# Spectrum of A Class of Delay Differential Equations and Its Solution Expansion

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*Abstract:* In this paper we study the spectrum and solution expansion of the differential equation with multiple delays. Firstly, we present explicitly the asymptotic expressions of the eigenvalues under certain conditions. Then we prove that the root vectors of the system fail to form a basis for the state Hilbert space. However, by a trick, we expand the solution of the system according to the root vectors. As an application, we explain how to apply solution expansion to the numerical simulation of this kind of delay differential equations.

*Key–Words:* delay differential equation, multiple delays, spectrum, root vector, expansion of solution, numerical simulation

## 1 Introduction

It is well-known that the delay differential equations (DDEs) play an important role in the research of various applied science, such as control theory, biology ([6]–[8], [17]), economy, physics, life science ([22]) and engineering ([17]), etc. Since delay is a common phenomena in these fields, DDEs are usually used as their mathematical models for precise description. However, the occurrence of the delays may destroy the stability of the whole system and cause periodic oscillations. So the dynamics and the stability of models with delay terms have become attractive and challenging topics (see [1]–[8], [10], [11], [14], [16], [18]–[20]). Note that the DDEs are usually large-scale and highly complicated nonlinear dynamical systems. In most cases, the stability of them can be studied by the linearized models at their equilibriums. The readers are referred to [14] for recent advances.

If the system is stable, an important problem is how to get its solution, especially the numerical one. The simplest model of DDEs is

$$\begin{cases} \dot{x}(t) = \alpha x(t) + \beta x(t-\tau) \\ x(\theta) = \phi(\theta) \end{cases}, \quad \theta \in (0, -\tau). \tag{1}$$

Even for this simple model, the numerical treatment leads to a very complex stability problem (e.g. see [22], [12]). This is because the DDEs are essentially infinite dimensional. In the available literatures, there is a lack of theories which can guarantee that the error between the numerical solution and the accurate solution can be ignored when time is sufficiently large.

On the other hand, if we can obtain all the zeros, denoted by $\{\lambda_n, n \in \mathbb{Z}\}$, of the transcendental function:

$$f(z) = z - \alpha - \beta e^{-z\tau},$$

then we can get the analytic solution

$$x(t) = \sum_{n\in\mathbb{Z}} e^{\lambda_n t} c_k \tag{2}$$

of (1), provided that $\phi(\theta) = \sum_{n\in\mathbb{Z}} e^{\lambda_n \theta} c_k$. It seems that (2) allows us to present the numerical solution by appropriately taking finite many terms of the series $\sum_{n\in\mathbb{Z}} e^{\lambda_n t} c_k$. However, there are still two difficulties. One is whether $\phi(\theta)$ can be expanded according to the exponentials $\{e^{\lambda_n t}, n \in \mathbb{X}\}$, or equivalently, whether the family $\{e^{\lambda_n t}, n \in \mathbb{Z}\}$ forms a basis for some function space. The other is whether the series in (2) converges absolutely.

If we extend the above structure method to differential equations with multiple delays, it may arise some more difficulties. To the best of the author's knowledge, there is no result concerning the solution expansion of delay differential equations according to the root vectors. Nevertheless, the solution expansion is a powerful tool for the dynamical analysis of the system and the numerical simulation of the solution. Moreover, with the help of the solution expansion, we can stabilize the delay system by removing the first finite eigenvalues in the right-half complex plane to the left-half complex plane.

In the present paper, we shall study a class of differential equations with multiple delays as follows:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum\limits_{j=1}^{p} A_j x(t-h_j), \quad t>0, \\ x(0)=r, \\ x(\theta)=\phi(\theta), -h_p \le \theta \le 0, \end{cases} \tag{3}$$

where $A_j \in \mathcal{L}(\mathbb{C}^n), j=0,1,2,\cdots,p$, are real matrices, $0<h_1<h_2<\cdots<h_p$ represent the different constant delays, and $r \in \mathbb{C}^n$ and $\phi(\theta) \in L^2([-h_p,0],\mathbb{C}^n)$ are initial values. We shall provide the solution expansion of (3) and some related theoretical results.

Here we take the function space $L^2([-h_p,0],\mathbb{C}^n)$ as the state space, since one can discuss the convergence of the expansion series in the sense of $L^2$-norm, which is weaker than uniform convergence.

Set

$$\begin{cases} x_0(t) = x(t) \\ x_j(t,s) = x(t-h_j s) \end{cases}, \quad s \in [0,1], j=1,2,\cdots,p.$$

Then the delay system (3) is equivalent to the following system:

$$\begin{cases} \frac{dx_0(t)}{dt} = A_0 x_0(t) + \sum\limits_{j=1}^{p} A_j x_j(t,1), \ t>0, \\ \frac{\partial x_j(t,s)}{\partial t} = -\frac{1}{h_j}\frac{\partial x_j(t,s)}{\partial s}, \ j=1,2,\cdots,p, \\ x_0(0)=r, \\ x_j(0,s)=\phi(-h_j s), \ j=1,2,\cdots,p, \end{cases} \tag{4}$$

where $s \in [0,1]$.

Our aim is to present the expansion of the solution of the system (4) according to the root vectors by means of the semigroup theory. To this end, we shall first give the explicit asymptotic expressions of all the eigenvalues and their corresponding eigenvectors by the standard results in [2] and [3]. However, it seems that we cannot expand the solution, since by the estimation of the spectrum projection we show that the root vectors do not form a basis for the state Hilbert space. Fortunately, using a result in [21] we get the solution expansion according to the root vectors without checking the stability of the delay system and the completeness of the root vectors. Furthermore, we prove that the expansion series converges absolutely, which provides efficient estimation of the order of the error.

The rest of the paper is as follows. In section 2, we formulate the system (4) into an abstract evolutionary equation in an appropriate state Hilbert space. For the sake of the integrity of the paper, we also list some basic results, namely the well-posedness of the system and the explicit expressions of the eigenvalues and their corresponding eigenvectors. In section 3, we prove that the root vectors of the system fail to form a basis for the state space. However, in section 4, we assert that one can expand the solution according to the root vectors for certain time by verifying the results in [21]. Finally, in section 5, we analyze the convergence of the expansion and its simulation and conclude the paper.

## 2 Preliminaries

In this section, we shall present some basic results about the system (4) for further study of the solution expansion in section 3 and 4. The readers are referred to [2] and [3] for a detailed discussion. It is worth-mentioned that the state space we construct here is different from that in those two references and is more comfortable to investigate the solution expansion.

First, we formulate (4) into an appropriate state Hilbert space. For this aim, take the state space $\mathcal{H}$ as

$$\mathcal{H} = \mathbb{C}^n \times \prod_{j=1}^{p} L^2([0,1],\mathbb{C}^n) \tag{5}$$

endowed with inner product

$$[X,Y]_{\mathcal{H}} = (x,y)_{\mathbb{C}^n} + \sum_{j=1}^{p}\int_0^1 (f_j(s),g_j(s))_{\mathbb{C}^n} ds, \tag{6}$$

where $X=(x,f(s))^T, Y=(y,g(s))^T \in \mathcal{H}$.

Define an operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{D}(\mathcal{A}) = \left\{ X \in \mathcal{H} \ \middle| \ \begin{array}{l} X=(r,f_1,f_2,\cdots,f_p) \\ \frac{df_j}{ds} \in L^2([0,1],\mathbb{C}^n) \\ f_j(0)=r, j=1,2,\cdots,p \end{array} \right\}, \tag{7}$$

and

$$\mathcal{A}X = \begin{pmatrix} A_0 & A_1\delta_1 & \cdots & A_p\delta_1 \\ 0 & -\frac{1}{h_1}\frac{d}{ds} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -\frac{1}{h_p}\frac{d}{ds} \end{pmatrix} X, \tag{8}$$

where $\delta_1\phi=\phi(1), \forall\phi(s)\in C[0,1]$.

Set

$$X(t) = (x_0(t), x_1(t,s), x_2(t,s), \cdots, x_p(t,s))^T$$

and

$$X(0) = X_0 = (r, \phi(-h_1 s), \phi(-h_2 s), \cdots, \phi(-h_p s))^T.$$

Then we rewrite (4) into the following equivalent form:

$$\begin{cases} \dfrac{dX(t)}{dt} = \mathcal{A}X(t), \quad t > 0, \\ X(0) = X_0. \end{cases} \tag{9}$$

Similar with the analysis in [3], we can prove the following theorem.

**Theorem 1** *Let $\mathcal{A}$ be defined by (7) and (8). For any $\lambda \in \mathbb{C}$, set*

$$\Delta(\lambda) = \lambda I_n - A_0 - \sum_{j=1}^{p} A_j e^{-\lambda h_j}, \tag{10}$$

*where $I_n$ is the identity matrix on $\mathbb{C}^n$. If $\det \Delta(\lambda) \neq 0$, then $\lambda \in \rho(\mathcal{A})$, and the resolvent of $\mathcal{A}$ is compact and given by*

$$\begin{cases} (\lambda I - \mathcal{A})^{-1} Y = X = (x_0, x_1, x_2, \cdots, x_p)^T, \\ x_0 = \Delta(\lambda)^{-1} \Big[ y_0 + \sum\limits_{j=1}^{p} A_j \int_0^1 e^{\lambda h_j (s-1)} h_j y_j(s) ds \Big], \\ x_j(s) = e^{-\lambda h_j s} x_0 + \int_0^s e^{-\lambda h_j (s-\tau)} h_j y_j(\tau) d\tau, \end{cases} \tag{11}$$

*where $Y = (y_0, y_1, y_2 \cdots, y_p)^T \in \mathcal{H}$. In particular, we have*

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\}. \tag{12}$$

The theorem below indicates the well-posedness of (9). It follows immediately by the verification of Lumer-Phillips Theorem (see [13]) or one can refer to [3] for similar results.

**Theorem 2** *Let $\mathcal{A}$ be defined by (7) and (8). Then $\mathcal{A}$ generates a $C_0$ semigroup on $\mathcal{H}$.*

Next, we exhibit the explicit asymptotic expressions of the eigenvalues of the system operator $\mathcal{A}$ and their corresponding eigenvectors. To get the eigenvalues, we have only to analyze the zeros of $\det \Delta(\lambda)$ due to Theorem 1. Our asymptotic analysis is similar with [2].

Since $\mathcal{A}$ generates a $C_0$ semigroup, there is a constant $M > 0$ such that $\Re\lambda \leq M, \forall \lambda \in \sigma(\mathcal{A})$. Let $N > 0$ be any real number fixed. We consider the spectrum of $\mathcal{A}$ in the strip $-N \leq \Re\lambda \leq M$. Noting that function $e^{-h_j \lambda}$ is bounded in the strip, we have

$$\begin{aligned} & \det \Delta(\lambda) \\ = \ & \lambda^n \det \left( I_n - \lambda^{-1} A_0 - \lambda^{-1} \sum_{j=1}^{p} A_j e^{-h_j \lambda} \right) \\ \simeq \ & \lambda^n \left[ 1 - O(\lambda^{-1}) \right] \end{aligned}$$

when $|\lambda|$ is sufficiently large. Hence there is no zero of $\det \Delta(\lambda)$ in this strip when $|\lambda|$ is large enough. As $\det \Delta(\lambda)$ is an entire function on $\mathbb{C}$, there are at most finite zeros of $\det \Delta(\lambda)$ in the strip $-N \leq \Re\lambda \leq M$.

Now we consider the asymptotic distribution of zeros of $\det \Delta(\lambda)$ as $\Re\lambda \to -\infty$. For the sake of simplification, we assume that $\det(A_p) \neq 0$ and $A_p$ has $n$ distinct eigenvalues: $\mu_1, \mu_2, \cdots, \mu_n$. In this case, we have

$$\begin{aligned} & \det \Delta(\lambda) \\ = \ & e^{-n h_p \lambda} \det \Big[ \lambda e^{h_p \lambda} I_n - A_p - \sum_{j=0}^{p-1} e^{(h_p - h_j)\lambda} A_j \Big]. \end{aligned}$$

Since $0 = h_0 < h_1 < h_2 < \cdots < h_p$, $h_p - h_j > 0$, so

$$|e^{(h_p - h_j)\lambda}| = o\left( \frac{1}{|\Re\lambda|} \right) \quad \text{as} \quad \Re\lambda \to -\infty.$$

Thus we have

$$\begin{aligned} & e^{n h_p \lambda} \det \Delta(\lambda) \\ = \ & \det(\lambda e^{h_p \lambda} I_n - A_p) + O(e^{(h_p - h_{p-1})\lambda}). \end{aligned}$$

Hence, the asymptotic spectra of $\mathcal{A}$ are determined by

$$\det(\lambda e^{h_p \lambda} I_n - A_p) = \prod_{j=1}^{n} (\lambda e^{h_p \lambda} - \mu_j) = 0.$$

From above we see that the spectrum of $\mathcal{A}$ has $n$ asymptotic branches, and each branch is determined by

$$\lambda e^{h_p \lambda} = \mu_j, \quad j = 1, 2, 3, \cdots, n. \tag{13}$$

Now we are in a position to discuss the zeros of the entire function of the form

$$F(z) = z e^{hz} - a, \quad h > 0, \tag{14}$$

where $a \in \mathbb{C}$ is a non-zero number fixed. Note that $F(z)$ in (14) is of the same type treated in [2]. By virtue of [2, Theorem 12.8], we obtain the asymptotic explicit expressions of $\lambda_n$ with large modulus:

$$\lambda_k = \frac{1}{h} \ln \frac{|a|}{|\omega_k|} + i\omega_k + o\left( \frac{\ln k}{k} \right), \ k \in \mathbb{Z},$$

where

$$\omega_k = \begin{cases} \frac{1}{h} [\arg(a) + \frac{(4k-1)\pi}{2}], \ \arg(a) + 2k\pi > \frac{\pi}{2} \\ \frac{1}{h} [\arg(a) + \frac{(4k+1)\pi}{2}], \ \arg(a) + 2k\pi < -\frac{\pi}{2}. \end{cases}$$

Further, we have the following theorem about the asymptotic expression of the spectrum of $\mathcal{A}$.

**Theorem 3** *Assume that $\mathcal{A}$ is defined as before, and that $\det(A_p) \neq 0$ and $\mu_j, j = 1, 2, \cdots, n$ are the distinct eigenvalues of $A_p$. Then the following assertions are true.*

*(1). The spectra of $\mathcal{A}$ distribute symmetrically with respect to the real axis;*

*(2). The asymptotic spectra of $\mathcal{A}$ are given by*

$$\Lambda = \{\xi_k^{(j)} \mid k \in \mathbb{Z},\ j = 1, 2, \cdots, p\}$$

*with*

$$\xi_k^{(j)} = \frac{1}{h_p} \ln \frac{|\mu_j|}{|\omega_k^{(j)}|} + i\omega_k^{(j)} + o\left(\frac{\ln k}{k}\right),$$

*where*

$$\omega_k^{(j)} = \frac{1}{h_p}[\arg(\mu_j) + \frac{(4k-)\pi}{2}]$$

*if* $\arg(\mu_j) + 2k\pi > \frac{\pi}{2}$*; and*

$$\omega_k^{(j)} = \frac{1}{h_p}[\arg(\mu_j) + \frac{(4k+1)\pi}{2}]$$

*if* $\arg(\mu_j) + 2k\pi < -\frac{\pi}{2}$*;*

*(3). Suppose that $\det(A_p) \neq 0$, then the spectra of $\mathcal{A}$ have $n$ asymptotic branches, which are given by*

$$\lambda_k^{(j)} = \xi_k^{(j)} + o\left(k^{-\frac{h_p - h_{p-1}}{h_p}}\right) \tag{15}$$

*where $j = 1, 2, \cdots, n,\ k \in \mathbb{Z}$.*

**Proof:** Note that $A_j = 0, 1, 2 \cdots, p$, are real matrices. When $\lambda \in \mathbb{C}$ such that $\det \Delta(\lambda) = 0$, we also have $\det \Delta(\overline{\lambda}) = 0$. Therefore, $\sigma(\mathcal{A})$ distributes symmetrically with respect to the real axis. The assertion (2) directly comes from the discussions above Theorem 3. Now, let $\lambda_k^{(j)} = \xi_k^{(j)} + \varepsilon_k^{(j)}$ be a zero of $\det \Delta(\lambda) = 0$. Then when $|\Re\lambda|$ is sufficiently large, we have

$$\begin{aligned} 0 &= e^{nh_p\lambda_k^{(j)}} \det \Delta(\lambda_k^{(j)}) \\ &= \prod_{s=1}^{n} \left[\lambda_k^{(j)} e^{h_p \lambda_k^{(j)}} - \mu_s\right] \\ &\quad + O(e^{(h_p - h_{p-1})\lambda_k^{(j)}}), \end{aligned}$$

which is equivalent to

$$\prod_{s=1}^{n} \left[\lambda_k^{(j)} e^{h_p \lambda_k^{(j)}} - \mu_s\right] = O(e^{(h_p - h_{p-1})\lambda_k^{(j)}}).$$

Since

$$\begin{aligned} \lambda_k^{(j)} e^{h_p\lambda_k^{(j)}} &= \xi_k^{(j)} e^{h_p \xi_k^{(j)}} e^{h_p \varepsilon_k^{(j)}} + \varepsilon_k^{(j)} e^{h_p(\xi_k^{(j)} + \varepsilon_k^{(j)})} \\ &= \mu_j e^{h_p\varepsilon_k^{(j)}} + \frac{\mu_j \varepsilon_k^{(j)}}{\xi_k^{(j)}} e^{h_p \varepsilon_k^{(j)}} \end{aligned}$$

and

$$|e^{(h_p - h_{p-1})\xi_k^{(j)}}| = \left|\frac{\mu_j}{\omega_k^{(j)}}\right|^{\frac{h_p - h_{p-1}}{h_p}} \cdot e^{(h_p - h_{p-1})o(\frac{\ln k}{k})},$$

we have

$$\begin{aligned} &\left(\left[\mu_j + \tfrac{\mu_j \varepsilon_k^{(j)}}{\xi_k^{(j)}}\right] e^{h_p \varepsilon_k^{(j)}} - \mu_j\right) \prod_{\substack{s=1,\\ s \neq j}}^{n} \left[\lambda_k^{(j)} e^{h_p \lambda_k^{(j)}} - \mu_s\right] \\ &= O\left(\left|\omega_k^{(j)}\right|^{-\frac{h_p - h_{p-1}}{h_p}}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} &\left(\left[\mu_j + \tfrac{\mu_j \varepsilon_k^{(j)}}{\xi_k^{(j)}}\right] e^{h_p \varepsilon_k^{(j)}} - \mu_j\right) \\ =\ & O\left(\left|\omega_k^{(j)}\right|^{-\frac{h_p - h_{p-1}}{h_p}}\right). \end{aligned}$$

From above we get

$$\mu_j \varepsilon_k^{(j)} \left[\frac{h_p \xi_k^{(j)} + 1}{\xi_k^{(j)}}\right] = O\left(\left|\omega_k^{(j)}\right|^{-\frac{h_p - h_{p-1}}{h_p}}\right).$$

Note that

$$\omega_k^{(j)} = O(k),$$

so we have

$$\varepsilon_k^{(j)} = O\left(k^{-\frac{h_p - h_{p-1}}{h_p}}\right).$$

The desired result follows. □

The following result shows the multiplicity of the eigenvalues of $\mathcal{A}$ and the form of the corresponding eigenvectors (For the similar form of eigenvectors, the readers are also referred to [3]).

**Theorem 4** *Let $\mathcal{A}$ be defined as (7) and (8). Assume that $\det(A_p) \neq 0$ and $A_p$ has $n$ distinct eigenvalues, then $\sigma(\mathcal{A}) = \{\lambda_k^{(j)}, k \in \mathbb{Z}, j = 1, 2, \cdots, n\}$ are simple eigenvalues except finite many eigenvalues. For each $\lambda_k^{(j)}$, a corresponding eigenvector is*

$$\Phi_k^{(j)} = \left(x_k^{(j)}, e^{-sh_1\lambda_k^{(j)}} x_k^{(j)}, \cdots, e^{-sh_p\lambda_k^{(j)}} x_k^{(j)}\right)^T,$$

*where $x_k^{(j)}$ is a non-zero solution to equation $\Delta(\lambda_k^{(j)})x = 0$.*

**Proof:** Since $\det(A_p) \neq 0$ and $A_p$ has $n$ distinct eigenvalues, when $|\lambda|$ is sufficiently large,

$$\begin{aligned} & e^{nh_p\lambda} \det \Delta(\lambda) \\ =\ & \det(\lambda e^{h_p\lambda} I_n - A_p) + O(e^{(h_p - h_{p-1})\lambda}) \end{aligned}$$

has simple zero, so $\mathrm{rank}(\Delta(\lambda_k^{(j)}) = n-1$. This means that $\lambda_k^{(j)}$ is a simple eigenvalue of $\mathcal{A}$. Let $x_k^{(j)}$ be a non-zero solution of equation

$$\Delta(\lambda_k^{(j)})x = 0.$$

Then according to Theorem 1, it holds that

$$\Phi_k^{(j)} = \left(x_k^{(j)}, e^{-sh_1\lambda_k^{(j)}}x_k^{(j)}, \cdots, e^{-sh_p\lambda_k^{(j)}}x_k^{(j)}\right)^T,$$

$\Phi_k^{(j)} \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}\Phi_k^{(j)} = \lambda_k^{(j)}\Phi_k^{(j)}$. □

## 3 Non-Basis Property of the Root Vectors of $\mathcal{A}$

In this section we shall prove that the root vectors of $\mathcal{A}$ fail to form a basis for $\mathcal{H}$. To this end, we shall estimate the norm of the spectral projection $E(\lambda;\mathcal{A})$ corresponding to eigenvalue $\lambda \in \sigma(\mathcal{A})$.

First, let us discuss the adjoint operator of $\mathcal{A}$. For any $F = (x,f)^T \in \mathcal{D}(\mathcal{A})$ and $G = (y,g)^T \in \mathbb{C}^n \times [H^1(0,1)]^p$, we have

$$\begin{aligned}
[\mathcal{A}F, G]_{\mathcal{H}} &= \left(A_0x + \sum_{j=1}^{p} A_jf_j(1), y\right)_{\mathbb{C}^n} \\
&\quad - \sum_{j=1}^{p}\int_0^1 \frac{1}{h_j}(f_j'(s), g_j(s))_{\mathbb{C}^n}ds \\
&= (x, A_0^*y)_{\mathbb{C}^2} + \sum_{j=1}^{p}(f_j(1), A_j^*y)_{\mathbb{C}^n} \\
&\quad - \sum_{j=1}^{p}\frac{1}{h_j}(f_j(1), g_j(1))_{\mathbb{C}^n} \\
&\quad + \sum_{j=1}^{p}\frac{1}{h_j}(x, g_j(0))_{\mathbb{C}^n} \\
&\quad + \sum_{j=1}^{p}\int_0^1 \frac{1}{h_j}(f_j(s), g_j'(s))_{\mathbb{C}^n}ds \\
&= \left(x, A_0^*y + \sum_{j=1}^{p}\frac{1}{h_j}g_j(0)\right)_{\mathbb{C}^n} \\
&\quad + \sum_{j=1}^{p}\left(f_j(1), A_j^*y - \frac{1}{h_j}g_j(1)\right)_{\mathbb{C}^n} \\
&\quad + \sum_{j=1}^{p}\int_0^1 \frac{1}{h_j}(f_j(s), g_j'(s))_{\mathbb{C}^n}ds.
\end{aligned}$$

Therefore we have

$$\mathcal{D}(\mathcal{A}^*) = \left\{ Y \in \mathcal{H} \,\middle|\, \begin{array}{l} Y = (y, g_1, g_2\cdots, g_p)^T \\ \frac{dg_j}{ds} \in L^2([0,1],\mathbb{C}^n) \\ g_j(1) = h_jA_j^*y \\ j = 1,2,\cdots,p \end{array} \right\} \tag{16}$$

and

$$\mathcal{A}^*Y = \begin{pmatrix} A_0^* & \frac{1}{h_1}\delta_0 & \cdots & \frac{1}{h_p}\delta_0 \\ 0 & \frac{1}{h_1}\frac{d}{ds} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{h_p}\frac{d}{ds} \end{pmatrix} Y, \tag{17}$$

where $\delta_0\psi = \psi(0), \forall\psi(s) \in C[0,1]$ and $A_j^*$ denotes the conjugate transpose of matrix $A_j, j = 0,1,\cdots,p$.

In the same way as the analysis of the eigenvectors $\mathcal{A}$, a direct calculation and the theory of adjoint operator deduce the following theorem.

**Theorem 5** *Let $\mathcal{A}^*$ be defined by (16)–(17). Then the spectrum of $\mathcal{A}^*$ is given by*

$$\sigma(\mathcal{A}^*) = \overline{\sigma(\mathcal{A})} = \{\overline{\lambda} \mid \det\Delta(\lambda)^* = 0\},$$

*where $\Delta(\lambda)^* = \overline{\lambda}I_n - A_0^\tau - \sum_{j=1}^{p} A_j^\tau e^{-\overline{\lambda}h_j}$. For $\overline{\lambda} \in \sigma(\mathcal{A}^*)$, a corresponding eigenvector is*

$$\Psi_\lambda = \left(y, h_1e^{-h_1\overline{\lambda}(1-s)}A_1^\tau y, \cdots, h_pe^{h_p\overline{\lambda}(1-s)}A_p^\tau y\right)^T,$$

*where $y$ is a non-zero solution to $\Delta(\lambda)^*y = 0$.*

The next theorem estimate the norm of the spectrum projection, which implies the non-basis property of the root vectors of $\mathcal{A}$.

**Theorem 6** *Let $\lambda \in \sigma(\mathcal{A})$ be a simple eigenvalue of $\mathcal{A}$ and $E(\lambda;\mathcal{A})$ be the corresponding Riesz projection. Then for any $X \in \mathcal{H}$, we have*

$$E(\lambda;\mathcal{A})X = [X, \Psi_\lambda]_{\mathcal{H}}\Phi_\lambda,$$

*where $\Phi_\lambda$ and $\Psi_\lambda$ are given in Theorem 4 and Theorem 5 respectively such that $[\Phi_\lambda, \Psi_\lambda]_{\mathcal{H}} = 1$. Moreover, when $\Re\lambda \to -\infty$, $E(\lambda;\mathcal{A})$ has the estimate*

$$\begin{aligned}
||E(\lambda;\mathcal{A})|| &= ||\Psi_\lambda||_{\mathcal{H}}||\Phi_\lambda||_{\mathcal{H}} \\
&\approx \frac{e^{h_p|\Re\lambda|}||A_p^\tau y_{\overline{\lambda}}||}{2h_p|\Re\lambda||(A_px_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n}|}.
\end{aligned} \tag{18}$$

*Therefore, the eigenvector sequence $\{\Phi_\lambda \mid \lambda \in \sigma(\mathcal{A})\}$ does not form a basis for $\mathcal{H}$.*

**Proof:** For any $\lambda, \zeta \in \sigma(A)$, $\lambda \neq \zeta$, let $\Phi_\lambda$ be an eigenvector of $\mathcal{A}$ corresponding to $\lambda$, and let $\Psi_\zeta$ be an eigenvector of $\mathcal{A}^*$ corresponding to $\overline{\zeta}$. It holds that

$$\begin{aligned}\lambda[\Phi_\lambda, \Psi_\zeta]_{\mathcal{H}} &= [\mathcal{A}\Phi_\lambda, \Psi_\zeta]_{\mathcal{H}} = [\Phi_\lambda, \mathcal{A}^*\Psi_\zeta]_{\mathcal{H}} \\ &= \zeta[\Phi_\lambda, \Psi_\zeta]_{\mathcal{H}}.\end{aligned}$$

So when $\lambda \neq \zeta$, we have $[\Phi_\lambda, \Psi_\zeta]_{\mathcal{H}} = 0$.

Let $\lambda \in \sigma(\mathcal{A})$ be a simple eigenvalue of $\mathcal{A}$ and $E(\lambda; \mathcal{A})$ be the corresponding Riesz spectral projection. Then $E(\lambda; \mathcal{A})\mathcal{H}$ is a 1-dimensional subspace of $\mathcal{H}$ spanned by $\Phi_\lambda$. Let $\Psi_\lambda$ be an eigenvector of $\mathcal{A}^*$ corresponding to $\overline{\lambda}$ such that $[\Phi_\lambda, \Psi_\lambda]_{\mathcal{H}} = 1$, then for any $X \in \mathcal{H}$, we have

$$E(\lambda; \mathcal{A})X = [X, \Psi_\lambda]_{\mathcal{H}}\Phi_\lambda.$$

According to Theorem 4 and Theorem 5, $\Phi_\lambda$ and $\Psi_\lambda$ are of the form

$$\Phi_\lambda = \Big(x(\lambda), e^{-h_1\lambda s}x(\lambda), \cdots, e^{-h_p\lambda s}x(\lambda)\Big)^T,$$

where $\Delta(\lambda)x(\lambda) = 0$; and

$$\begin{aligned}\Psi_\lambda &= \Big(y(\overline{\lambda}), h_1 e^{-h_1\overline{\lambda}(1-s)}A_1^\tau y(\overline{\lambda}), \\ &\quad \cdots, h_p e^{-h_p\overline{\lambda}(1-s)}A_p^\tau y(\overline{\lambda})\Big)^T,\end{aligned}$$

where $\Delta(\lambda)^* y(\overline{\lambda}) = 0$.

In order that $[\Phi_\lambda, \Psi_\lambda]_{\mathcal{H}} = 1$, we shall choose suitable vectors $x(\lambda)$ and $y(\overline{\lambda})$. In what follows, we will finish the proof by two steps.

*Step 1.* We choose $x_\lambda \in \mathbb{C}^n$ such that $\Delta(\lambda)x_\lambda = 0$ and $||x_\lambda||_{\mathbb{C}^n} = 1$.

Set

$$x(\lambda) = \sqrt{|\Re\lambda|}e^{\lambda h_p}x_\lambda$$

and

$$\Phi(\lambda) = \Big(x(\lambda), e^{-h_1\lambda s}x(\lambda), \cdots, e^{-h_p\lambda s}x(\lambda)\Big)^T.$$

Then we have

$$||\Phi(\lambda)||^2_{\mathcal{H}} = |\Re\lambda|e^{2\Re\lambda h_p}\left(1 + \sum_{j=1}^p \frac{e^{-2\Re\lambda h_j} - 1}{-2\Re\lambda h_j}\right). \tag{19}$$

From (19) we see that when $\Re\lambda \to -\infty$,

$$||\Phi(\lambda)||^2_{\mathcal{H}} \approx \frac{1}{2h_p}.$$

*Step 2.* We choose $y_{\overline{\lambda}} \in \mathbb{C}^n$ such that $\Delta(\lambda)^* y_{\overline{\lambda}} = 0$ and $||y_{\overline{\lambda}}||_{\mathbb{C}^n} = 1$.

Set

$$y(\overline{\lambda}) = \eta(\lambda)y_{\overline{\lambda}}$$

and

$$\begin{aligned}\Psi(\lambda) &= \Big(y(\overline{\lambda}), h_1 e^{-h_1\overline{\lambda}(1-s)}A_1^\tau y(\overline{\lambda}), \\ &\quad \cdots, h_p e^{-h_p\overline{\lambda}(1-s)}A_p^\tau y(\overline{\lambda})\Big)^T,\end{aligned}$$

where

$$\begin{aligned}\overline{\eta(\lambda)} = \Big[\sqrt{|\Re\lambda|}\Big(&(x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n}e^{\lambda h_p} \\ &+ \sum_{j=1}^{p-1} h_j e^{\lambda(h_p - h_j)}(A_j x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n} \\ &+ h_p(A_p x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n}\Big)\Big]^{-1}.\end{aligned} \tag{20}$$

Then we have

$$\begin{aligned}&[\Phi(\lambda), \Psi(\lambda)]_{\mathcal{H}} \\ =\ & (x(\lambda), y(\overline{\lambda}))_{\mathbb{C}^n} \\ &+ \sum_{j=1}^p h_j \int_0^1 e^{-\lambda h_j s}(x(\lambda), A_j^\tau y(\overline{\lambda}))_{\mathbb{C}^n} e^{-h_1\lambda(1-s)}ds \\ =\ & (x(\lambda), y(\overline{\lambda}))_{\mathbb{C}^n} + \sum_{j=1}^p h_j e^{-\lambda h_j}(A_j x(\lambda), y(\overline{\lambda}))_{C^n} \\ =\ & \sqrt{|\Re\lambda|}\,\overline{\eta(\lambda)}\Big[e^{\lambda h_p}(x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n} \\ &+ \sum_{j=1}^p h_j e^{\lambda(h_p - h_j)}(A_j x(\lambda), y(\overline{\lambda}))_{C^n}\Big] \\ =\ & 1.\end{aligned}$$

Furthermore,

$$\begin{aligned}&||\Psi(\lambda)||^2_{\mathcal{H}} \\ =\ & ||y(\overline{\lambda})||^2_{\mathbb{C}^n} \\ &+ \sum_{j=1}^p h_j^2 ||A_j^\tau y(\overline{\lambda})||^2_{\mathbb{C}^n} \int_0^1 e^{-2\Re\lambda h_j(1-s)}ds \\ =\ & |\eta(\lambda)|^2 \\ &+ |\eta(\lambda)|^2 \sum_{j=1}^p h_j^2 ||A_j^\tau y_{\overline{\lambda}}||^2_{\mathbb{C}^n}\frac{e^{-2\Re\lambda h_j} - 1}{-2\Re\lambda h_j} \\ =\ & |\eta(\lambda)|^2\left(1 + \sum_{j=1}^p h_j^2 ||A_j^\tau y_{\overline{\lambda}}||^2_{\mathbb{C}^n}\frac{e^{-2\Re\lambda h_j} - 1}{-2\Re\lambda h_j}\right).\end{aligned}$$

So when $\Re\lambda \to -\infty$, we get from above that

$$||E(\lambda; \mathcal{A})|| \approx \frac{e^{h_p|\Re\lambda|}||A_p^\tau y_{\overline{\lambda}}||_{\mathbb{C}^n}}{2h_p|\Re\lambda||(A_p x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n}|} \to \infty.$$

According to the necessary condition for a sequence to be a basis, the eigenvector sequence $\{\Phi(\lambda) \mid \lambda \in \sigma(\mathcal{A})\}$ does not form a basis for $\mathcal{H}$ (see [15]). $\square$

# 4 Expansion of Solution of the System

In this section we expand of solution of (9) according to the root vectors although we have proven that they are not a basis for $\mathcal{H}$. We begin with recalling some basic notations and results in [21].

Let $\{S(t)\}_{t\geq 0}$ be a $C_0$ semigroup on a Banach space $\mathbb{X}$ and $\mathfrak{A}$ be its generator. Assume that $\mathfrak{A}$ has the discrete spectra, that is, $\sigma(\mathfrak{A}) = \sigma_p(\mathfrak{A}) = \{\lambda_n; n \in \mathbb{Z}\}$ consists of all isolated eigenvalues of finite multiplicity. For each $\lambda_n \in \sigma(\mathfrak{A})$, denote by $E(\lambda_n; \mathfrak{A})$ its Riesz projection on $\mathbb{X}$. We define the $S(t)-$invariant spectral-subspace of $\mathbb{X}$ by

$$Sp(\mathfrak{A}) = \overline{\operatorname{span}\left\{\sum_{j=1}^{m} E(\lambda_j; \mathfrak{A})x \;\middle|\; x \in \mathbb{X}; \forall\, m \in \mathbb{Z}\right\}},$$

and another $S(t)-$invariant subspace by

$$\mathcal{M}_\infty = \{x \in \mathbb{X} \mid E(\lambda; \mathfrak{A})x = 0, \forall \lambda \in \sigma(\mathfrak{A})\}.$$

Clearly, $Sp(\mathfrak{A}) \cap \mathcal{M}_\infty = \{0\}$, and $\overline{Sp(\mathfrak{A}) + \mathcal{M}_\infty} \subseteq \mathbb{X}$.

For each $\lambda_n \in \sigma(\mathfrak{A})$, we denote the algebraic multiplicity of $\lambda_n$ by $m_n$, and define operators

$$D_n = (\mathfrak{A} - \lambda_n I)E(\lambda_n; \mathfrak{A}), \quad D_n^0 = E(\lambda_n; \mathfrak{A}).$$

Then for each $n \in \mathbb{N}$, $D_n$ is a bounded linear operator with the property that

$$D_n^k = (\mathfrak{A} - \lambda_n I)^k E(\lambda_n; \mathfrak{A}), \quad D_n^{m_n} = 0.$$

The following two Lemma have been proved in [21].

**Lemma 7** *Let $S(t)$ be a $C_0$ semigroup on a Banach space $\mathbb{X}$ and $\mathfrak{A}$ be its generator. Suppose that $\mathfrak{A}$ satisfies the following conditions:*

*(c1). there exist positive constants $M_1$ , $\rho_1$ and $\rho_3$ such that*

$$\sum_{k=0}^{m_n} \frac{t^k \|D_n^k\|}{k!} \leq M_1 e^{-\rho_1 \Re\lambda_n} e^{\rho_3 t}, \quad \forall n \in \mathbb{N},\ t \geq 0; \tag{21}$$

*(c2). there exists a $\tau_0 > 0$ such that the series $\sum\limits_{n=1}^{\infty} e^{\Re\lambda_n \tau_0}$ converges.*

*Then we can define two families of operators parameterized on $[\tau_0 + \rho_1, \infty)$*

$$S_1(t) : \mathbb{X} \to Sp(\mathfrak{A}), \quad S_2(t) : \mathbb{X} \to \mathcal{M}_\infty,$$

*such that*

*1). $S_1(t)$ is a compact operator; $S_1(t)$ and $S_2(t)$ are strongly continuous;*

*2). $S_j(t)S(s) = S(s)S_j(t) = S_j(t+s)$, for $t \geq \tau_0 + \rho_1,\ s \geq 0,\ j = 1, 2;$*

*3). $S(t)$ has a decomposition*

$$S(t) = S_1(t) + S_2(t), \quad t \geq \tau_0 + \rho_1.$$

*In addition, if the following condition on the spectrum of $\mathfrak{A}$ holds:*

*(c3). there exist constants $M_2 > 0$ and $\rho_2 > 0$ such that*

$$|\Im\lambda_n| \leq M_2 e^{-\rho_2 \Re\lambda_n},$$

*then, $S_1(t)x$ is differentiable in $(\tau_0 + \rho_1 + \rho_2, \infty)$ for each $x \in \mathbb{X}$.*

**Lemma 8** *Let $S(t)$ be a $C_0$ semigroup on a Banach space $\mathbb{X}$ and $\mathfrak{A}$ be its generator. Suppose that the conditions (c1)–(c3) in Lemma 7 hold. In addition, if one of the following conditions is fulfilled:*

*1). the generalized eigenvectors of $\mathfrak{A}$ are complete in $\mathbb{X}$;*

*2). the restriction of the resolvent of $\mathfrak{A}$ to $\mathcal{M}_\infty$ is an entire function with values in $\mathbb{X}$ of finite exponential type $h$.*

*Then $S(t)$ is a differentiable semigroup for $t > \tau_1$, where*

$$\tau_1 := \max\{\tau_0 + \rho_1 + \rho_2, \tau_0 + \rho_1 + h\}. \tag{22}$$

In what following, we shall verify the conditions in Lemma 7 and Lemma 8.

**Proposition 9** *Let $\mathcal{A}$ be defined by (7) and (8), then for $\lambda \in \sigma(\mathcal{A})$ with $|\lambda|$ sufficiently large, $\lambda$ is a simple eigenvalue of $\mathcal{A}$, and the corresponding Riesz projector $E(\lambda; \mathcal{A})$ has the estimate*

$$\|E(\lambda; \mathcal{A})\| \leq M e^{-h_p \Re\lambda}, \tag{23}$$

*where $M > 0$ is a constant. Hence the condition (c1) in Lemma 7 holds.*

**Proof:** According to Theorem 5, $E(\lambda; \mathcal{A})$ has the estimate

$$||E(\lambda; \mathcal{A})|| \approx \frac{e^{h_p|\Re\lambda|}||A_p^\tau y_{\overline{\lambda}}||_{\mathbb{C}^n}}{2h_p|\Re\lambda||(A_p x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n}|}.$$

So we only need to show that $|(A_p x_\lambda, y_\lambda)_{\mathbb{C}^n}| \geq \delta$ for some constant $\delta > 0$. Since

$$\Delta(\lambda)x_\lambda = 0, \quad \Delta(\lambda)^* y_{\overline{\lambda}} = 0,$$

$$||x_\lambda||_{\mathbb{C}^n} = 1, \quad ||y_{\overline{\lambda}}||_{\mathbb{C}^n} = 1,$$

we have $(x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n} \neq 0, \forall \lambda \in \sigma(\mathcal{A})$. Hence we can set

$$c = \inf_{\lambda\in\sigma(\mathcal{A})} |(x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n}|.$$

Note that

$$\begin{aligned} 0 &= \big(\Delta(\lambda)x_\lambda, y_{\overline{\lambda}}\big)_{\mathbb{C}^n} \\ &= \lambda(x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n} - (A_0 x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n} \\ &\quad -\sum_{j=1}^{p} e^{-h_j\lambda}(A_j x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} 0 &= \lambda e^{h_p\lambda}(x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n} - (A_p x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n} \\ &\quad -\sum_{j=0}^{p-1} e^{(h_p-h_j)\lambda}(A_j x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n}, \end{aligned}$$

where $h_0 = 0$. When $\Re\lambda \to -\infty$, $e^{(h_p-h_j)\lambda} \to 0$, $j = 0, 1, \cdots, p-1$, so we have

$$(A_p x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n} = \lambda e^{h_p\lambda}(x_\lambda, y_{\overline{\lambda}})_{\mathbb{C}^n} + o(1).$$

For $\lambda = \lambda_k^{(j)} \in \sigma(\mathcal{A})$, we have the asymptotic expression $\lambda_k^{(j)} e^{h_p\lambda_k^{(j)}} = \mu_j + o(k^{-\frac{h_p-h_{p-1}}{h_p}})$. Thus

$$\begin{aligned} |(A_p x_{\lambda_k^{(j)}}, y_{\overline{\lambda_k^{(j)}}})_{\mathbb{C}^n}| &= |\mu_j (x_{\lambda_k^{(j)}}, y_{\overline{\lambda_k^{(j)}}})_{\mathbb{C}^n}| + o(1) \\ &\geq c|\mu_j| + o(1). \end{aligned}$$

For $|\Re\lambda| > d$, taking

$$M = \frac{||A_p^\tau||}{2h_p dc \min\limits_{1\leq j\leq n} |\mu_j|},$$

we have

$$\begin{aligned} \frac{||A_p^\tau y_\lambda||_{\mathbb{C}^n}}{2h_p|\Re\lambda||(A_p x_\lambda, y_\lambda)_{\mathbb{C}^n}|} &\leq \frac{||A_p^\tau||}{2h_p dc \min\limits_{1\leq j\leq n} |\mu_j|} \\ &:= M. \end{aligned}$$

Therefore, we get

$$||E(\lambda;\mathcal{A})|| \leq M e^{-h_p\Re\lambda}$$

for all $\lambda \in \sigma(\mathcal{A})$ with $|\Re\lambda|$ sufficiently large. The proof is then complete. □

**Proposition 10** *Let $\mathcal{A}$ be defined as (7) and (8). Then for $\tau_0 > h_p$, we have*

$$\sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} \sum_{j=1}^{n} e^{\Re\lambda_k^{(j)}\tau_0} < \infty. \tag{24}$$

*Further, there is a constant $M > 0$ such that*

$$|\lambda_k^{(j)}| \leq M e^{-h_p\Re\lambda_k^{(j)}}, \quad k \in \mathbb{Z}, j \in \{1, 2, \cdots, n\}. \tag{25}$$

*Hence the conditions (c2) and (c3) in Lemma 7 are fulfilled.*

**Proof:** According to Theorem 3, the spectrum of $\mathcal{A}$ has the asymptotic expression

$$\lambda_k^{(j)} = \frac{1}{h_p}\ln\left|\frac{\mu_j}{\omega_k^{(j)}}\right| + i\omega_k^{(j)} + o(\frac{1}{\ln k}),$$

where

$$\omega_k^{(j)} = \frac{1}{h_p}[\arg\mu_j + \frac{(4k-1)\pi}{2}],$$

if $\arg\mu_j + 2k\pi - \frac{\pi}{2} > 0$; and

$$\omega_k^{(j)} = \frac{1}{h_p}[\arg\mu_j + +\frac{(4k+1)\pi}{2}],$$

if $\arg\mu_j + 2k\pi + \frac{\pi}{2} < 0$.

Thus we have

$$e^{\Re\lambda_k^{(j)}\tau_0} = \left|\frac{\mu_j}{\omega_k^{(j)}}\right|^{\frac{\tau_0}{h_p}} \leq \frac{C}{|k|^{\frac{\tau_0}{h_p}}}.$$

When $\tau_0 > h_p$, it holds that

$$\sum_{k=-\infty, k\neq 0}^{+\infty} \frac{1}{|k|^{\frac{\tau_0}{h_p}}} < \infty,$$

and hence

$$\sum_{k=-\infty,k\neq 0}^{+\infty} \sum_{j=1}^{n} e^{\Re\lambda_k^{(j)}\tau_0} < \infty.$$

Since $\lambda_k^{(j)}, \forall k \in \mathbb{Z}$ are the solutions of the equation

$$\lambda e^{h_p\lambda} = \mu_j + o(\frac{1}{\ln k}),$$

we have

$$|\lambda_k^{(j)}| \leq (|\mu_j| + 1)e^{-h_p\Re\lambda_k^{(j)}}, \quad j = 1, 2, \cdots, n$$

for sufficient large $k$. Therefore, there is a constant $M > \max\limits_j\{|\mu_j| + 1\}$ such that

$$|\lambda_k^{(j)}| \leq M e^{-h_p\Re\lambda_k^{(j)}}, \quad \forall k \in \mathbb{Z}, \forall j = 1, 2, \cdots, n.$$

The desired results follow. □

**Proposition 11** *Let $\mathcal{A}$ be defined as (7) and (8). Then its resolvent is a meromorphic function on $\mathbb{C}$ of finite exponential type at most $2h_p$. Therefore, the condition 2) in Lemma 8 holds.*

**Proof:** According to (11), the resolvent of $\mathcal{A}$ have the expression

$$
\begin{aligned}
& R(\lambda,\mathcal{A})\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix} \\
= & \begin{pmatrix} \Delta(\lambda)^{-1}y_0 \\ e^{-\lambda h_1 s}x_0 \\ e^{-\lambda h_2 s}x_0 \\ \vdots \\ e^{-\lambda h_p s}x_0 \end{pmatrix} \\
+ & \begin{pmatrix} +\Delta(\lambda)^{-1}\sum\limits_{j=1}^{p} A_j \int_0^1 e^{\lambda h_j(s-1)}h_j y_j(s)ds \\ \int_0^s e^{-\lambda h_1(s-\tau)}h_1 y_1(\tau)d\tau \\ \int_0^s e^{-\lambda h_2(s-\tau)}h_2 y_2(\tau)d\tau \\ \vdots \\ \int_0^s e^{-\lambda h_p(s-\tau)}h_p y_p(\tau)d\tau \end{pmatrix}.
\end{aligned}
$$

For each $j$ and $s\in(0,1)$, we have

$$
\begin{aligned}
& \left\| e^{-\lambda h_j s}x_0 + \int_0^s e^{-\lambda h_j(s-\tau)}h_j y_j(\tau)d\tau \right\|_{\mathbb{C}^n} \\
\leq\ & e^{-\Re\lambda h_j s}||x_0||_{\mathbb{C}^n} \\
& + h_j||y_j||_{L^2[0,1]}\sqrt{\frac{1-e^{-2\Re\lambda h_j s}}{2\Re\lambda h_j}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \left\| e^{-\lambda h_j s}x_0 + \int_0^s e^{-\lambda h_j(s-\tau)}h_j y_j(\tau)d\tau \right\|_{L^2[0,1]} \\
\leq\ & \Big[\int_0^1 \Big\| e^{-\lambda h_j s}x_0 \\
& + \int_0^s e^{-\lambda h_j(s-\tau)}h_j y_j(\tau)d\tau\Big\|^2_{\mathbb{C}^n} ds\Big]^{\frac{1}{2}} \\
\leq\ & \Big[\int_0^1 \Big[ e^{-\Re\lambda h_j s}\|x_0\|_{\mathbb{C}^n} \\
& + h_j\|y_j\|_{L^2[0,1]}\sqrt{\frac{1-e^{-2\Re\lambda h_j s}}{2\Re\lambda h_j}}\Big]^2 ds\Big]^{\frac{1}{2}} \\
\leq\ & \sqrt{2}\Big[\int_0^1 e^{-2\Re\lambda h_j s}\|x_0\|^2_{\mathbb{C}^n}ds \\
& + \int_0^1 h_j^2\|y_j\|^2_{L^2[0,1]}\frac{1-e^{-2\Re\lambda h_j s}}{2\Re\lambda h_j}ds\Big]^{\frac{1}{2}} \\
\leq\ & \sqrt{2}\|x_0\|_{\mathbb{C}^n}\Big[\frac{1-e^{-2\Re\lambda h_j}}{2\Re\lambda h_j}\Big]^{\frac{1}{2}} \\
& + \sqrt{2}h_j\|y_j\|_{L^2[0,1]}\Big[\frac{1}{2\Re\lambda h_j} - \frac{1-e^{-2\Re\lambda h_j}}{(2\Re\lambda h_j)^2}\Big]^{\frac{1}{2}} \\
\leq\ & M_1 e^{h_p|\Re\lambda|}(||x_0||_{\mathbb{C}^n} + ||y_j||_{L^2[0,1]}),
\end{aligned}
$$

where $M_1>0$ is a constant.

For $x_0$, we have

$$
\begin{aligned}
& ||x_0||_{\mathbb{C}^n} \\
\leq\ & ||\Delta^{-1}(\lambda)\|_2 \\
& \cdot\left\| y_0 + \sum_{j=1}^{p} A_j \int_0^1 e^{-\lambda h_j(1-s)}h_j y_j(s)ds \right\|_{\mathbb{C}^n} \\
\leq\ & ||\Delta^{-1}(\lambda)\|_2\Big[\|y_0\|_{\mathbb{C}^n} \\
& + \sum_{j=1}^{p} h_j||A_j||_2 \int_0^1 e^{-\Re\lambda h_j(1-s)}\|y_j(s)\|_{\mathbb{C}^n}ds\Big] \\
\leq\ & ||\Delta^{-1}(\lambda)\|_2\Big[\|y_0\|_{\mathbb{C}^n} \\
& + \sum_{j=1}^{p} h_j||A_j||_2\|y_j\|_{L^2[0,1]}\sqrt{\frac{1-e^{-2\Re\lambda h_j}}{2\Re\lambda h_j}}\Big] \\
\leq\ & M_2 e^{|\Re\lambda|h_p}||\Delta^{-1}(\lambda)\|_2 \\
& \cdot\Big[\|y_0\|_{\mathbb{C}^n} + \sum_{j=1}^{p}\|y_j\|_{L^2[0,1]}\Big],
\end{aligned}
$$

where $M_2>0$ is a constant.

According to [9, 4.12, pp28], if $T$ is an invertible matrix, then there is a constant $\gamma>0$, which is independent of $T$ and dependent only on the norm of $\mathbb{C}^n$, such that

$$||T^{-1}|| \leq \gamma\frac{||T||^{n-1}}{|\det(T)|}.$$

Since $\Delta(\lambda)=\lambda I_n - A_0 - \sum\limits_{j=1}^{p} e^{-h_j\lambda}A_j$ is an invertible matrix, we have

$$||\Delta^{-1}(\lambda)|| \leq \gamma\frac{||\Delta(\lambda)||^{n-1}}{|\det\Delta(\lambda)|}.$$

Note that $||T||_2 \leq ||T||_F$, where $||T||_F$ is the frobenious norm of matrix. Now we calculate

$||\det\Delta(\lambda)||_F$. Since

$$\Delta(\lambda) = \begin{pmatrix} \lambda - a_{11}^{(0)} - \sum\limits_{j=1}^{p} e^{-h_j\lambda}a_{11}^{(j)} & -a_{12}^{(0)} - \sum\limits_{j=1}^{p} e^{-h_j\lambda}a_{12}^{(j)} & \cdots & -a_{1n}^{(0)} - \sum\limits_{j=1}^{p} e^{-h_j\lambda}a_{1n}^{(j)} \\ -a_{21}^{(0)} - \sum\limits_{j=1}^{p} e^{-h_j\lambda}a_{21}^{(j)} & \lambda - a_{22}^{(0)} - \sum\limits_{j=1}^{p} e^{-h_j\lambda}a_{22}^{(j)} & \cdots & -a_{2n}^{(0)} - \sum\limits_{j=1}^{p} e^{-h_j\lambda}a_{2n}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1}^{(0)} - \sum\limits_{j=1}^{p} e^{-h_j\lambda}a_{n1}^{(j)} & -a_{n2}^{(0)} - \sum\limits_{j=1}^{p} e^{-h_j\lambda}a_{n2}^{(j)} & \cdots & \lambda - a_{nn}^{(0)} - \sum\limits_{j=1}^{p} e^{-h_j\lambda}a_{nn}^{(j)} \end{pmatrix},$$

so

$$\begin{aligned} & ||\Delta(\lambda)||_F^2 \\ = & \sum_{k=1}^{n}\left|\lambda - a_{kk}^{(0)} - \sum_{j=1}^{p} e^{-h_j\lambda}a_{kk}^{(j)}\right|^2 \\ & + \sum_{i,k=1,i\neq k}^{n}\left|a_{ik}^{(0)} + \sum_{j=1}^{p} e^{-h_j\lambda}a_{ik}^{(j)}\right|^2 \\ \leq & \sum_{k=1}^{n}\left[1 + |a_{kk}^{(0)}|^2 + \sum_{j=1}^{p}|a_{kk}^{(j)}|^2\right] \\ & \cdot\left[|\lambda|^2 + \sum_{j=0}^{p} e^{-2h_j\Re\lambda}\right] \\ & + \sum_{i,k=1,i\neq k}^{n}\sum_{j=0}^{p}|a_{ik}^{(j)}|^2\sum_{j=0}^{p} e^{-2h_j\Re\lambda} \\ \leq & \left(|\lambda|^2 + \sum_{j=0}^{p} e^{-2h_j\Re\lambda}\right)\left[n + \sum_{j=0}^{p}||A_j||_F^2\right]. \end{aligned}$$

For $\Delta(\lambda)$, when $\Re\lambda > 0$, we have

$$\begin{aligned} \det\Delta(\lambda) &= \det\left(\lambda I_n - A_0 - \sum_{j=1}^{p} e^{-h_j\lambda}A_j\right) \\ &= \det(\lambda I_n - A_0) + o(1). \end{aligned}$$

Thus

$$\begin{aligned} \|\Delta^{-1}(\lambda)\|^2 &\leq \gamma^2\frac{||\Delta(\lambda)||_F^{2(n-1)}}{|\det\Delta(\lambda)|^2} \\ &\leq \gamma^2\left[n + \sum_{j=0}^{p}||A_j||_F^2\right]^{n-1} \\ &\quad\cdot\frac{\left(|\lambda|^2 + \sum\limits_{j=0}^{p} e^{-2h_j\Re\lambda}\right)^{n-1}}{|\det(\lambda I_n - A_0) + o(1)|^2} \\ &\leq M_3, \end{aligned}$$

where $M_3 > 0$ is a constant.

Indeed, for any constants $c < d$, when $c \leq \Re\lambda \leq d$, function $e^{-h_j\lambda}$ is uniformly bounded. As $|\lambda| > \sum\limits_{j=0}^{p}||A_j||e^{h_p|\Re\lambda|}$, we have

$$\begin{aligned} \det\Delta(\lambda) &= \det\left(\lambda I_n - A_0 - \sum_{j=1}^{p} e^{-h_j\lambda}A_j\right) \\ &= \lambda^n(1 + o(1)). \end{aligned}$$

Thus

$$\begin{aligned} \|\Delta^{-1}(\lambda)\|^2 &\leq \gamma^2\frac{||\Delta(\lambda)||_F^{2(n-1)}}{|\det\Delta(\lambda)|^2} \\ &\leq \gamma^2\left[n + \sum_{j=0}^{p}||A_j||_F^2\right]^{n-1} \\ &\quad\cdot\frac{\left(|\lambda|^2 + \sum\limits_{j=0}^{p} e^{-2h_j\Re\lambda}\right)^{n-1}}{|\lambda^n(1 + o(1))|^2} \\ &\leq M_3. \end{aligned}$$

For $\Re\lambda < 0$ with $|\lambda|$ sufficiently large, we have

$$\det(\Delta(\lambda)) = e^{-nh_p\lambda}\left(\prod_{j=1}^{n}(\lambda e^{h_p\lambda} - \mu_j) + o(1)\right).$$

Therefore, for $|\lambda e^{h_p\lambda} - \mu_j| > \delta > 0$, we have

$$\begin{aligned} & \|\Delta^{-1}(\lambda)\|^2 \\ \leq & \gamma^2\frac{||\Delta(\lambda)||_F^{2(n-1)}}{|\det\Delta(\lambda)|^2} \\ \leq & \gamma^2\left[n + \sum_{j=0}^{p}||A_j||_F^2\right]^{n-1} \end{aligned}$$

$$
\begin{aligned}
& \cdot \frac{\left(|\lambda|^2 + \sum\limits_{j=0}^{p} e^{-2h_j \Re\lambda}\right)^{n-1}}{e^{-2nh_p\Re\lambda}\left|\prod\limits_{j=1}^{n}(\lambda e^{h_p\lambda} - \mu_j) + o(1)\right|^2} \\
= \ & \gamma^2 \left[n + \sum_{j=0}^{p} ||A_j||_F^2\right]^{n-1} \\
& \cdot \frac{\left(|\lambda|^2 e^{2h_p\Re\lambda} + \sum\limits_{j=0}^{p} e^{2(h_p-h_j)\Re\lambda}\right)^{n-1}}{e^{-2h_p\Re\lambda}\left|\prod\limits_{j=1}^{n}(\lambda e^{h_p\lambda} - \mu_j) + o(1)\right|^2} \\
\leq \ & M_3.
\end{aligned}
$$

From the above discussion, we see that for all $\lambda \in \mathbb{C}$ and $|\lambda e^{h_p\lambda} - \mu_j| > \delta > 0, j = 1, 2, \cdots, n$, there is a constant $M_3 > 0$ such that

$$||\Delta^{-1}(\lambda)|| \leq M_3.$$

Thus we have

$$||R(\lambda, A)|| \leq M e^{2h_p|\lambda|}.$$

The desired result follows. $\square$.

By now we have checked all the conditions of Lemma 8, where $\rho_1 = \rho_2 = h_p$ and $\tau_0 > h_p$, and the type of resolvent is at most $2h_p$. According to Lemma 8 we have the following result.

**Theorem 12** *Let $\mathcal{A}$ be defined by (7) and (8) and $T(t)$ be the $C_0$ semigroup generated by $\mathcal{A}$. Suppose that the matrix $A_p$ is invertible and has $n$ distinct eigenvalues. Then when $t > 4h_p$, the solution of system (9) can be expanded as*

$$
\begin{aligned}
& X(t) \\
= \ & T(t)X_0 \\
= \ & \sum_{k=1}^{N}\sum_{j=1}^{n} E(\lambda_k^{(j)}; \mathcal{A})T(t)X_0 \\
& + \sum_{k=N+1}^{\infty}\sum_{j=1}^{n} e^{t\lambda_k^{(j)}}(X_0, \Psi(\lambda_k^{(j)}))_{\mathcal{H}}\Phi(\lambda_k^{(j)})
\end{aligned} \tag{26}
$$

*where $N$ is the number of non-simple eigenvalues, and $\Phi(\lambda), \Psi(\lambda)$ are defined in Theorem 4 and 5, respectively.*

## 5 Conclusion

In the present paper, we have proved that the solution $X(t)$ of system (9) can be expanded according to the root vectors of the system operator $\mathcal{A}$ although they do not form a basis for the state space $\mathcal{H}$. Here we should note that the conditions of Theorem 12 are sufficient for $t > 4h_p$. In fact, the expression (26) is exactly the solution of (9) for $t > 0$. However, the present conditions do not ensure the convergence of the series in (26) in the sense of norm on $\mathcal{H}$. Return to equation (3), its solution is exactly given by the first component of $X(t)$.

Although our result is obtained under the assumption that $\det(A_p) \neq 0$ and $A_p$ has $n$ distinct eigenvalues, the method used in this paper can be extended to the general case that $\det(A_p) = 0$. Similarly, we can get the explicit expansion of the solution according to their root vectors.

Here we address that the formula (26) gives a theoretical support for the high accuracy of the simulation when $t$ is large. Note that when $N$ is sufficiently large, the second term in (26) has the estimate

$$
\begin{aligned}
& \left\|\sum_{k=N+1}^{\infty}\sum_{j=1}^{n} e^{t\lambda_k^{(j)}}(X_0, \Psi(\lambda_k^{(j)}))_{\mathcal{H}}\Phi(\lambda_k^{(j)})\right\| \\
\leq \ & \sum_{k=N+1}^{\infty}\sum_{j=1}^{n} e^{t\Re\lambda_k^{(j)}}|(X_0, \Psi(\lambda_k^{(j)}))_{\mathcal{H}}|||\Phi(\lambda_k^{(j)})|| \\
\leq \ & M\sum_{k=N+1}^{\infty}\sum_{j=1}^{n} e^{t\Re\lambda_k^{(j)}}||X_0|||e^{-h_p\Re\lambda_k^{(j)}},
\end{aligned}
$$

where we have used Proposition 9.

Proposition 10 shows that

$$\sum_{k=N+1}^{\infty}\sum_{j=1}^{n} e^{\Re\lambda_k^{(j)}\tau_0} < \infty$$

as $t - h_p > \tau_0 > h_p$. Therefore, when $t > 2h_p$, the remainder term of the infinite sum converges absolutely to zero in the sense of norm on $\mathcal{H}$. Particularly, as $t \to \infty$, the decay rate of the remainder is determined by $\Re\lambda_{N+1}$ and the remainder has the estimate

$$
\begin{aligned}
& \left\|\sum_{k=N+1}^{\infty}\sum_{j=1}^{n} e^{t\lambda_k^{(j)}}(X_0, \Psi(\lambda_k^{(j)}))_{\mathcal{H}}\Phi(\lambda_k^{(j)})\right\| \\
\leq \ & M||X_0||e^{\Re\lambda_{N+1}(t-2\tau_0)}\sum_{k=N+1}^{\infty}\sum_{j=1}^{n} e^{\tau_0\Re\lambda_k^{(j)}}.
\end{aligned}
$$

Therefore, the formula (26) provides an approximate solution to (9):

$$X(t) \approx \widehat{X}(t) = \sum_{k=1}^{N}\sum_{j=1}^{n} E(\lambda_k^{(j)}; \mathcal{A})T(t),$$

and the errors are estimated by

$$\mathcal{E}(t) \leq M||X_0||e^{\Re\lambda_{N+1}(t-2\tau_0)} \sum_{k=N+1}^{\infty} \sum_{j=1}^{n} e^{\tau_0 \Re\lambda_k^{(j)}}.$$

As applications of Theorem 12 in the numerical solution, we calculate some concrete models in [17] and [22] (e.g., see [23]). Comparing with the traditional ways of numerical solution, the approximate solution $\widehat{X}(t)$ has higher accuracy. For different initial value, it takes much shorter time than the traditional ways to get the numerical solution since the eigenvalues and the root vectors remain unchanged as long as the system parameters are unchanged. We omit the details of these simulations.

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