Turing instability and wave patterns for a symmetric discrete competitive Lotka-Volterra system

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Abstract: In this paper, Turing instability of a symmetric discrete competitive Lotka-Volterra system is considered. To this end, conditions for producing Turing instability of a general discrete system is attained and this conclusion is applied to the discrete competition Lotka-Volterra system. Then a series of numerical simulations of the discrete model are performed with different parameters. Results show that the discrete competitive Lotka-Volterra system can generate a large variety of wave patterns in the Turing instability region. Particularly, the diffusion coefficients can be equivalent, that is, there is neither "activator" nor "inhibitor". Similar results can not be obtained for the corresponding continuous models. On the other hand, the number of the eigenvalues is illuminated by calculation and the unstable spaces can be clearly expressed. Thus, the Turing mechanism is also explained.

Key–Words: Turing instability; Diffusion; Discrete system; Eigenvalue; Lotka-Volterra system; Wave pattern

1 Introduction

Reaction-diffusion systems have been proposed as mechanisms for biological pattern formation in embryological and ecological context [1]. All such works are based on Turing's pioneering work [2] in [3], the authors were the first to call attention to the fact that Turing's idea would be applicable in ecological situation also. They conjectured that the nature of the equations which describe chemical interaction does not seem fundamentally different from the nature of those which describe ecological interaction among the species. Again, the idea that dispersal could give rise to instabilities and hence to spatial pattern was due to a number of authors, see [4], for review.

In [2], Turing specifically considered two chemicals, an activator and an inhibitor. The activator enhances the production rate of its own and of the inhibitor, whereas the inhibitor suppresses both the activator and the inhibitor. The diffusion coefficient of the inhibitor is much larger than that of the activator. Without diffusion, the local reaction of the two substances is stable and converges to the equilibrium. However, with diffusion, the uniform steady state is unstable. A nearly uniform initial distribution spontaneously gives rise to a spatially heterogeneous pattern, and this distribution is referred to as the "Turing instability". This simple mechanism suggests that the reaction of a small number of chemicals and their random diffusion can create stationary non-uniform patterns in a perfectly homogeneous field.

In reaction-diffusion systems, studies of spatially-distributed active media have demonstrated the ubiquity of self-organized spatiotemporal patterns, in particular spiral waves, which emerge in excitable systems. Standing waves, spiral wave and target wave have been found in chemical system, such as the Belousov Zhabotinsky (BZ) reaction in which mathematical modeling predicts Turing structures [5]-[7]. Maybe, one may expect that there are systems that possess a range of parameters where the wave patterns exist in the Turing instability region.

It is well known that the competitive Lotka-Volterra system can be described as

$$\begin{cases} \frac{du}{dt} = u(r_1 - a_{11}u - a_{12}v) = f(u,v) \\ \frac{dv}{dt} = v(r_2 - a_{21}u - a_{22}v) = g(u,v) \end{cases} \tag{1}$$

where $r_i (i = 1, 2)$ and $a_{ij}, i, j = 1, 2$ are positive constants and $A = [a_{ij}]$. There are four equilibria, $E_0 = (0, 0)$, $E_1 = (r_1/a_{11}, 0)$ representing the absence of species 2, E_2 $(0, r_2/a_{22})$ representing the absence of species 1, and

$$E = (u^*, v^*) = \left(\frac{r_1 a_{22} - r_2 a_{12}}{\det A}, \frac{r_2 a_{11} - r_1 a_{21}}{\det A} \right)$$

which is in the interior of the positive quadrant of the (u^*, v^*) plane if

$$\frac{a_{12}}{a_{22}} < \frac{r_1}{r_2} < \frac{a_{11}}{a_{21}} \tag{2}$$

Otherwise, there is no equilibrium with positive coordinates representing the coexistence of the two species. One can show by phase plane methods or by Lyapunov functions that the condition (2) implies that the unique positive equilibrium E is globally asymptotically stable, for instants, see [2] or [3].

When we add diffusion part to the system, then the model transform into the form

$$\begin{cases} \frac{\partial u}{dt} = u(r_1 - a_{11}u - a_{12}v) + D_1\nabla^2 u, \\ \frac{\partial v}{dt} = v(r_2 - a_{21}u - a_{22}v) + D_2\nabla^2 v. \end{cases} \tag{3}$$

So the Eqs. (3) becomes

$$w_t = J_1 w + D\nabla^2 w$$

in which

$$w = \left(\begin{array}{c} u - u^* \\ v - v^* \end{array} \right)$$

and

$$J_1 = \left[\begin{array}{cc} f_u & f_v \\ g_u & g_v \end{array} \right]_{(u^*,v^*)}$$

and

$$D = \left[\begin{array}{cc} D_1 & 0 \\ 0 & D_2 \end{array} \right]$$

Let k^2 be the eigenvalue of the diffusion part and W be the eigenfunction. Then

$$\nabla^2 W + k^2 W = 0 \tag{4}$$

So, Jacobian Matrix of (3) should be of the following form

$$J_1^{'} = J_1 - Dk^2$$

If anyone of the eigenvalues λ of the $J_1^{'}$ have positive real part, the system (3) is unstable at (u^*, v^*). This is because

$$\left|\lambda I - J_1^{'}\right| = \left| \begin{array}{cc} D_1k^2 - f_u + \lambda & -f_v \\ -g_u & D_2k^2 - g_v + \lambda \end{array} \right| = 0$$

i.e.

$$(D_1k^2 - f_u + \lambda)(D_2k^2 - g_v + \lambda) - f_v g_u = 0.$$

Let

$$h(k^2) = |J_1| - k^2(f_u D_2 + g_v D_1) + D_1 D_2 k^4,$$

then

$$\lambda^2 - [f_u + g_v - (D_1 + D_2)k^2]\lambda - h(k^2) = 0.$$

Thus

$$\begin{aligned} \lambda_{1,2} &= \pm\{[f_u + g_v - (D_1 + D_2)k^2]^2 - 4h(k^2)\}^{\frac{1}{2}} + \\ &\quad [f_u + g_v - (D_1 + D_2)k^2]. \end{aligned}$$

As we discussed above, if the system (1) is stable at (u^*, v^*), there must be $f_u + g_v = (-a_{11} * u^*) - (-a_{22} * v^*) < 0$, i.e. $f_u + g_v - (D_1 + D_2)k^2 < 0$. So if we want the system to remain unstable, there must be a $k^2 \neq 0$, lead to the condition

$$\begin{cases} [f_u + g_v - (D_1 + D_2)k^2]^2 - 4h(k^2) > 0 \\ \{[f_u + g_v - (D_1 + D_2)k^2]^2 - 4h(k^2)\}^{\frac{1}{2}} + \\ \quad [f_u + g_v - (D_1 + D_2)k^2] > 0 \end{cases} \tag{5}$$

which ensure that there is at least one eigenvalue has positive real part. From (5) we can obtain that $h(k^2) < 0$ while $k^2 \neq 0$.

Since

$$\begin{aligned} h(k^2) &= |J_1| - k^2(f_u D_2 + g_v D_1) + D_1 D_2 k^4 \\ &= D_1 D_2[k^4 - \frac{1}{2}k^2(\frac{f_u}{D_1} + \frac{g_v}{D_2}) \\ &\quad + \frac{1}{D_1 D_2}(f_u g_v - f_v g_u)] \\ &= D_1 D_2[(k^2 - \frac{1}{2}(\frac{f_u}{D_1} + \frac{g_v}{D_2})]^2 \\ &\quad - \frac{D_1 D_2}{4}(\frac{f_u}{D_1} + \frac{g_v}{D_2})^2 + |J_1| \end{aligned}$$

we have

$$\begin{aligned} h_{\min} &= -\frac{D_1 D_2}{4}(\frac{f_u}{D_1} + \frac{g_v}{D_2})^2 + |J_1| \\ &= -\frac{D_1 D_2}{4}(\frac{f_u^2}{D_1^2} + \frac{2 f_u g_v}{D_1 D_2} + \frac{g_v^2}{D_2^2}) + |J_1| \\ &= -\frac{D_2 f_u^2}{4D_1} - \frac{f_u g_v}{2} - \frac{D_1 g_v^2}{4D_2} + |J_1| \\ &= -\frac{D_1 D_2}{4}(\frac{f_u}{D_1} + \frac{g_v}{D_2})^2 + |J_1| \end{aligned}$$

It is clear that only as long as $h_{\min} < 0$, when $k^2 = \frac{1}{2}(\frac{f_u}{D_1} + \frac{g_v}{D_2})$, we can have $h(k^2) < 0$. Thus it must satisfy the conditions of

$$\begin{cases} \frac{1}{2}(\frac{f_u}{D_1} + \frac{g_v}{D_2}) > 0 \\ -\frac{D_1 D_2}{4}(\frac{f_u}{D_1} + \frac{g_v}{D_2})^2 + |J_1| < 0. \end{cases}$$

The above equations are equal to

$$\begin{cases} D_2 f_u + D_1 g_v > 0 \\ D_1 D_2(\frac{f_u}{D_1} + \frac{g_v}{D_2})^2 - 4(f_u g_v - f_v g_u) > 0 \end{cases} \tag{6}$$

In conclusion, take (6) and the discussion about the system without the diffusion part into consideration, the unstable conditions for (3) in order to generate Turing Pattern

$$\begin{cases} \frac{a_{12}}{a_{22}} < \frac{r_1}{r_2} < \frac{a_{11}}{a_{21}} \\ D_2 f_u + D_1 g_v > 0 \\ D_1 D_2(\frac{f_u}{D_1} + \frac{g_v}{D_2})^2 - 4(f_u g_v - f_v g_u) > 0 \end{cases} \tag{7}$$

However, in this competition model, we have the limitation that $r_i(i = 1, 2)$ and $A = [a_{ij}](i, j = 1, 2)$ are positive, which implies that the condition $D_2 f_u + D_1 g_v > 0$ can never be satisfied. Thus, this continuous model can never generate Turing Pattern.

Now, we have a problem. Can the discrete systems produce Turing instability? In this paper, we will consider Turing instability at the positive fixed point for a discrete competition Lotka-Volterra system. Typical spiral pattern will be obtained by numerical simulations. Relevant research on the discrete competition models for spiral structure have also been established when the coexistence is no longer possible and exclusion of one of the competitors takes place (e.g., see [8]-[10]).

However, the present paper is only concerned with the coexistence case. Thus, we can say Turing instability, wave pattern in the Turing space, etc.

This paper is organized as follows: in the next section, we demonstrate the general theory of Turing instability for a discrete system by using linearized technique. In particular, the necessary and sufficient conditions of stability or instability will be obtained. The obtained results are very simple, however, they are new. In Section 3, a symmetric discrete competitive Lotka-Volterra model will be introduced and the conditions of Turing instability will be achieved. Similar results can not be obtained for the corresponding continuous models. Particularly, the diffusion coefficients can be equivalent, that is, there is neither "activator" nor "inhibitor". In Section 4, some numerical simulations will be given for the symmetric discrete competition Lotka-Volterra system with different parameters. Typical spiral Turing patterns are observed. The pattern information associated with the time evolution and the change of parameter will be also given. Section 5 is concerned with the number of the eigenvalues and the unstable spaces. Turing mechanism hopes to be revealed. Some conclusions will be proposed in the final section.

2 Turing Instability for Discrete System

In this section, we firstly consider a general discrete reaction-diffusion system without diffusion part of the form

$$\begin{cases} u_{t+1} = f(u_t, v_t) \\ v_{t+1} = g(u_t, v_t) \end{cases} \tag{8}$$

for $t \in Z^+ = \{0, 1, 2, \cdots\}$ and let (u^*, v^*)[1] denote the fixed point of the system (8). The linearized form of (8) is then

$$\begin{cases} u_{t+1} = f_u u_t + f_v v_t \\ v_{t+1} = g_u u_t + g_v v_t \end{cases} \tag{9}$$

which has the Jacobian matrix

$$J_1 = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}_{(u^*, v^*)}. \tag{10}$$

To obtain the stable result of (9), we consider the algebraic equation

$$\lambda^2 + b\lambda + c = 0. \tag{11}$$

It is well known that it has two roots of the form

$$\lambda_{1,2} = (-b \pm \sqrt{b^2 - 4c})/2$$

By simple calculation, we can obtain the following result:

Proposition 1 *The roots $\lambda_{1,2}$ of the algebraic equation (11) satisfy the condition $|\lambda_{1,2}| < 1$ if, and only if $b - 1 < c$, $-b - 1 < c$ and $c < 1$.*

The following fact can be immediately deduced from Proposition 1.

Proposition 2 *The system (8) at the fixed point (u^*, v^*) is local asymptotically stable when the conditions*

$$f_{u^*} g_{v^*} - g_{u^*} f_{v^*} > -(f_{u^*} + g_{v^*}) - 1, \tag{12}$$

$$f_{u^*} g_{v^*} - g_{u^*} f_{v^*} > (f_{u^*} + g_{v^*}) - 1 \tag{13}$$

and

$$f_{u^*} g_{v^*} - g_{u^*} f_{v^*} < 1. \tag{14}$$

hold.

In the following, we consider the reaction-diffusion system of the form

$$\begin{cases} u_{t+1}^{ij} = f(u_t^{ij}, v_t^{ij}) + D_1 \nabla^2 u_t^{ij} \\ v_{t+1}^{ij} = g(u_t^{ij}, v_t^{ij}) + D_2 \nabla^2 v_t^{ij} \end{cases} \tag{15}$$

with the periodic boundary conditions

$$u_t^{i,0} = u_t^{i,m},\ u_t^{i,1} = u_t^{i,m+1} \tag{16}$$

and

$$u_t^{0,j} = u_t^{m,j},\ u_t^{1,j} = u_t^{m+1,j} \tag{17}$$

for $i, j \in \{1, 2, \cdots, m\} = [1, m]$ and $t \in Z^+$, where m is a positive integer,

$$\nabla^2 u_t^{ij} = u_t^{i+1,j} + u_t^{i,j+1} + u_t^{i-1,j} + u_t^{i,j-1} - 4u_t^{ij}$$

[1] In fact, u^* and v^* can be zero or negative, however, in the present paper, we will concentrate on the positive fixed point.

and

$$\nabla^2 v_t^{ij} = v_t^{i+1,j} + v_t^{i,j+1} + v_t^{i-1,j} + v_t^{i,j-1} - 4v_t^{ij}.$$

In order to study Turing instability of (15)-(17), we firstly consider eigenvalues of the following equation

$$\nabla^2 X^{ij} + \lambda X^{ij} = 0 \tag{18}$$

with the periodic boundary conditions

$$X^{i,0} = X^{i,m},\ X^{i,1} = X^{i,m+1} \tag{19}$$

and

$$X^{0,j} = X^{m,j},\ X^{1,j} = X^{m+1,j}. \tag{20}$$

In view of [11], the eigenvalue problem (18)-(20) has the eigenvalues

$$\lambda_{l,s} = 4\left(\sin^2\left(\frac{(l-1)\pi}{m}\right) + \sin^2\left(\frac{(s-1)\pi}{m}\right)\right) = k_{ls}^2 \quad \text{for } l, s \in [1, m]. \tag{21}$$

The linearized form of (15) is

$$\begin{cases} u_{t+1}^{ij} = f_{u^*}u_t^{ij} + f_{v^*}v_t^{ij} + D_1\nabla^2 u_t^{ij} \\ v_{t+1}^{ij} = g_{u^*}u_t^{ij} + g_{v^*}v_t^{ij} + D_2\nabla^2 v_t^{ij} \end{cases} \tag{22}$$

Then respectively taking the inner product of (22) with the corresponding eigenfunction X_{ls}^{ij} of the eigenvalue $\lambda_{l,s}$, we see that

$$\begin{cases} \sum\limits_{i,j=1}^{m} X_{ls}^{ij}u_{t+1}^{ij} = f_{u^*}\sum\limits_{i,j=1}^{m} X_{ls}^{ij}u_t^{ij} \\ +f_{v^*}\sum\limits_{i,j=1}^{m} X_{ls}^{ij}v_t^{ij} + D_1\sum\limits_{i,j=1}^{m} X_{ls}^{ij}\nabla^2 u_t^{ij} \\ \sum\limits_{i,j=1}^{m} X_{ls}^{ij}v_{t+1}^{ij} = g_{u^*}\sum\limits_{i,j=1}^{m} X_{ls}^{ij}u_t^{ij} \\ +g_{v^*}\sum\limits_{i,j=1}^{m} X_{ls}^{ij}v_t^{ij} + D_2\sum\limits_{i,j=1}^{m} X_{ls}^{ij}\nabla^2 v_t^{ij} \end{cases} \tag{23}$$

Let

$$U_t = \sum_{i,j=1}^{m} X_{ls}^{ij}u_t^{ij} \quad \text{and} \quad V_t = \sum_{i,j=1}^{m} X_{ls}^{ij}v_t^{ij}$$

and use the periodic boundary conditions (16) and (17), then it follows that

$$\begin{cases} U_{t+1} = f_{u^*}U_t + f_{v^*}V_t - D_1k_{ls}^2U_t \\ V_{t+1} = g_{u^*}U_t + g_{v^*}V_t - D_2k_{ls}^2V_t \end{cases}$$

or

$$\begin{cases} U_{t+1} = \left(f_{u^*} - D_1k_{ls}^2\right)U_t + f_{v^*}V_t, \\ V_{t+1} = g_{u^*}U_t + \left(g_{v^*} - D_2k_{ls}^2\right)V_t. \end{cases} \tag{24}$$

Conversely, if (U_t, V_t) is a solution of the system (24), then

$$\left(u_t^{ij} = U_tX_{ls}^{ij}, v_t^{ij} = V_tX_{ls}^{ij}\right)$$

is also clearly a solution of (22) with the periodic boundary conditions (16) and (17). Thus, the following fact can be obtained

Proposition 3 *If $\left(u_t^{ij}, v_t^{ij}\right)$ is a solution of the problem of (15)-(17), then*

$$\left(U_t = \sum_{i,j=1}^{m} X_{ls}^{ij}u_t^{ij}, V_t = \sum_{i,j=1}^{m} X_{ls}^{ij}v_t^{ij}\right)$$

is a solution of (24), where k_{ls}^2 is some eigenvalue of (18)-(20) and X_{ls}^{ij} is the corresponding eigenfunction; For some eigenvalue k_{ls}^2 of (18)-(20), if (U_t, V_t) is a solution of the system (24), then

$$\left(u_t^{ij} = U_tX_{ls}^{ij}, v_t^{ij} = V_tX_{ls}^{ij}\right)$$

is a solution of (22) with the periodic boundary conditions (16) and (17).

From above we see that the unstable system (24) will induce that the problem (15)-(17) is unstable. On the other hand, the following fact can also be obtained by using Proposition 1.

Proposition 4 *If there exist positive numbers D_1, D_2, and the eigenvalue k_{ls}^2 of the problem (18)-(20) such that one of the following conditions*

$$h(k_{ls}^2) < (k_{ls}^2(D_1 + D_2) - (f_{u^*} + g_{v^*})) - 1, \tag{25}$$

$$h(k_{ls}^2) < -(k_{ls}^2(D_1 + D_2) - (f_{u^*} + g_{v^*})) - 1 \tag{26}$$

or

$$h(k_{ls}^2) > 1 \tag{27}$$

holds, then the problem (15)-(17) at the fixed point (u^*, v^*) is unstable, where

$$\begin{aligned} h(k_{ls}^2) \ &= \ D_1D_2k_{ls}^4 - (D_1g_{v^*} + D_2f_{u^*})k_{ls}^2 \\ &\quad +(f_{u^*}g_{v^*} - f_{v^*}g_{u^*}). \end{aligned}$$

Propositions 2 and 4 imply that the problem (15)-(17) is diffusion-driven unstable or Turing unstable.

3 Discrete Competitive Lotka-Volterra System

A continuous competition Lotka-Volterra system can be described as follows

$$\begin{cases} x'(t) = x(t)(r_1 - a_{11}x(t) - a_{12}y(t)) \\ y'(t) = y(t)(r_2 - a_{21}x(t) - a_{22}y(t)) \end{cases} \tag{28}$$

for $t \in [0, +\infty)$, where $x(t)$ and $y(t)$ are the quantities of the two species at time t, $r_1 > 0$ and $r_2 > 0$ are growth rates of the respective species, a_{11} and a_{22} represent the strength of the intra-specific competition, and a_{12} and a_{21} being the strength of the interspecific competition. One can show by phase plane methods or by Lyapunov functions that when:

$$a_{12}/a_{22} < r_1/r_2 < a_{11}/a_{21}$$

the unique positive equilibrium (x^*, y^*) is globally asymptotically stable see [12], where (x^*, y^*) satisfies

$$\begin{cases} a_{11}x^* + a_{12}y^* = r_1, \\ a_{21}x^* + a_{22}y^* = r_2. \end{cases}$$

Unfortunately, by simple calculation the corresponding reaction diffusion system of (28) can not exhibit diffusion-driven instability or Turing instability.

Now, we consider the corresponding discrete system of the form

$$\begin{cases} u_{t+1} = r_1 u_t (1 - u_t - a_{12}v_t) \\ v_{t+1} = r_2 v_t (1 - a_{21}u_t - v_t) \end{cases} \tag{29}$$

for $t \in Z^+$, where r_1, r_2, a_{12} and a_{21} are positive constants. We will here consider the completely symmetric case, that is,

$$\begin{cases} u_{t+1} = r u_t (1 - u_t - a v_t), \\ v_{t+1} = r v_t (1 - a u_t - v_t), \end{cases} \tag{30}$$

where r and a are positive constants. At the same time, we also assume that the diffusion coefficients are equal, that is, $D_1 = D_2 = D$. Thus, we also consider the reaction diffusion system of the form

$$\begin{cases} u_{t+1}^{ij} = r u_t^{ij}\left(1 - u_t^{ij} - a v_t^{ij}\right) + D\nabla^2 u_t^{ij} \\ v_{t+1}^{ij} = r v_t^{ij}\left(1 - a u_t^{ij} - v_t^{ij}\right) + D\nabla^2 v_t^{ij} \end{cases} \tag{31}$$

with the boundary conditions (16) and (17).

Clearly, the system (30) has four possible steady states, i.e. $P_0 = (0,0)$, exclusion points $P_1 = ((1-1/r), 0)$, $P_2 = (0, (1-1/r))$ and nontrivial coexistence point $P_c = (u^*, v^*)$, where

$$u^* = v^* = (r-1)/r(a+1).$$

The linear stability analysis leads to the stable domains defined by the sets

$$S(P_0) = \{(r,a) \,|\, 0 < r < 1, a > 0\},$$

$$S(P_{1,2}) = \{(r,a) \,|\, a > 1, 1 < r < 3\},$$

and

$$S(P_c) = \{(r,a) \,|\, 0 < a < 1, 1 < r < 3\}.$$

For the cases $S(P_{1,2})$, Turing-like structure or chaotic Turing structure have been considered by Ricard and Bascompte(e.g., [8]-[10]). In this paper, we are only concerned with the coexistence case, that is, $0 < a < 1$ and $1 < r < 3$.

In this case, the systems (22) and (24) are respectively reduced to

$$\begin{cases} u_{t+1}^{ij} = \tau\left(k_{ls}^2\right) u_t^{ij} - d v_t^{ij} \\ v_{t+1}^{ij} = -d u_t^{ij} + \tau\left(k_{ls}^2\right) v_t^{ij} \end{cases} \tag{32}$$

and

$$\begin{cases} U_{t+1} = \tau\left(k_{ls}^2\right) U_t - dV_t, \\ V_{t+1} = -dU_t + \tau\left(k_{ls}^2\right) V_t, \end{cases} \tag{33}$$

where

$$\tau\left(k_{ls}^2\right) = 1 - \frac{r-1}{a+1} - Dk_{ls}^2$$

and

$$d = \frac{a(r-1)}{(a+1)}.$$

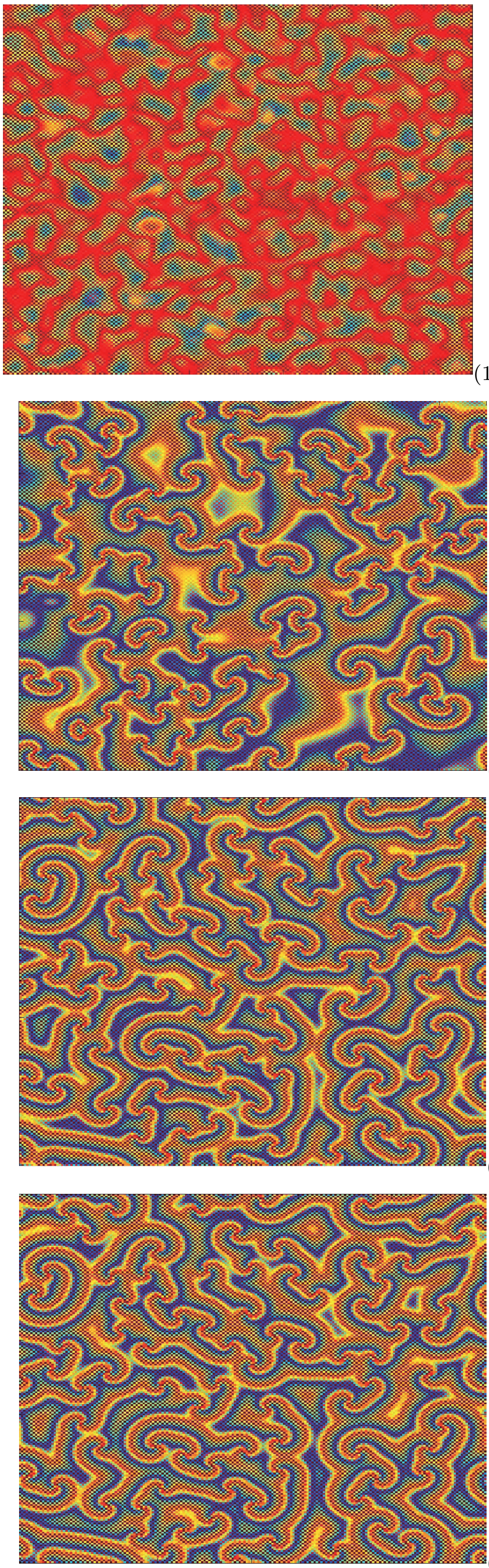
In view of Proposition 4, it follows immediately that the corresponding reaction-diffusion system (31) with the conditions (16) and (17) is unstable when the condition $Dk_{ls}^2 > 3 - r$ holds for some positive number D and the eigenvalue k_{ls}^2 of the problem (18)-(20).

4 Discrete Competitive Lotka-Volterra System

In this section, a series of numerical simulations will be performed so that we can explore the dynamical behavior of the discrete competition Lotka-Volterra reaction-diffusion system (31) with the conditions (16) and (17).

In all of the following simulations, the small amplitude random perturbation is 1% around the steady state, the size of the lattice is chosen to be 200×200, periodic boundary conditions are applied and which implies zero flux on the square boundary.

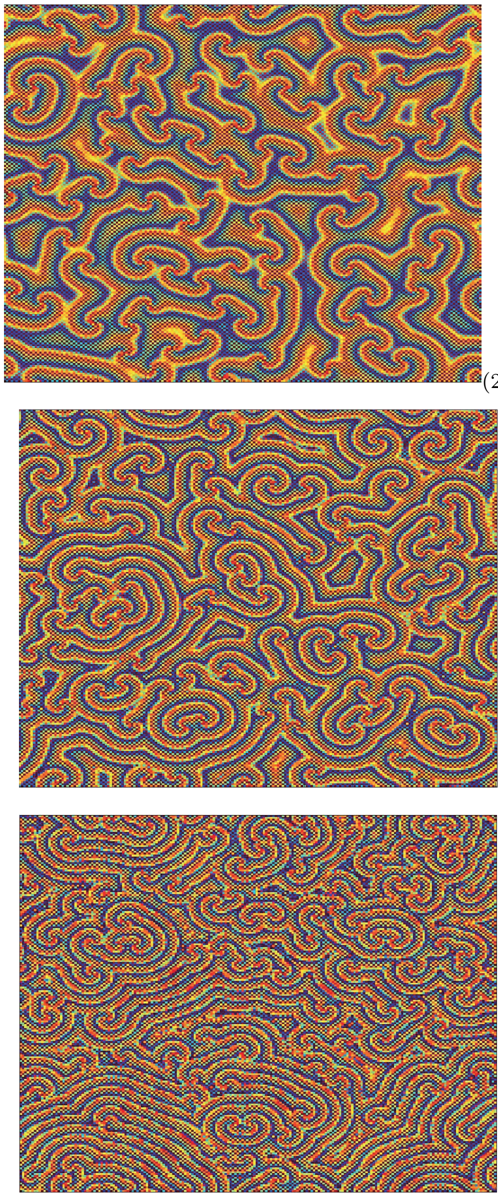
Simulations of pattern development at $t = 50, 500, 99900, 100000$ (from Fig.1(1.1) to Fig.1(1.4)) which shows the evolution in the stripe spacing as the interaction time proceeds, for the



(1.1) (1.2) (1.3) (1.4)

Figure 1: Simulations of pattern development. (1.1):t=50; (1.2):t=500; (1.3):t=99950; (1.4):t=100000.

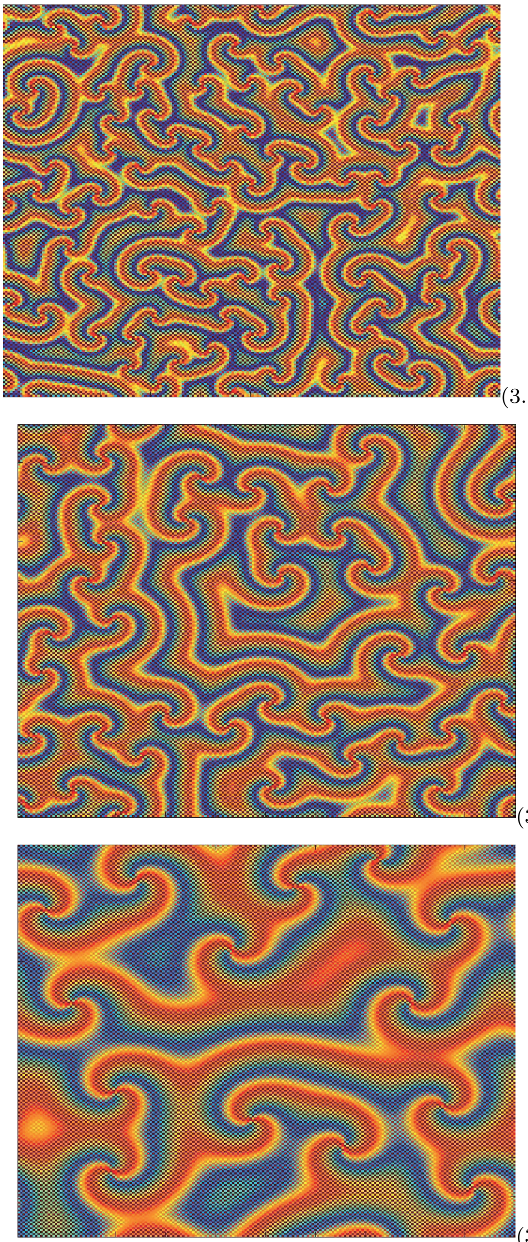
following parameters $r = 2.98$, $a = 0.5$, $D = 0.1$. In Fig.1(1.1), the symmetry break around the fixed point is shown and Fig.1(1.2) is the state at the moment $t = 500$ which exhibits the self-organization process of the system. Spiral pattern emerge in Fig.1(1.3)(1.4) which are similar to the one in [13], see Fig.2). Another interesting situation is depicted in Fig.1(1.3) and Fig.1(1.4) where periodic-like pattern are observed.



(2.1) (2.2) (2.3)

Figure 2: Spiral information with the decrease of r. (2.1): r=2.98; (2.2): r=2.95; (2.3): r=2.92.

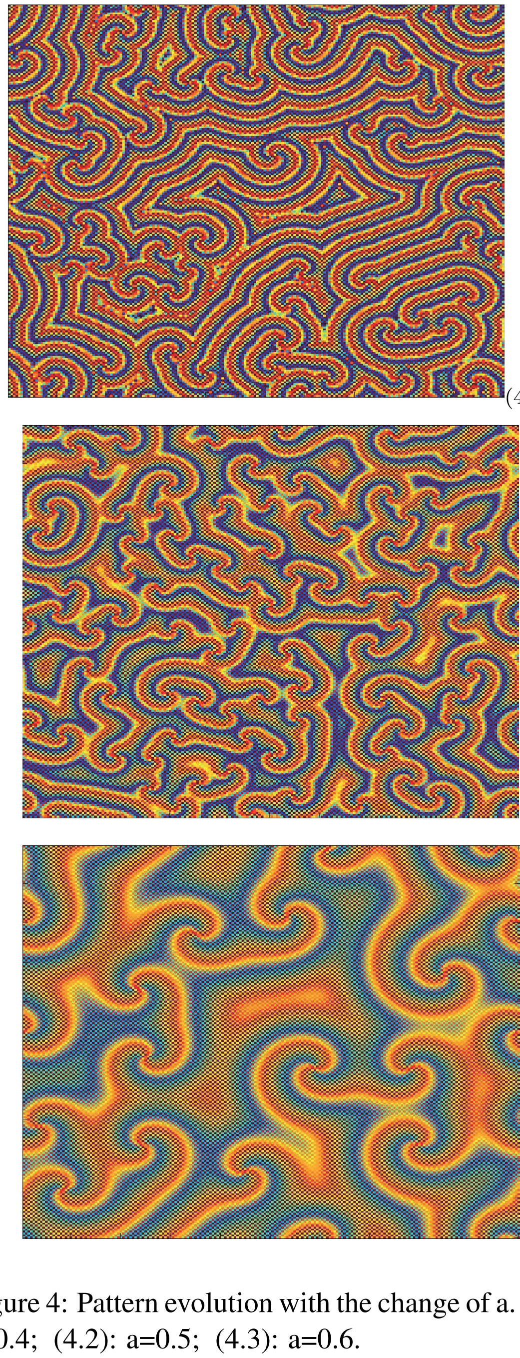
Turing instability is diffusion-driven instability, thus the diffusion rate of the species is vital to the pattern formation. For investigating the effect of diffusion coefficients on patterns, by keeping all the other parameters of the system fixed, we increase the diffusion coefficient D in step of $\triangle D = 0.05$ ($D = 0.1$, $D = 0.105$, $D = 0.11$) and find that this can change the emerging pattern dramatically, as depicted in Fig.2. From Fig.2(2.1) to Fig.2(2.3) at $t = 100000$, the wavelength of the traveling waves becomes shorter with the increase of D. This reveals that fast diffusive motion of two competitive species of the discrete system is advantageous for the occurrence of stripe waves.



(3.1) (3.2) (3.3)

Figure 3: Pattern evolution with the increase of D. (3.1):D=0.1 ; (3.2):D=0.105; (3.3):D=0.11.

Similarly, we search for the impact of the discrete intrinsic growth rates of the species on the interaction, we decrease only r in step of $\triangle r = 0.03$ ($r = 2.92$, $r = 2.95$, $r = 2.98, t = 100000$), the other parameters are the same as that of Fig.1, different wave waves are shown in Fig.3($3.1, 3.2, 3.3$), and this indicates that the intrinsic growth rates of the two competitive species can affect the distribution of the species.



(4.1) (4.2) (4.3)

Figure 4: Pattern evolution with the change of a. (4.1): a=0.4; (4.2): a=0.5; (4.3): a=0.6.

We have also explored other regions of parameter space to look for new complex patterns, change the parameter a in step of $\triangle a = 0.1$ ($a = 0.4, a = 0.5, a = 0.6$), keeping the other parameters fixed, that is $r = 2.98$, $D = 0.1$. Distinct spiral patterns emerge in Fig.4($4.1, 4.2, 4.3$) at $t = 100000$, and this demonstrates that different interspecific competition rates play an important role in pattern formation.

Here, we only choose a part of number simulations. In fact, the similar patterns can be observed for the different parameters r, a, and D when they satisfy the conditions of Turing instability.

5 Discussion

In the following, eigenvalues will be analyzed in order to better illuminate the unstable spaces which are the root to produce wave patterns.

The coefficient matrix of (33) has the eigenvalues

$$\lambda_{1,2}\left(k_{ls}^2\right) = \tau\left(k_{ls}^2\right) \pm d.$$

Thus, we can let

$$\begin{cases} U_t = c_1\left(k_{ls}^2\right) x_t + c_2\left(k_{ls}^2\right) y_t \\ V_t = c_3\left(k_{ls}^2\right) x_t + c_4\left(k_{ls}^2\right) y_t \end{cases}$$

such that

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda_1\left(k_{ls}^2\right) & 0 \\ 0 & \lambda_2\left(k_{ls}^2\right) \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} \tag{34}$$

which has a general solution

$$x_t = \lambda_1^t\left(k_{ls}^2\right) x_0 \text{ and } y_t = \lambda_2^t\left(k_{ls}^2\right) y_0.$$

Thus, the system (33) has a solution

$$\begin{cases} U_t = c_1\left(k_{ls}^2\right) \lambda_1^t\left(k_{ls}^2\right) x_0 + c_2\left(k_{ls}^2\right) \lambda_2^t\left(k_{ls}^2\right) y_0, \\ V_t = c_3\left(k_{ls}^2\right) \lambda_1^t\left(k_{ls}^2\right) x_0 + c_4\left(k_{ls}^2\right) \lambda_2^t\left(k_{ls}^2\right) y_0. \end{cases}$$

It follows from Proposition 3 that

$$\begin{cases} u_{ij}^t = \left[c_1\left(k_{ls}^2\right) \lambda_1^t\left(k_{ls}^2\right) x_0 + c_2\left(k_{ls}^2\right) \lambda_2^t\left(k_{ls}^2\right) y_0\right] X_{ls}^{ij} \\ v_{ij}^t = \left[c_3\left(k_{ls}^2\right) \lambda_1^t\left(k_{ls}^2\right) x_0 + c_4\left(k_{ls}^2\right) \lambda_2^t\left(k_{ls}^2\right) y_0\right] X_{ls}^{ij} \end{cases} \tag{35}$$

is a solution of (32).

From (21), we know that the number of eigenvalues for the eigenvalue problem (18)-(20) is m^2, where $k_{11}^2 = 0$ is a unique simple eigenvalue. When m is even, the eigenvalue problem (18)-(20) has a unique maximum eigenvalue 8 and the multiplicity of the other eigenvalues is 2 or 4; When m is odd, the multiplicity of the other eigenvalues except $k_{11}^2 = 0$ is 4, the maximum eigenvalue is

$$8\sin^2((m-1)\pi/2m).$$

which can be clearly shown in Fig.5 (for example, m is even, $m = 200$).

It is well known that a linear combination of two solutions of (32) is also its solution. Thus, we may say that the asymptotic action of solutions of (32) is completely confirmed by the eigenvalues of the eigenvalue problem (18)-(20) and (35). Particularly, in Fig.1, the condition

$$Dk_{ls}^2 > 3 - r,$$

$$k_{ls}^2 = 4(\sin^2((l-1)\pi/m) + \sin^2((s-1)\pi/m))$$

can be interpreted as

$$\begin{aligned} &\sin^2((l-1)\pi/m) + \sin^2((s-1)\pi/m) \\ &> (3-r)/4 * D = 0.05 \end{aligned} \tag{36}$$

by calculation, we obtain that there are 39343 pairs of (l, s) satisfied the condition (36), and this means there exist 39343 eigenvalues whose norms are larger than 1 for such parameters, and the linear combination of the corresponding eigenvectors can generate different unstable spaces, thus, theoretically speaking, the above discrete system can produce a large variety of wave patterns.

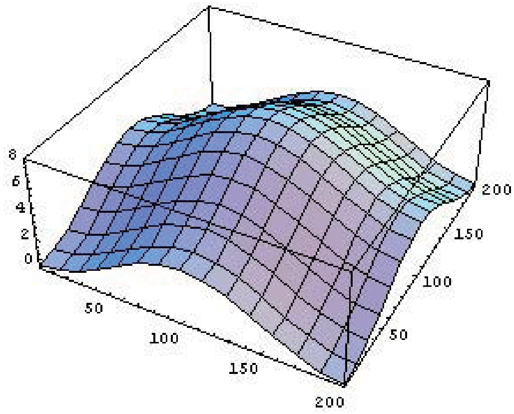


Figure 5: Eigenvalue distribution.

6 Conclusion

Firstly, we have presented a theoretical analysis of Turing instability for a general discrete system and the conditions of Turing instability for a competitive Lotka-Volterra system follows immediately which indicates the destabilization of the homogeneous distribution of two competitive species and the emergence of diffusion-induced pattern formation.

Secondly, a large variety of wave pattern are obtained by numerical simulations which is consistent with the predictions drawn from the analysis of the discrete competitive system. As given by Fig.1 ~ 4, the discrete model can produce spiral patterns. It is worth mentioning here that the continuous competitive system described in this paper can not generate Turing instability, furthermore, for some other continuous system which can produce Turing instability, the diffusion coefficient of the inhibitor must be much larger than that of the activator, but this condition is not necessary for discrete system any more.

Finally, the eigenvalue of the system (33) is analyzed in detail. For some fixed parameters of the discrete system, the number of the eigenvalues whose norms are larger than 1 is obtained by calculation, and the unstable spaces can be expressed clearly.

The conclusions drawn from this paper may motivate us to pay more attention to the difference of wave patterns between the continuous system and discrete one, and help us to better understand the population dynamics in a real competitive environment.

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