mathematical model for a variety of problems in mechanics, physics and biology, one can see [1], [2] and [3] etc.

A classical example of (1) is the heat equation,

$$u_t = \triangle u, \quad (3)$$

its theory is well known, among its features we find C^∞ smoothness of solutions, infinite speed of propagation of disturbances and the strong maximum principle. These properties are able to be generalized to a number of related evolution equations, notably those which are linear and uniformly parabolic. Other well-known examples of (1) include the porous medium equation

$$u_t = \triangle u^m, \ \ m > 1, \quad (4)$$

and evolutionary p-Laplacian equation

$$u_t = \text{div}(|\nabla u|^{p-2}\nabla u), \ \ p > 2. \quad (5)$$

Clearly, compared with the heat equation, a marked departure occurs. These equations are degenerate parabolic and there are generally no classical solutions. Moreover, instead of the infinite speed of propagation of disturbances, the weak solutions of the Cauchy problem to (4) or (5) have the property of finite propagation. One can see [3], [10] et al.

The existence of nonnegative solution of some special cases of (1), defined in some weak sense, is well established (see [4] and [5] et al.). Here we quote the following definition.

Definition 1 A nonnegative function $u(x,t)$ is called a weak solution of (1)-(2) if u satisfies

(i)

$$u \in C(0,T; L^1(R^N)) \cap L^\infty(R^N \times (\tau,T)),$$

$$u^m \in L^p_{loc}(0,T; W^{1,p}(R^N)), \quad (6)$$

$$u_t \in L^1(R^N \times (\tau,T)), \quad \forall \tau > 0; \quad (7)$$

(ii)

$$\int_S [u(x,t)\varphi_t(x,t) - \mid Du^m \mid^{p-2} Du^m \cdot D\varphi$$

$$-u^{q_1} \mid Du^m \mid^{p_1} \varphi]dxdt = 0, \quad \forall\varphi \in C_0^1(S); \quad (8)$$

(iii)

$$\lim_{t\to 0} \mid u(x,t) - u_0(x) \mid dx = 0. \quad (9)$$

In this paper, we always assume that the solutions of the corresponding equations are nonnegative. Similar to the proof of [4], we can prove the existence of the solution of (1)-(2) in the sense of Definition 1, we will published this result in another paper. However, we have found that the proof of the uniqueness of the solutions in [4] is not able to be generalized to our equation (1). It seems that the uniqueness of the solutions of (1)-(2) is hard to be proved.

In this paper, we are interested in the behavior of solutions as $t \to \infty$. Generally, there are three methods to study this problem.

I) Elliptic equation method. This method is base on the existence of the weak solutions of corresponding degenerate elliptic equation

$$\text{div}(v^{m-1} \mid Dv \mid^{p-2} Dv) - u^{q_1} \mid Dv^m \mid^{p_1} = 0. \quad (10)$$

Then, one is able to consider whether the weak solution $u(x,t)$ of (1) is asymptotic to the weak solution of (10) $v(x)$ or not. By this method, several papers (see e.g. [6], [7]) were devoted to the study of the asymptotic behavior of the solutions of the porous medium equations and the evolutionary $p-$ Laplacian equations. Also by elliptic method, J. Manfredi and V. Vespri had studied the large time behavior of the solutions of the initial boundary problem without absorption term $u^{q_1} \mid Du^m \mid^{p_1}$ in [8].

II) Fundamental solution method. This method bases on comparing the large time behavior of the weak solution of (1)-(2) to the Barenblatt-type solution of (1).

It is not difficult to verify that

$$E_c = t^{\frac{-1}{\mu}}\{[b - \frac{m(p-1)-1}{mp}(N\mu)^{\frac{-1}{(p-1)}} \times (\mid x \mid t^{\frac{-l}{N\mu}})^{\frac{p}{p-1}}]_+\}^{\frac{p-1}{m(p-1)-1}}$$

is the Barenblatt-type solution of the Cauchy problem

$$u_t = \text{div}(\mid Du^m \mid^{p-2} Du^m), \text{ in } S = R^N \times (0,\infty), \quad (11)$$

$$u(x,0) = c\delta(x), \text{ on } R^N, \quad (12)$$

where

$$\mu = m(p-1) - 1 + \frac{p}{N}), c = \int_{R^N} u_0(x)dx,$$

b is a constant such that

$$b = \int_{R^N} E_c(x,t)dx,$$

and δ denotes the usual Dirac mass centered at the origin.

If there is not the absorption term $u^{q_1} \mid Du^m \mid^{p_1}$ in (1), using the idea of asymptotic radial symmetry, [14] and [15] established the large time behavior of solutions of evolutionary p-Laplacian equation (i.e. m=1 in (11)) and the porous medium equation (i.e. p=2 in (11)) respectively. Using Morse substitution technique, by the assumption of that the uniqueness of the Barenblatt-type solution of (11) is true, [16] had established the large time behavior of solutions of (1) when $p_1 = 0$. When $m = 1$ or $p = 2$ the uniqueness had been solved, however, the uniqueness of the solution of general case of (11)-(12) is still open to this date.

III) Singular solution method. This method bases on comparing the large time behavior of the general solutions of (1)-(2) to the very singular solutions of (1). Now, we give the related concepts.

Let

$$u(x,t) = t^{-\alpha} f(|x|t^{-\beta}). \quad (13)$$

The constants

$$\alpha = \frac{p - p_1}{p(q_1 + (p_1 - p + 1)m) - (1 + m - mp)(p - p_1)},$$

$$\beta = \frac{q_1 + (p_1 - p + 1)m}{p(q_1 + (p_1 - p + 1)m) - (1 + m - mp)(p - p_1)},$$

clearly, $\alpha > 0, \beta > 0$ because of that $p > p_1$, $q_1 + p_1 m > m(p-1) > 1$. The equation (1) is equivalent to the following equation

$$(|(f^m)'|^{p-2}(f^m)')' + \frac{n-1}{r}|(f^m)'|^{p-2}(f^m)' + \beta r f' + \alpha f - f^{q_1}|(f^m)'|^{p_1} = 0, \quad (14)$$

where $r = |x|t^{-\beta}$, with the initial condition

$$f(0) = a > 0, f'(0) = 0. \quad (15)$$

A weak solution of (14) has the form of (13), as usual, is called a self-similar solution of (1).

By a singular solution of (1) it means that a notrivial nonnegative function $U \in C(\bar{S}\backslash(0))$, if U satisfies (1) in the sense of distribution in S and

$$\lim_{t\to 0} \sup_{|x|>\varepsilon} U(x,t) = 0, \forall\varepsilon > 0. \quad (16)$$

Further, if the singular solution U satisfies the following formula

$$\lim_{t\to 0} \int_{|x|\le\varepsilon} U(x,t) = \infty, \forall\varepsilon > 0, \quad (17)$$

then U is called a very singular solution.

Clearly, (16) is equivalent to

$$\lim_{t\to 0} r^{\frac{\alpha}{\beta}} f(r) = 0. \quad (18)$$

and (17) is equivalent to

$$\lim_{t\to 0} r^{n\beta-\alpha} \int_{r\leq \varepsilon t^{-\beta}} f(r)dr = 0.\forall\varepsilon > 0. \quad (19)$$

If $n\beta < \alpha$, the solution f of equation (14) satisfies (18), then $f \in L^1(0,\infty; r^{n-1}dr)$, f satisfies(19). Thus the self-similar solution $u(x,t)$ defined as (13) satisfies (16) and (17), and so $u(x,t)$ is a very singular solution of equation (1). Recently, the author have got the existence of the self-similar solutions and the singular solutions of (1) in [17].

The main results of this paper are the following

Theorem 2 Let $m(p-1) > 1$. If E_c is a unique solution of (11)-(12), then the solution u of (1)-(2) satisfies

$$t^{\frac{l}{\mu}} \mid u(x,t) - E_c(x,t) \mid \to 0, \text{ as } t \to \infty, \quad (20)$$

uniformly on the sets $\{x \in R^N :\mid x \mid < at^{\frac{-l}{\mu N}}, a > 0\}$, where

$$c = \int_{R^N} u_0(x)dx - \int_0^\infty \int_{R^N} u^{q_1} \mid Du^m \mid^{p_1} dxdt.$$

Theorem 3 Suppose $m(p-1) > 1, q_1 + mp_1 > m(p-1) - 1$ and

$$\mid x \mid^\alpha u_0(x) \leq B, \quad \lim_{|x|\to\infty} \mid x \mid^\alpha u_0(x) = C,$$

where $\alpha, B,$ and C are constants with $\alpha \in (0, \frac{p-p_1}{q_1+mp_1})$. If the solution $u(x,t)$ of (1)-(2) satisfies

$$\mid Du^m \mid \geq 1, (x,t) \in S, \quad (21)$$

then

$$t^{\frac{1}{q_1-1}} u(x,t) \to C^*, \text{ as } t \to \infty, \quad (22)$$

uniformly on the sets

$$\{x \in R^N : \mid x \mid \leq at^{\frac{1}{\beta}}, a > 0\},$$

where

$$C^* = (\frac{1}{q_1 - 1})^{\frac{1}{q_1-1}}$$

and

$$\beta = \frac{p(q_1 + mp_1 - 1) - p_1(m(p-1) - 1)}{q_1 + mp_1 - m(p-1)}.$$

Theorem 4 Suppose $1 < m(p-1) < q_1 + mp_1 < m(p-1) + \frac{p}{N}$ and $\alpha > \frac{p-p_1}{q_1+mp_1-m(p-1)}$,

$$\mid x \mid^\alpha u_0(x) \leq B, \int_{R^N} u_0(x)dx > 0.$$

Assume that (1) has a unique very singular solution $U(x,t)$. Then the solution of (1)-(2) satisfies

$$t^{\frac{1}{q_1-1}} \mid u(x,t) - U(x,t) \mid \to 0 \text{ as } t \to \infty, \quad (23)$$

uniformly on the sets

$$\{x \in R^N : \mid x \mid \leq at^{\frac{1}{\beta}}\}.$$

Remark 5 For $m = 1$, the uniqueness of solutions of (11)-(12) is known (see [9]). For $m = 1, p = 2$, the uniqueness of the very singular solution of (1) is know too (see [12]).

Remark 6 For all $(x,t) \in S$, the condition (21) supposes that

$$\mid Du^m \mid \geq 1,$$

this condition seems so strong that the conclusion (22) is not so interesting. However, according to the proof of our paper, we can deduce that for any given $t > 0$, $x \in R^N$, whether (22) is true, or we have

$$\mid Du^m(x,t) \mid \leq 1. \quad (24)$$

As we have said before, that the uniqueness of the solutions of (1)-(2) is still an open problem, according to our original studying, a essential difficulty comes from that it is still difficult to prove that (24) is true for all $(x,t) \in S$. So, not only that Theorem 3 includes some information in the asymptotic behavior of the solutions, but also it includes some information in the uniqueness of the solutions of the problem.

2 Proof of Theorem 2

Let u be a solution of (1). We define the family of functions

$$u_k = k^N u(kx, k^{N\mu}t), k > 0.$$

It is easy to see that they are the solutions of the problems

$$u_t = \text{div}(\mid Du^m \mid^{p-2} Du^m) - k^\upsilon u^{q_1} \mid Du \mid^{p_1}, \quad (25)$$

$$u(x,0) = u_{0k}(x), \text{ on } R^N, \quad (26)$$

where $\mu = m(p-1) + \frac{p}{N} - 1$ as before and

$$\upsilon = p - p_1 + N(m(p-1) - q_1 - mp_1),$$

$$u_{0k}(x) = k^N u_0(x).$$

Lemma 7 For any $s \in (0, m(p-1)), u_k$ satisfies

$$\int_0^T \int_{B_R} \frac{u_k^{s-m}}{(1+u_k^s)^2} \mid Du_k \mid^2 dxdt \leq c(s, R, \mid u_0 \mid_{L^1}), \quad (27)$$

$$\int_0^T \int_{B_R} u^{m(p-1)+\frac{p}{N}-s} dxdt \le c(s, R, | u_0 |_{L^1}), \tag{28}$$

where $B_R(x_0) = \{x \in R^N : | x - x_0 | < R\}$ as usual, and if $x_0 = 0$, we denote it as B_R simply.

Proof: From Definition 1, we are able to deduce that (see [10]): for $\forall \varphi \in C^1(\bar{S})$, $\varphi = 0$ when $| x |$ is large enough,

$$\int_{R^N} u_k(x,t)\varphi dx - \int_0^T \int_{R^N} (u_k\varphi_t - | Du_k^m |^{p-2} Du_k^m \cdot D\varphi) dxdt + k^{\upsilon} \int_0^T \int_{R^N} u_k^{q_1} | Du_k^m |^{p_1} \varphi dxdt = \int_{R^N} u_{0k}(x)\varphi(x,0)dx,$$

so,

$$\int_{R^N} u_k(x,t)\varphi dx - \int_0^T \int_{R^N} (u_k\varphi_t - | Du_k^m |^{p-2} Du_k^m \cdot D\varphi) dxdt \le \int_{R^N} u_{0k}(x)\varphi(x,0)dx. \tag{29}$$

Let

$$\psi_R \in C_0^{\infty}(B_{2R}), 0 \le \psi_R \le 1, \psi_R = 1, \text{ on } B_R, \quad | D\psi_R | \le cR^{-1}. \tag{30}$$

By an approximate procedure, we can choose $\varphi = \frac{u_k^s}{1+u_k^s}\psi_R^p$ in (29), then

$$\int_{R^N} \int_0^{u_k(x,t)} \frac{z^s}{1+z^s} dz \psi_R^p(x)dx + s\int_h^t \int_{R^N} \frac{u_k^{s-m}}{(1+u_k^s)^2} | Du_k^m |^p \psi_R^p(x) dxd\tau \le -p\int_h^t \int_{R^N} \frac{u_k^s}{1+u_k^s} | Du_k^m |^{p-2} \psi_R^{p-1}(x) Du_k \cdot D\psi_R dxd\tau + \int_{R^N} \int_0^{u_k(x,h)} \frac{z^s}{1+z^s} dz \psi_R^p(x)dx, \tag{31}$$

where $0 < h < t$. Noticing that

$$| \int_h^t \int_{R^N} \frac{u_k^s}{1+u_k^s} | Du_k^m |^{p-2} \psi_R^{p-1}(x) Du_k^m \cdot D\psi_R dxd\tau |$$

$$\le \int_h^t \int_{R^N} [\varepsilon(\frac{u_k^{(s+m-2)\frac{p-1}{p}}}{(1+u_k^s)^{2\frac{p-1}{p}}} | Du_k |^{p-1} \psi_R^{p-1})^{\frac{p}{p-1}} + c(\varepsilon)(\frac{u_k^{s+m-1-(s+m-2)\frac{p-1}{p}}}{(1+u_k^s)^{1-2\frac{p-1}{p}}} | D\psi_R |)^p] dxdt$$

$$= \int_h^t \int_{R^N} [\varepsilon(\frac{u_k^{s+m-2}}{(1+u_k^s)^2} | Du_k |^p \psi_R^p + c(\varepsilon)\frac{u_k^{p+m-2}}{(1+u_k^s)^{2-p}} | D\psi_R |^p] dxdt, \tag{32}$$

$$\int_{R^N} \int_0^{u_k(x,h)} \frac{z^s}{1+z^s} dz \psi_R^p(x)dx \le \int_{R^N} u_k(x, k^{N\mu}h)dx, \tag{33}$$

then by (31)-(33), we obtain

$$\sup_{0<t<T} \int_{R^N} \int_0^{u_k(x,t)} \frac{z^s}{1+z^s} dzdx + \int_h^t \int_{R^N} \frac{u_k^{s+m-2}}{(1+u_k^s)^2} | Du_k |^p \psi_R^p dxd\tau \le c\int_{R^N} u_k(x, k^{N\mu}h)dx + c\int_h^t \int_{R^N} \frac{u_k^{p+s+m-2}}{(1+u_k^s)^{2-p}} | D\psi_R |^p dxd\tau. \tag{34}$$

Since $u_k \in L^{\infty}(R^N \times (h,T)) \cap L^1(S_T)$, $m(p-1)-1 > 0$, we have

$$\lim_{R\to\infty} \int_h^t \int_{R^N} \frac{u_k^{p+s+m-2}}{(1+u_k^s)^{2-p}} | D\psi_R |^p dxd\tau = 0. \tag{35}$$

Let $R \to \infty, h \to 0$ in (34).

$$\sup_{0<t<T} \int_{R^N} \int_0^{u_k(x,t)} \frac{z^s}{1+z^s} dzdx + \int\int_{S_t} \frac{u_k^{s-m}}{(1+u_k^s)^2} | Du_k^m |^p dxd\tau \le c\int_{R^N} u_{0k}(x)dx. \tag{36}$$

Thus

$$\sup_{0<t<T} \int_{B_{2R}} u_k(x,t)dx + \int_0^T \int_{B_{2R}} \frac{u_k^{s-m}}{(1+u_k^s)^2} | Du_k^m |^p dxd\tau \le c(R). \tag{37}$$

Let

$$u_1 = \max\{u_k(x,t), 1\}, \quad w = u_1^{\frac{m(p-1)-s}{p}}.$$

By Sobolev's imbedding inequality (see [11]), for $\xi \in C_0^1(B_{2R})$, $\xi \ge 0$, we have

$$(\int_{R^N} \xi^p w^r dx)^{\frac{1}{r}} \le c(\int_{R^N} | D(\xi w) |^p)^{\frac{s}{p}}$$

$$\times(\int_{B_{2R}} w^{\frac{p}{m(p-1)-s}}dx)^{\frac{(1-\theta)(m(p-1)-s}{p}},$$

where

$$\theta=(\frac{m(p-1)-s}{p}-\frac{1}{r})(\frac{1}{N}-\frac{1}{p}+\frac{m(p-1)-s}{p})^{-1},$$

$$r=\frac{p(m(p-1)+\frac{p}{N}-s)}{m(p-1)-s}.$$

It follows that

$$\int\int_{S_T}\xi^p w^r dxdt\leq c\int\int_{S_T}\mid D(\xi w)\mid^p dxdt$$

$$\times\sup_{t\in(0,T)}(\int_{B_{2R}} w^{\frac{p}{m(p-1)-s}}dx)^{\frac{(r-p)(m(p-1)-s)}{p}}. \quad (38)$$

Since

$$\mid Dw\mid^p\leq c\frac{u_k^{s-m}}{(1+u_k^s)^2}\mid Du_k^m\mid^p$$

a.e. on $\{u_k\geq 1\}$, and

$$\mid Dw\mid=0,$$

a.e. on $\{u_k\leq 1\}$, we have

$$\int\int_{S_T}\mid D(\xi w)\mid^p dxdt$$

$$\leq c\int\int_{S_T}(\xi^p\mid Dw\mid^p+w^p\mid D\xi\mid^p)dxdt$$

$$\leq c[\int\int_{S_T}\mid D\xi\mid^p u_1^{m(p-1)-s}dxdt$$

$$+\int_0^T\int_{B_{2R}}\frac{u_k^{s-m}}{(1+u_k^s)^2}\mid Du_k^m\mid^p dxdt]. \quad (19)$$

Hence, by (38), (39) and (37), we get

$$\int\int_{S_T}\xi^p u_1^{m(p-1)+\frac{p}{N}-s}dxdt$$

$$\leq c(s,R,\mid u_0\mid_{L^1})(1+\int\int_{S_T}\mid D\xi\mid^p u_1^{m(p-1)-s}dxdt).$$

Let $\xi=\psi_R^b$. Where ψ_R is the function satisfies (30) and $b=\frac{N(m(p-1)+\frac{p}{N}-s)}{p}$. Then

$$\int\int_{S_T}\psi_R^{pb}u_1^{m(p-1)+\frac{p}{N}-s}dxdt\leq c(s,R,\mid u_0\mid_{L^1})$$

$$\times(1+\int\int_{S_T}\psi_R^{pb}u_1^{m(p-1)+\frac{p}{N}-s}dxdt)^{\frac{m(p-1)-s}{m(p-1)-s+\frac{p}{N}}},$$

which implies (28) is true.

Let $Q_\rho=B_\rho(x_0)\times(t_0-\rho^p,t_0)$ with $t_0>(2\rho)^p$.

Lemma 8 u_k satisfies

$$\sup_{Q_\rho}u_k\leq c(\rho,s_1)(\int\int_{Q_{2\rho}}u_{k1}^{m(p-1)-1+s_1}dxdt)^{1/s_1},$$

where $c(\rho,s_1)$ depends on ρ and s_1, and s_1 can be any number satisfying $0<s_1<1+\frac{p}{N}$.

Proof: For $\forall\varphi\in C^1(\bar{S})$, $\varphi=0$ when $\mid x\mid$ is large enough, we have

$$\int_{R^N}u_k(x,t)\varphi dx$$

$$-\int_0^T\int_{R^N}(u_k\varphi_t-\mid Du_k^m\mid^{p-2}Du_k^m\cdot D\varphi)dxdt$$

$$+k^\upsilon\int_0^T\int_{R^N}u_k^{q_1}\mid Du_k^m\mid^{p_1}\varphi dxdt$$

$$=\int_{R^N}u_{0k}(x)\varphi(x,0)dx. \quad (40)$$

Let ξ be the cut function on Q_ρ, i.e.

$$0\leq\xi\leq 1,\xi\mid_{Q_\rho}=1,\xi\mid_{R^N\setminus Q_\rho}=0.$$

We choose the testing function in (40) as $\varphi=\xi^p u_k^{2\gamma-1}$, where $\gamma>\frac{1}{2}$ is a constant, and notice that

$$k^\upsilon\int_0^T\int_{R^N}u_k^{q_1}\mid Du_k^m\mid^{p_1}\varphi dxdt\geq 0.$$

Then

$$\frac{1}{2\gamma}\int_{B_{2\rho}}\xi^p u_k^{2\gamma}(x,t)dx$$

$$+\frac{2\gamma-1}{m}\int_0^t\int_{B_{2\rho}}\xi^p u_k^{2\gamma-1-m}\mid\nabla u_k^m\mid^p dxds$$

$$\leq p\int_0^t\int_{B_{2\rho}}\xi^{p-1}\mid\nabla\xi\mid u_k^{2\gamma-1}\mid\nabla u_k^m\mid^{p-1}dxds$$

$$+\frac{p}{2\gamma}\int_0^t\int_{B_{2\rho}}\xi^{p-1}\mid\xi_t\mid u_k^{2\gamma}dxds. \quad (41)$$

Using Schwartz inequality

$$\xi^{p-1}\mid\nabla\xi\mid u_k^{2\gamma-1}\mid\nabla u_k^m\mid^{p-1}$$

$$=u^{2\gamma-1-m}\xi^{p-1}\mid\nabla u_k^m\mid^{p-1}\mid\nabla\xi\mid u_k^m$$

$$\leq u_k^{2\gamma-1-m}(\varepsilon\xi^p\mid\nabla u_k^m\mid^p+c(\varepsilon)u_k^{mp}\mid\nabla\xi\mid^p),$$

from (41), we have

$$\frac{1}{2\gamma}\int_{B_{2\rho}}\xi^p u_k^{2\gamma}(x,t)dx$$

$$+[(\frac{2\gamma-1}{m}-\varepsilon)]\int_0^t\int_{B_{2\rho}}\xi^p u_k^{2\gamma-1-m}\mid\nabla u_k^m\mid^p dxds$$

$$\leq c\int_0^t\int_{B_{2\rho}} u_k^{2\gamma-1+m(p-1)} \mid \nabla\xi \mid^p dxds$$

$$+\frac{p}{2\gamma}\int_0^t\int_{B_{2\rho}} \xi^{p-1} \mid \xi_t \mid u_k^{2\gamma} dxds. \qquad (42)$$

By the fact of that

$$\mid \nabla(\xi u_k^{\frac{2\gamma-1+m(p-1)}{p}}) \mid^p$$

$$=\mid u_k^{\frac{2\gamma-1+m(p-1)}{p}} \nabla\xi$$

$$+\frac{2\gamma-1+m(p-1)}{mp}\xi u_k^{\frac{2\gamma-1-m}{p}} \nabla u_k^m \mid^p$$

$$\leq c \mid \nabla\xi \mid^p u_k^{2\gamma-1+m(p-1)} + c\xi^p \mid \nabla u_k^m \mid^p u_k^{2\gamma-1-m},$$

from (42), we have

$$\sup_{t_0-2\rho^p<t<t_0}\int_{B_{2\rho}} \xi^p u_k^{2\gamma} dxds$$

$$+\int\int_{Q_{2\rho}} \mid \nabla(\xi u_k^{\frac{2\gamma-1+m(p-1)}{p}}) \mid^p dxds$$

$$\leq c\int_0^t\int_{B_{2\rho}} u_k^{2\gamma-1+m(p-1)} \mid \nabla\xi \mid^p dxds$$

$$+c\int_0^t\int_{B_{2\rho}} \xi^{p-1} \mid \xi_t \mid u_k^{2\gamma} dxds. \qquad (43)$$

Let

$$\beta = \max\{1, \frac{2\gamma-1+m(p-1)}{\gamma}\},$$

and

$$w = \xi^\beta u_k^{\frac{2\gamma-1+m(p-1)}{p}}.$$

By the embedding theorem, from (43), we have

$$\int\int_{Q_{2\rho}} w^h dxdt$$

$$\leq c\left\{\sup_{t_0-2\rho^p<t<t_0}\int_{B_{2\rho}} w^{\frac{2\gamma p}{2\gamma-1+m(p-1)}} dx\right\}^{\frac{2\gamma-1+m(p-1)}{2\gamma p}(1-\delta)h}$$

$$\cdot\int_{t_0-(2\rho^p)}\left(\int_{B_{2\rho}} \mid \nabla w \mid^p dx\right)^{\frac{\delta h}{p}} dx, \qquad (44)$$

where

$$\delta = (\frac{2\gamma-1+m(p-1)}{2\gamma p} - \frac{1}{h})$$

$$\cdot(\frac{1}{N} - \frac{1}{p} + \frac{2\gamma-1+m(p-1)}{2\gamma p})^{-1}.$$

In particular, we choose

$$h = p[1 + \frac{2\gamma p}{N(2\gamma-1+m(p-1))}],$$

then from (43), we have

$$\int\int_{Q_{2\rho}} \xi^{\beta h} u_k^{2\gamma-1+m(p-1)+\frac{2\gamma p}{N}} dxdt$$

$$\leq c_1\left(\sup_{t_0-2\rho^p<t<t_0}\int_{B_{2\rho}} \xi^{\frac{2\gamma p\beta}{2\gamma-1+m(p-1)}} u_k^{2\gamma} dx\right)^{\frac{p}{N}}$$

$$\cdot\int\int_{Q_{2\rho}} \mid \nabla(\xi^\beta u_k^{\frac{2\gamma-1+m(p-1)}{p}}) \mid^p dxdt$$

$$\leq c_2\{\sup_{t_0-2\rho^p<t<t_0}\int_{B_{2\rho}} \xi^{\frac{2\gamma p\beta}{2\gamma-1+m(p-1)}} u_k^{2\gamma} dx$$

$$+\int\int_{Q_{2\rho}} \mid \nabla(\xi^\beta u_k^{\frac{2\gamma-1+m(p-1)}{p}}) \mid^p dxdt\}^{1+\frac{p}{N}}. \quad (45)$$

Now, for $\tau \in [\frac{1}{2}, 1]$, we denote that

$$\rho^l = 2\rho(\tau + \frac{1-\tau}{2^l}), l = 1, 2, \cdots,$$

and choose the cut functions $\xi_l(x,t)$ of $Q_{\rho l}$, such that on $Q_{\rho(l+1)}$, $\xi_l = 1$.

Denote

$$K = 1 + \frac{p}{N}, 2\gamma = K^l.$$

and let

$$u_{1k} = \max\{1, u_k\}.$$

Then, by (44)(45), we have

$$\int\int_{Q_{\rho(l+1)}} u_k^{m(p-1)-1+K^{l+1}} dxdt$$

$$\leq \int\int_{Q_{\rho(l+1)}} u_{1k}^{m(p-1)-1+K^{l+1}} dxdt$$

$$\leq \int\int_{Q_{\rho(l+1)}} u_k^{m(p-1)-1+K^{l+1}} dxdt + \text{mes}Q_{\rho(l+1)}$$

$$\{\frac{cc_1^l}{((1-\tau)\rho)^p}\int\int_{Q_{\rho l}} u_{1k}^{m(p-1)-1+K^l} dxdt\}^K.$$

Using Morse interaction technique, we have

$$\sup_{2\tau\rho} u_{1k} \leq \{\frac{1}{((1-\tau)\rho)^{N+p}}\int\int_{Q_{2\rho}} u_{1k}^{m(p-1)-1+K} dxdt\}^{\frac{1}{K}}.$$

Then, we have

$$\sup_{2\tau\rho} u_{1k} \leq (\sup_{Q_{2\rho}} u_{1k})^{\frac{K-r}{K}}$$

$$\cdot\{\frac{1}{((1-\tau)\rho)^{N+p}}\int\int_{Q_{2\rho}} u_{1k}^{m(p-1)-1+r}dxdt\}^{\frac{1}{K}}.$$

By Schwartz inequality,

$$\sup_{2\tau\rho} u_{1k} \le \frac{1}{2}\sup_{2\rho} u_{1k}$$

$$+c(r)\{\frac{1}{((1-\tau)\rho)^{N+p}}\int\int_{Q_{2\rho}} u_{1k}^{m(p-1)-1+r}dxdt\}^{\frac{1}{r}}.$$

By the lemma 3.1 in [13], for any $\tau \in [\frac{1}{2}, 1)$, we have

$$\sup_{2\tau\rho} u_{1k} \le c(r,p)\{\int\int_{Q_{2\rho}} u_{1k}^{m(p-1)-1+r}dxdt\}^{\frac{1}{r}},$$

from this inequality, we get the conclusion.

Lemma 9 u_k satisfies

$$\int_\tau^T\int_{B_R} | Du_k^m |^p dxdt \le c(\tau, R), \quad (46)$$

$$\int_\tau^T\int_{B_R} | u_{kt} |^p dxdt \le c(\tau, R). \quad (47)$$

Proof: By Lemma 7 and 8, u_k are uniformly bounded on every compact set $K \subset S_T$. Let ψ_R be a function ion satisfying (30) and $\xi \in C_0^1(0, T+1)$ with $0 \le \xi \le 1, \xi = 1$ if $t \in (\tau, T)$. We choose $\eta = \psi_R^p \xi u_k^m$ in (29) to obtain

$$\frac{1}{m+1}\int_{R^N} u_k^{m+1}(x,T)\psi_R^p dx$$

$$+\int\int_{S_T} | Du_k^m |^p \psi_R^p \xi dxdt$$

$$\le \frac{1}{m+1}\int\int_{S_T} u_k^{m+1}\xi\psi_R^p dxdt$$

$$-p\int\int_{S_T} u_k^m | Du_k^m |^{p-2} Du_k^m \cdot D\psi_R \psi_R^{p-1}\xi dxdt. \quad (48)$$

At the same time, noticing

$$\int\int_{S_T} u_k^m | Du_k^m |^{p-1} | D\psi_R | \psi_R^{p-1}\xi dxdt$$

$$\le \varepsilon\int\int_{S_T} | Du_k^m |^p \psi_R^p \xi dxdt$$

$$+c(\varepsilon)\int\int_{S_T} u_k^{pm} | D\psi_R |^p \xi dxdt, \quad (49)$$

we know that (46) is true.

Now, we will prove (47). Let

$$v(x,t) = u_{kr}(x,t) = ru_k(x, r^{m(p-1)-1}t), \ \ r \in (0,1).$$

Then

$$v_t(x,t) = \operatorname{div}(| Dv^m |^{p-2} Dv^m)$$

$$-r^{m(p-1)-q_1-mp_1}k^\upsilon v^{q_1} | Dv^m |^{p_1}, \quad (50)$$

$$v(x,0) = ru_k(x,0), \quad (51)$$

Noticing that

$$mp_1 + q_1 > m(p-1), \ 0 < r < 1,$$

which implies that

$$r^{m(p-1)-q_1-mp_1}k^\upsilon > k^\upsilon,$$

using the argument similar to that in the proof Theorem 1 of [4], we can prove

$$u_k \ge u_{kr}.$$

It follows that

$$\frac{u_k(x, r^{m(p-1)-1}t) - u_k(x,t)}{(r^{m(p-1)-1}-1)t}$$

$$\ge \frac{r-1}{(1-r^{m(p-1)-1})t}u_k(x, r^{m(p-1)-1}t).$$

Letting $r \to 1$, we get

$$u_{kt} \ge -\frac{u_k}{(m(p-1)-1)t}. \quad (52)$$

Denote $w = t^\beta u_k(x,t), \beta = \frac{1}{m(p-1)-1}$. By (52), $w_t \ge 0$. By (25),

$$\int_\tau^T\int_{B_{2R}} t^{-\beta}w_t\psi_R dxdt$$

$$= -\int_\tau^T\int_{B_{2R}} | Du_k^m |^{p-2} Du_k^m \cdot D\psi_R dxdt$$

$$-\int_\tau^T\int_{B_{2R}} k^\epsilon u_k^{q_1} | Du^m |^{p_1} \psi_R dxdt$$

$$+\beta\int_\tau^T\int_{B_{2R}} t^{-1}u_k(x)\psi_R dxdt$$

$$\le \frac{\beta}{\tau}\int_\tau^T\int_{B_{2R}} u_k dxdt$$

$$+(\int_\tau^T\int_{B_{2R}} | Du_k^m |^p dxdt)^{\frac{p-1}{p}}$$

$$\cdot(\int_\tau^T\int_{B_{2R}} | D\psi_R |^p dxdt)^{\frac{1}{p}}. \quad (53)$$

From (37), (53) and Lemma 8, we obtain (47).

Proof of Theorem 2 By Lemmas 7-9 and [9], there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ and a function v such that on every compact set $K \subset S$

$$u_{k_j} \to u, \text{ in } C(K),$$

$$Du_k^m \rightharpoonup Du^m, \text{in } L^p_{loc}(S_T),$$

$$| u_{kt} |_{L^1_{loc}(S_T)} \leq c.$$

Similar to what was done in the proof of Theorem 2 in [4], we can prove u satisfies (1) in the sense of distribution.

We now prove $v(x, 0) = c\delta(x)$. Let $\chi \in C^1_0(B_R)$. Then we have

$$\int_{R^N} u_k(x,t)\chi dx - \int_{R^N} \varphi_k \chi dx$$

$$= -\int_0^t \int_{R^N} | Du_k^m |^{p-2} Du_k^m \cdot D\chi dxds$$

$$-k^v \int_0^t \int_{R^N} u_k^{q_1} | Du_k^m |^{p_1} \chi dxds. \quad (54)$$

To estimate $\int_0^t \int_{R^N} | Du_k^m |^{p-2} Du_k^m \cdot D\chi dxds$, without losing generality, one can assume that $u_k > 0$. By Hölder inequality and Lemma 7,

$$| \int_0^t \int_{R^N} | Du_k^m |^{p-2} Du_k^m \cdot D\chi dxds |$$

$$\leq c[\int_0^T \int_{B_{2R}} \frac{u_k^{s-m}}{(1+u_k^s)^2} | Du_k^m |^p dxdt]^{\frac{p-1}{p}}$$

$$\cdot(\int_0^T \int_{B_{2R}} (1+u_k^s)^{2(p-1)} u_k^{(p-1)(m-s)} dxd\tau)^{\frac{1}{p}}$$

$$\leq c[\int_0^t \int_{B_{2R}} (u_{k1}^{(p-1)(m-s)} + u_{k1}^{(p-1)(s+m)}) dxd\tau]^{\frac{1}{p}}$$

$$\leq c(\int_0^t \int_{B_{2R}} u_{k1}^{m(p-1)+\frac{p}{N}-s} dxdt)^{\frac{(p-1)(s+m)}{m(p-1)+\frac{p}{N}-s}\frac{1}{p}} t^d, \quad (55)$$

where $s \in (0, \frac{1}{N})$, and

$$d = \frac{1-Ns}{Nm(p-1)+p-Ns} < \frac{1}{p}.$$

Hence from (54), we get

$$| \int_{R^N} u_k(x,t)\chi dx - \int_{R^N} \varphi_k \chi dx$$

$$+k^v \int_0^t \int_{R^N} u_k^{q_1} | Du_k^m |^{p_1} \chi dxds |$$

$$=| \int_{R^N} u_k(x,t)\chi dx - \int_{R^N} \varphi_k \chi(k^{-1}x) dx$$

$$+\int_0^{N\mu t} \int_{R^N} u_k^{q_1} | Du_k^m |^{p_1} \chi(k^{-1}x) dxd\tau | \leq ct^d. \quad (56)$$

Letting $k \to \infty, t \to 0$ in turn, we obtain

$$\lim_{t\to 0} \int_{R^N} v(x,t)\chi dx$$

$$= \chi(0)(\int_{R^N} \varphi(x)dx - \int_0^\infty \int_{R^N} u_k^{q_1} | Du_k^m |^{p_1} dxdt).$$

Thus

$$v(x, 0) = c\delta(x),$$

where

$$c = \int_{R^N} \varphi(x)dx - \int_0^\infty \int_{R^N} u^q dxdt.$$

$v(x, t)$ is a solution of (6)-(7). By the assumption on uniqueness of solution, we have $v(x, t) = E_c(x, t)$ and the entire sequence $\{u_k\}$ converges to E_c as $k \to \infty$. Set $t = 1$. Then

$$u_k(x, 1) = k^N u(kx, k^{N\mu}) \to E_c(x, 1)$$

uniformly on every compact subset of R^N. Thus writing $kx = k', k^{N\mu} = t'$, and dropping the prime again, we see that

$$t^{\frac{1}{\mu}} u(x,t) \to E_c(xt^{\frac{1}{N\mu}}, 1) = t^{\frac{1}{\mu}} E_c(x,t)$$

uniformly on the sets $\{x \in R^N :| x |\leq at^{\frac{1}{N\mu}}\}, a > 0$. Thus Theorem 2 is true.

3 Proofs of Theorem 3 and 4

Let be a solution of (1)-(2) and

$$u_k(x,t) = k^\delta u(kx, k^\beta t), k > 0.$$

If

$$\delta = \frac{p-p_1}{q_1+mp_1-m(p-1)},$$

$$\beta = \frac{p(q_1+mp_1-1)-p_1(m(p-1)-1)}{q_1+mp_1-m(p-1)},$$

then

$$u_{kt} = \text{div}(| Du_k^m |^{p-2} Du_k^m) - u_k^{q_1} | Du_k^m |^{p_1}, \quad (57)$$

$$u_k(x, 0) = \varphi_k(x) = k^\delta \varphi(kx). \quad (58)$$

Lemma 10 If the nonnegative solution u_k of (57)-(58) satisfies

$$| Du_k^m |\geq 1, \quad (59)$$

then

$$u_k(x,t) \le C^* t^{-\frac{1}{q_1-1}}, \ \ C^* = (\frac{1}{q_1-1})^{\frac{1}{q_1-1}}. \quad (60)$$

Proof: We consider the regularized problem of (57), say,

$$u_{kt} = \text{div}((\mid Du_k^m \mid^2 + \varepsilon)^{\frac{p-2}{2}} Du_k^m) - u_k^{q_1} \mid Du_k^m \mid^{p_1}, \quad (61)$$

By the assumption of the uniqueness of the solution of (57)-(58), we can prove that

$$u_{k\varepsilon} \to u_k \ \text{ as } \varepsilon \to 0, \ \text{in } C(K)$$

on every compact set $K \subset S$, where $u_{k\varepsilon}$ are the solutions of (61)-(58). By computation, it is easy show $C^*(t-t_0)^{-\frac{1}{q_1-1}}$ is a solution of the following equation

$$u_{kt} = \text{div}((\mid Du_k^m \mid^2 + \varepsilon)^{\frac{p-2}{2}} Du_k^m) - u_k^{q_1}, \quad (62)$$

in $R^N \times (t_0, \infty)$, $t_0 > 0$.

For any $\delta_1 > 0$, we choose $\delta_0 \in (0, \delta_1)$ such that

$$\mid u_{k\varepsilon}(x, \delta_1) \mid_{L^\infty(R^N)} \le C^*(\delta_1 - \delta_0)^{-\frac{1}{q_1-1}}.$$

Hence by the comparison principle, noticing that (59) implies that

$$-u_k^{q_1} \mid Du_k^m \mid^{p_1} \le -u_k^{q_1},$$

we have

$$u_{k\varepsilon}(x,t) \le C^*(t-t_0)^{-\frac{1}{q_1-1}}, t > \delta_1$$

The proof of Lemma 10 is completed by letting $\delta_1 \to 0$ and $\varepsilon \to 0$.

Lemma 11 u_k satisfies

$$\int_\tau^T \int_{B_R} \mid Du_k^m \mid^p \le c(\tau, R), \quad (63)$$

$$\int_\tau^T \int_{B_R} \mid u_t \mid dxdt \le c(\tau, R), \quad (64)$$

where $\tau \in (0, T)$.

The proof of Lemma 11 is similar to that of Lemma 9, we omit details here.

Proof of Theorem 3 By Lemma 10, $\{u_k\}$ are uniformly bounded on every compact set of S. Hence by [9], there exists a subsequence $\{u_{k_j}\}$ and a function $U \in C(S)$ such that

$$u_{k_j} \to U, \ \text{in } C(K)$$

and

$$U(x,t) \le C^* t^{-\frac{1}{q_1-1}}.$$

We now prove that $U(x,t) = C^* t^{-\frac{1}{q_1-1}}$. Let us introduce the function

$$\varphi_k^A = \min\{\varphi_k, A\} \quad (3.9)$$

and denote by $V_{K\varepsilon}^A$ the solution of (61) with initial value (65). By the comparison principle,

$$V_{K\varepsilon}^A \le u_{k\varepsilon}, \quad (66)$$

where $u_{k\varepsilon}$ is the solution of (61)-(58).

Define

$$V_A = C^*(t + \frac{A^{1-q_1}}{q_1-1})^{-\frac{1}{q_1-1}},$$

which is the solution of (62) with initial value

$$V_A(x,0) = A. \quad (67)$$

Noticing that

$$\lim_{k\to\infty} \varphi_k^A(x) = \lim_{k\to\infty} \min\{A, \frac{\varphi(kx) \mid kx \mid^\alpha k^{\delta-\alpha}}{\mid x \mid^\alpha}\} = A,$$

using the uniqueness of solution of (62)-(67), we can prove (see [11])

$$V_{k\varepsilon}^A \to V_A, \ \text{ as } k \to \infty \text{ in } C(K),$$

where K is a compact set in S. Moreover, by [9] and [4]

$$V_{k\varepsilon}^A \to V_k^A u_{k\varepsilon} \to u_k, \quad \text{as } k \to \infty \text{ in } \ C(K)$$

uniformly in K, where V_k^A is the solution of (1) with initial value (65). It follows that

$$V_k^A \to V_A, \quad \text{as } k \to \infty \ \text{ in } \ C(K).$$

Letting $\varepsilon \to 0$ and $k \to \infty$ in turn in (66), we get

$$V_A(x,t) \le V_\infty(x,t) = C^* t^{-\frac{1}{q_1-1}}, (x,t) \in S.$$

Since the lower bound holds for every $A > 0$, we conclude that

$$U(x,t) = V_\infty(x,t) = C^* t^{-\frac{1}{q_1-1}}, \ (x,t) \in S.$$

Thus

$$k^{\frac{p-p_1}{(q_1+mp_1-m(p-1))}} u(kx, k^\beta t) \to C^* t^{-\frac{1}{q_1-1}}, \ \text{ as } k \to \infty.$$

Set $t = 1$. Then

$$k^{\frac{p-p_1}{(q_1+mp_1-m(p-1))}} u(kx, k^\beta) \to C^*, \ \text{ as } k \to \infty,$$

uniformly on every compact subset on R^N. Therefore if we set $kx = x\acute{}$, $k^\beta = t\acute{}$, and omit the primes , then we obtain

$$t^{\frac{1}{q_1-1}} u(x,t) \to C^* \text{ as } t \to \infty$$

uniformly on sets $\{x \in R^N :| x |\leq \alpha t^{\frac{1}{\beta}}\}$ with $\alpha > 0$ for $t > 0$ and so Theorem 3 is proved.

Proof of Theorem 4 By Lemma 10 and [9], there exists a subsequence $\{u_{k_j}\}$ and a function $U \in C(S)$ such that

$$u_{k_j} \to U, (x,t) \in C(K). \qquad (68)$$

By Lemma 11, we can prove that U satisfies (1) in the sense of distribution in a manner similar way as Theorem 2 of [4].

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References:

[1] A. S. Kalashnikkov, Some problems of nonlinear parabolic equations of second order, *USSR. Math, Nauk, T.* 42(1987), pp. 135-176.

[2] O. A. Ladyzenskaja, New equations for the description of incompressible fluids and solvability in the large boundary value problem for them, *Proc. Steldov Inst. Math.* 102(1976), pp. 95-118.

[3] E. Di Benedetto, *Degenerate parabolic equations*, Universitext, Springer Verlag, 1993.

[4] J. Zhao and H. Yuan, The Cauchy problem of some doubly nonlinear degenerate parabolic equations, *Chinese Ann. Math.* A 16(2)(1995), pp. 179-194.

[5] T. Masayoshi, On solutions of some doubly nonlinear degenerate parabolic equations with absorption, *J. Math. Anal. Appl.* 132(1988), pp. 187-212.

[6] S. Kamin and J. L. Vazquez, Fundamental solutions and asymptotic behavior for the p-Laplacian equation, *Rev. Mat. Iberoamericana* 4(2) (1988), pp. 339-354.

[7] M. Winkler, Large time behavior of solutions to degenerate parabolic equations with absorption, *NoDEA.* 8(2001), pp. 343-361.

[8] J. Manfredi and V. Vespri, Large time behavior of solutions to a class of doubly nonlinear parabolic equations. *Electronic J. Diff. Equ.* (2)1994(1994), pp. 1-16.

[9] A. V. Ivanov, Hölder estimates for quasilinear parabolic equations, *J.Soviet Mat.* 56(2) (1991), pp. 2320-2347.

[10] Z. Wu, J. Zhao, J. Yin and H. Li, *Nonlinear diffusion equations*, Word Scientific Publishing, 2001.

[11] O. A. Ladyzanskauam, V. A. Solonilov and N. N. Uraltseva, Linear and quasilinear equation of parabolic type, *Trans. Math. Monographs* 23. *Amer. Math. Soc.*, Providence RTI. 1968.

[12] J. Yang and J. Zhao, The asymptotic behavior of solutions of some doubly degenerate nonlinear parabolic equations. *Northeastern Math. J.* (2)11(1995), pp. 241-252.

[13] M. Giaquinta, *Multiple integrals in the calculus of variations and nonliner elliptic systems*, Princeton Univ, Press., Princeton, NJ, 1983.

[14] M. Bertsch and L. A. Peletier, The asymptotic profile of a nonlinear diffusion equaation, *Arch. Rat. Mech. Anal.* 91(1985), pp. 207-229.

[15] E. Di Benedetto and M. A. Herrero, On the Cauchy problem and initian trace for a degenerate parabolic equation, *Trans. AM. Math. Soc.* 314(1989), pp. 187-224.

[16] D. G. Aronson and L. A. Peletier, Large time behavior of solutions of the porous medium equation in bounded domains, *J. Diff. Equations* 39(1981), pp. 1001-1022.

[17] H. Zhan, The self-similar solutions for a quasilinear doubly degenerate parabolic equation, *Chinese J. of Eng. Math.* 27(2010), pp. 1030-1034.

[18] H. Zhan, Large time behavior of solutions to a class of doubly nonlinear parabolic equations, *Appl. of Math.* (53)2008, pp. 521-333.