Michel-Penot subdifferential and Lagrange multiplier rule

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Abstract:-In this paper, we investigate some properties of Michel Penot subdifferentials of locally Lipschitz functions and establish Lagrange multiplier rule in terms of Michel-Penot subdifferentials for nonsmooth mathematical programming problem.

Key-Words: Nonsmooth optimization; approximate subdifferentials; generalized gradient; Michel Penot subdifferential; Banach space.

1 Introduction

In this paper, we consider a mathematical programming problem on a Banach space and derive necessary conditions in Lagrange multiplier form. The main tool in this paper is Michel-Penot subdifferential of Locally Lipschitz function defined on Banach space.

Most of extensions of Lagrange multiplier rules for various problems of nonsmooth optimization are given in terms of generalized gradient of Clarke or certain approximate subdifferentials (see [5], [2]). The extensions involving smaller, particularly convex valued, subdifferentials with certain calculus rules such as those introduced by Michel and Penot [14], Treiman [13], Dolecki [15] and Frankowaska [6] were obtained only for problems containing finitely many inequality constraints with no equality constraints at all. The past studies reveal that to obtain more precise and more selective first order necessary conditions, the size of subdifferential must be smaller.

The reason for this disparity is quite obvious that the small convex-valued subdifferentials lack upper semi-continuity, which is needed to handle equality constraints. Further the approximate subdifferentials and generalized gradients are smallest among upper semicontinuous and convex-valued upper semicontinuous subdifferentials with calculus (see [8]). Since the subdifferentials of Michel-Penot and Treiman are naturally connected with the Gateaux and Frechet derivatives (in the sense that the function is differentiable in the corresponding sense if the subdifferential is a singleton), it is reasonable to ask whether it is possible to obtain Lagrange multiplier rule involving these subdifferentials for problems with finitely many equality constraints. Ioffe [3] has given affirmative answer of this question which is a particular case of the Theorem 4 in [2], involving weak prederivative, a concept associated with Ioffe's fan theory (see [1]).

Ioffe [3] has shown that a Lagrange multiplier rule involving the Michel-Penot subdifferential is valid for the problem (P):

$$\left.\begin{aligned} &Min\ f(x) \\ &subject\ to\ g_i(x) \le 0; i = 1,2,..m, \\ &h_j(x) = 0; j = 1,2,..n, \\ &x \in C \end{aligned}\right\} \quad (P)$$

where X is a Banach space, $f, g_i\ (i = 1,2,..m.)$ and $h_j\ (j = 1,2,..n.)$ are functions from X to R and C is a closed convex subset of X.

We consider the above problem (P) with C closed but not necessarily convex subset. The Lagrangian function L for this problem is given by

$$L(x,\lambda,\mu,\nu,K)$$
$$=\lambda f(x)+\langle\mu,g(x)\rangle$$
$$+\langle\nu,h(x)\rangle+K\|(\lambda,\mu,\nu)\|\, d_C(x).$$

Where $d_C(.)$ denotes the distance function of closed set C, and $\|\cdot\|$ denotes the Euclidean norm on R^{1+m+n}.

We state the following Multiplier rule in terms of Michel-Penot subdifferential.

Theorem 1.1 Assume that $f, g_i, h_j\ (i=1,2,.., m;\ j=1,2,..n.)$ are locally Lipschitz at x. If x is a solution of (P), then for every K sufficiently large, there exist $\lambda\ge 0, \mu_i\ge 0$ and $\nu_j\in R$ not all zero such that

$$\mu_i\, g_i(x)=0$$

and

$$0\in\lambda\partial^{\Diamond} f(x)+\sum_{i=1}^{m}\mu_i\,\partial^{\Diamond} g_i(x)+\sum_{j=1}^{n}\nu_j\,\partial^{\Diamond} h_j(x)$$
$$+K\|(\lambda,\mu,\nu)\|\,\partial^{\Diamond} d_C(x)$$

where $\partial^{\Diamond}$ denotes the Michel-Penot subdifferential.

The Michel-Penot subdifferential of a locally Lipschitzian function is the principal part of Clarke subdifferential. It coincides with the Gateaux derivative at differentiable (in Gateaux sense) points. Here, it is interesting to note that a locally Lipschitz function can be determined by its Michel-Penot subdifferential uniquely up to an additive constant, though this can not be done by its Clarke subdifferential, if the set of abnormal point $\left(i.e.\ \partial^{\Diamond} f(x)\ne\partial^{\circ} f(x)\right)$ is not negligible.

The Michel-Penot subdifferential of Gateaux-differentiable function is singleton whereas Clarke subdifferential may have more than one point, e.g. the function $f:R\to R$ defined by

$$f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & if\ \ x\ne 0,\\ 0 & ,\ if\ \ x=0.\end{cases}$$

Then f is globally Lipschitz and differentiable with derivative

$$Df(x)=\begin{cases} 2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}, & if\ \ x\ne 0,\\ 0 & ,\ if\ \ x=0.\end{cases}$$

here $\partial^{\Diamond} f(0)=\{Df(0)\}=\{0\}$ and $\partial^{\circ} f(0)=[0,1]$.

Analogous to Clarke's subdifferential was motivated in stochastic programming (see Birge and Qi [12]). As the objective function of a stochastic programming problem is generally a multi integral of several variables (Birge and Qi [11]), and thus to avoid the computation of unnecessary extraneous subgradients, the study of Michel-Penot subdifferential of integral functionals is fruitful.

In some way one may consider the present work analogous to the existing general schemes for Mathematical Programming; (see e.g. Clarke [4], Halkin [7]) but the treatment of the problem is completely different. In fact in our present study not only results and realms of applicability are different but there is a fundamental difference in the approach too. Here, instead of postulating the existence of certain convex and/or linear approximations or use of Ioffe's fan theory [3], we need the Michel-Penot subdifferential which is intrinsic to the problem. Thus our work is closer in spirit to the classical theory (with derivatives) and to the convex analysis treatment (with subgradients).

The paper is organized as follows: In section 2, we discuss some fundamental notions of nonsmooth analysis and reproduce the basic properties of Michel-Penot directional derivative and Subdifferential to make the study self contained. In section 3, we investigate M-P regularity (semiregularity for locally Lipschitzian) and various calculus rules. Various inclusion relationship has been established, which in turn as equality under M-P regularity. It is noteworthy that by the generalized Rademacher's theorem equality holds almost everywhere in finite dimensions and in separable Banach space, relative to Haar measure. In sections 4 and 5, we discuss Michel-Penot subdifferential of integral functionals and a general formula for point wise maxima of locally Lipschitz function on infinite index set. In the last section, we employed these results to establish multiplier rule in terms of Michel-Penot subdifferential.

2 Preliminaries

In view of making the study self contained, we need to reproduce the following notions of nonsmooth analysis:

Definition2.1 Let $\phi : X \to R$ be a locally Lipschitz function at, $\bar{x}$ then the Clarke generalized directional derivative of ϕ at a point $\bar{x}$ and in the direction $d \in X$, denoted by $\phi^0(\bar{x}; d)$ is given as:

$$\phi^0(\bar{x}; d) = \limsup_{y \to \bar{x}, t \to 0} \frac{\phi(y + td) - \phi(y)}{t}$$

and the Clarke generalized gradient of ϕ at $\bar{x}$ is given by

$$\partial^{\circ}\phi(\bar{x}) = \left\{ x^* \in X^* : \phi^0(\bar{x}; d) \geq \left\langle x^*, x \right\rangle, \forall d \in X \right\}$$

where X^* denotes the topological dual of X $\langle .,. \rangle$ denotes the dual pairing between X^* and X.

Let C be a non empty subset of X and consider its distance functions $d_C : X \to R_+$, defined by

$$d_C(x) = \inf \left\{ \|x - c\| : c \in C \right\}.$$

This function is not everywhere differentiable but it is (globally) Lipschitz, with the Lipschitz constant equal to 1.

Let $\bar{x} \in C$ be given, we say that $v \in X$ is a tangent vector to C at $\bar{x}$ if $d_C{}^0(\bar{x}; v) = 0$.

The set of tangent vectors to C at $\bar{x}$ is a closed convex cone in X called Clarke tangent cone of C at $\bar{x}$ and denoted by $T(\bar{x}, C)$. That is

$$T(\bar{x}, C) = \left\{ v \in X \mid d_C{}^0(\bar{x}; v) = 0 \right\}$$

and the Clarke normal cone is polar to $T(\bar{x}, C)$, defined as

$$N(\bar{x}, C) = \left\{ x^* \in X^* \mid \left\langle x^*, v \right\rangle \leq 0, \forall v \in T(\bar{x}, C) \right\}.$$

Let $\bar{x} \in X$ and $\phi : X \to R$ be a locally Lipschitz function at $\bar{x}$. Then the Michel-Penot directional derivative of ϕ at the point $\bar{x}$ in the direction $\bar{x}$, denoted by $\phi^{\diamond}(\bar{x}; v)$, is given by

$$\phi^{\diamond}(\bar{x}; v) = \sup_{w \in X} \limsup_{t \downarrow 0} \frac{\phi(\bar{x} + tv + tw) - \phi(\bar{x} + tw)}{t}$$

and the Michel-Penot subdifferential of ϕ at $\bar{x}$ is given by

$$\partial^{\diamond}\phi(\bar{x}) = \left\{ x^* \in X^* : \phi^{\diamond}(\bar{x}; v) \geq \left\langle x^*, x \right\rangle, \forall v \in X \right\}$$

It is known (see [14]) that when a function f is Gateaux differentiable at $\bar{x}$,

$$\partial^{\diamond} f(\bar{x}) = \left\{ \nabla f(\bar{x}) \right\}.$$

The following properties of the Michel-Penot directional derivatives and Michel-Penot subdifferentials will be useful in the sequel.

Proposition2.1 Let $\bar{x} \in X$ and $f, g : X \to R$ be locally Lipschitz functions at $\bar{x}$, and the local Lipschitz constant of f be L_f ,then the following hold:

(i)The function $v \mapsto f^{\diamond}(\bar{x}; v)$ is finite, positively homogeneous and subadditive on X.

(ii) As a function of v, $f^{\diamond}(\bar{x}; v)$ is Lipschitz continuous with Lipschitz constant L_f.

(iii) $\partial^{\diamond} f(\bar{x})$ is a nonempty, convex, weak*-compact subset of X^* and $\|x^*\| \leq L_f$ for every $\bar{x} \in \partial^{\diamond} f(\bar{x})$, one has

$$f^{\diamond}(\bar{x}; v) = \max \left\{ \left\langle \xi^*, v \right\rangle \mid \xi^* \in \partial^{\diamond} f(\bar{x}) \right\}.$$

(iv) For every scalar λ, $\partial^{\diamond}(\lambda f)(\bar{x}) = \lambda \partial^{\diamond} f(\bar{x})$, and for every $v \in X$,

$f^{\diamond}(\bar{x}; -v) = (-f)^{\diamond}(\bar{x}; v)$.

(v) $\partial^{\diamond}(f + g)(\bar{x}) \subseteq \partial^{\diamond} f(\bar{x}) + \partial^{\diamond} g(\bar{x})$ and $(f + g)^{\diamond}(\bar{x}; v) \leq f^{\diamond}(\bar{x}; v) + g^{\diamond}(\bar{x}; v)$, the equalities hold if both f and g are MP-regular.

(vi) If $\bar{x}$ is a local minima of f, then $0 \in \partial^{\diamond} f(\bar{x})$ and $f^{\diamond}(\bar{x}; v) \geq 0 \,\forall\, v \in X$.

(vii) $f^{\diamond}(\bar{x}; v)$ is upper semi continuous as a function of $(\bar{x}; v)$.

Proof. The proof of (i) to (vi) may be seen in Michel and Penot [14] and Birge and Qi [12].

We prove (vii): Let x_i and v_i be arbitrary sequences converging to x and v respectively. For each i by definition of upper limit and supremum, there exist y_i in X and $t_i > 0$, such that

$$y_i = x + t_i z_i,\ \|z_i\| = 1 \text{ and } \|y_i - x_i\| + t_i < \frac{1}{i},$$

Then

$$f^{\lozenge}(x_i; v_i) - \frac{1}{i} \le \frac{f(x + t_i v_i + t_i z_i) - f(x + t_i z_i)}{t_i}$$
$$= \frac{f(x + t_i v_i + t_i z_i) - f(x + t_i z_i)}{t_i}$$
$$+ \frac{f(x + t_i v_i + t_i z_i) - f(x + t_i v + t_i z_i)}{t_i}$$

As the last term on right hand side of the above expression is bounded in magnitude by $K\|v_i - v\|$, thus taking upper limit, we get

$$\limsup_{i\to\infty} f^{\lozenge}(x_i; v_i)$$
$$\le \limsup_{i\to\infty} \frac{f(x + t_i v + t_i z_i) - f(x + t_i z_i)}{t_i}$$

Therefore

$$\limsup_{i\to\infty} f^{\lozenge}(x_i; v_i)$$
$$\le \sup_{z\in X} \limsup_{i\to\infty} \frac{f(x + t_i v + t_i z) - f(x + t_i z)}{t_i}$$
$$= f^{\lozenge}(x; v)$$

This shows that $f^{\lozenge}(\bar{x}; v)$ is upper semi-Continuous as a function of $(\bar{x}, v)$. □

Lemma 2.1 Let x_i and ξ_i be sequences in X and X^* such that $\xi_i \in \partial^{\lozenge} f(x_i)$. Suppose that x_i Converges to x, and that ξ is a cluster point of ξ_i in the weak*-topology. Then $\xi \in \partial^{\lozenge} f(x)$ (i.e. the multifunction $\partial^{\lozenge} f$ is weak*-closed).

Proof. Let $v \in X$ be given and there is a subsequence $\langle \xi_i, v\rangle$ which converges to $\langle \xi, v\rangle$, since $\xi_i \in \partial^{\lozenge} f(x_i)$, then $f^{\lozenge}(x_i; v) \ge \langle \xi_i, v\rangle$, which implies, by upper semi continuity of $f^{\lozenge}(.;.)$, that

$$f^{\lozenge}(x; v) \ge \langle \xi, v\rangle. \quad \square$$

3 M-P regularity and calculus rules

Analogous to Clarke regularity [5], semi-regularity, which is a weaker notion, was introduced by Birge and Qi [12] for locally Lipschitz function. Ye [9] extended this notion as M-P regularity to any function.

Definition 3.1 ([12], [9]) A function $f : X \to R$ is said to be M-P regular (or semiregular if f is locally Lipschitz) at x if (i) the usual directional derivative $f'(x; v)$ exists finitely for all v in X,

(ii) $f'(x; v) = f^{\lozenge}(x; v)$ for all v in X.

Remark 3.1 If there is no ambiguity between the two notions M-P regularity and semiregularity. We use the word M-P regularity for semiregularity (signifies local Lipschitzian property is present) also. Note that every convex function as well as every Gateaux differentiable function is M-P regular.

We can prove the following Mean Value Theorem, already proved by Borwein et. al [10], analogously as Theorem 2.3.7 in Clarke [5].Which is stronger than that of Clarke [5].

Theorem 3.1 (Mean-Value Theorem [10]). Suppose $f : U \subseteq X \to R$ be locally Lipschitz on the open set U. Let $[x, y]$ be a line segment in U. Then there exists a point u in $[x, y]$ such that

$$f(y) - f(x) \in \langle \partial^{\lozenge} f(u), y - x\rangle.$$

Chain Rules

We now intend to provide chain rules, for Michel-Penot subdifferentials. Let $h : U \to R^n$ and $g : R^n \to R$ are locally Lipschitz function at x, so that $f = g \circ h : U \to R$ is also locally Lipschitz at x. Then we have the following chain rules:

Theorem 3.2 (see [12]) Chain rule 1

$$\partial^{\Diamond} f(x) \subseteq \overline{\text{co}} \left\{ \sum_{i=1}^{n} \alpha_i \xi_i \mid \xi_i \in \partial^{\Diamond} h_i(x), \quad \alpha \in \partial^{\Diamond} g(h(x)) \right\},$$

(where $\overline{\text{co}}$ denotes weak*- closed convex hull) and equality holds under any one of the following additional hypotheses:

(i) g is regular at $h(x)$, each h_i is M-P regular at x and every element of $\partial^{\Diamond} g(h(x)$ has nonnegative component. In this case it follows that f is M-P regular.

(ii) g is M-P regular at $h(x)$, and h is Gateaux differentiable. In this case it follows that f is regular at x and $\overline{\text{co}}$ is superfluous.

(iii) g is Gateaux differentiable at $h(x)$, and $n = 1$ (in this case the $\overline{\text{co}}$ is superfluous). $(i)'$ g is M-P regular at $h(x)$, each h_i is M-P regular at x and every element of $\partial^{\Diamond} g(h(x))$ has nonnegative components, in this case also, it follows that f is M-P regular.

Remark 3.2 The condition $(i)'$ is weaker than (i), an easy consequence of Theorem 2.3.9 in Clarke [5] and the upper semicontinuity of multifunction $\partial^{\Diamond} g$ results in equality in this case.

Theorem 3.3 (see [12]) Chain rule 2.

Assume that $g : R^n \to R$ is differentiable (for locally Lipschitz functions on a finite dimensional spaces, Gateaux differentiability is the same as Frechet differentiability) and $h : X \to R^n$ is locally Lipschitz function then $f = g \circ h$:is Lipschitz near x, and one has:

$$\partial^{\Diamond} f(x) \subseteq g'(h(x)) \circ \partial^{\Diamond} h_i(x), \qquad (1)$$

i.e.

$$\partial^{\Diamond} f(x) \subseteq \left\{ \sum_{i=1}^{n} \alpha_i \xi_i \mid \xi_i \in \partial^{\Diamond} h_i(x), \quad \alpha = g'(h(x)) \right\}, \qquad (2)$$

and equality holds, if each $h_i(x)$ is M-P regular at x, and $g'(h(x))$ has nonnegative components. In this case, it follows that f is M-P regular at x.

(b) Let $F : X \to Y$ (Y is another Banach space), and let $g : Y \to R$. Suppose that F is Gateaux differentiable at x and g is Lipschitz near $F(x)$. Then $f = g \circ F$ is Lipschitz near x, and one has

$$\partial^{\Diamond} f(x) \subseteq \partial^{\Diamond} g(F(x)) \circ DF(x),$$

Equality holds if g or $-g$ is MP-regular at $F(x)$, in which case f or $-f$ is also MP-regular at x. Equality also holds if F maps every neighborhood of x to a set which is dense in a neighborhood of $F(x)$ (e. g. if $DF(x)$ is onto).

Corollary 3.1 Let $g : Y \to R$ be Lipschitz near x, and suppose that the space X, is continuously embedded in Y, is dense in Y and contains the point x. Then the restriction f of g to X is Lipschitz near x and

$$\partial^{\Diamond} f(x) = \partial^{\Diamond} g(x)$$

Proposition 3.1 (point wise maxima)

Suppose $\{f_i \mid i = 1,2,...n\}$ is a finite collection of functions, each of which is Lipschitz near x. The function $f : U \to R$ is defined as

$f(u) = \max\{f_i(u) \mid i = 1,2,..., n\}$.

Then for any $x \in U$,

$$\partial^{\Diamond} f(x) \subseteq co\{\partial^{\Diamond} f_i(x) \mid i \in I(x)\},$$

where $I(x) = \{i \mid f_i(x) = f(x)\}$. If f_i is M-P regular at x for each $i \in I(x)$, then equality holds and f is regular at x.

Proof. The proof follows from proposition 2.3.12 in [5]. The assertion regarding equality and regularity follows from Theorem (i) of chain rule 1.

Proposition 3.2 (Basic Calculus)

Suppose that f_1, f_2 be Lipschitz near x. Then $f_1 \cdot f_2$

and $\frac{f_1}{f_2}(f_2 \neq 0)$ are Lipschitz near x, and

$$\partial^\lozenge(f_1 \cdot f_2)(x) \subseteq f_1(x)\partial^\lozenge f_2(x) + f_2(x)\partial^\lozenge f_1(x) \quad (products)$$

$$\partial^\lozenge\left(\frac{f_1}{f_2}\right)(x) \subseteq \frac{f_2(x)\partial^\lozenge f_1(x) - f_1(x)\partial^\lozenge f_2(x)}{f_2^2(x)}$$

$$if \;\; f_2(x) \neq 0 \quad (quotients)$$

Also, if in addition $f_1(x) \geq 0, f_2(x) \geq 0$ and if f_1, f_2 are both M-P regular at x, then equality holds in Product rule and $f_1 \cdot f_2$ is M-P regular at x. If in addition and $f_1, -f_2$ are M-P regular at x, then equality holds in Quotient rule and $\frac{f_1}{f_2}$ is M-P regular at x.

4 Michel-Penot subdifferential of Integral Functionals

Suppose that U is an open subset of Banach space X. Let (T, Δ, μ) be a positive measure space. We consider a family of functions $\{f_t : U \to R \mid t \in T\}$ under following hypotheses:

(i) For each x in U, the map $t \mapsto f_t(x)$ is measurable;

(ii) For some integrable function $k : T \to R$, for all x and y in U and t in T, one has $|f_t(x) - f_t(y)| \leq k(t)\|x - y\|$.

We now invoke the Michel-Penot subdifferential of the integral functional f on X given by

$$f(x) = \int_T f_t(x)\,\mu(dt)$$

as

$$\partial^\lozenge f(x) \subseteq \int_T \partial^\lozenge f_t(x)\,\mu(dt) \qquad (3)$$

as given in [5], expression (3) (see [12]) has the following interpretation:

For each ξ in $\partial^\lozenge f(x)$ there is a map $t \mapsto \xi_t$ from T to X^* with $\xi_t \in \partial^\lozenge f_t(x)$ a.e. (almost everywhere relative to μ); such that for every v in X, the function $t \mapsto \langle \xi_t, v \rangle$ is in $L^1(X, R)$ and $\langle \xi, v \rangle = \int_T \langle \xi_t, v \rangle\,\mu(dt)$.

Now, consider three cases:

(a) T is countable.

(b) X is separable.

(c) T is a separable metric space, μ is a regular measure, and the map $t \mapsto \partial^\lozenge f_t(u)$ is weak*-upper semicontinuous for each u in U.

Theorem 4.1. Let f be defined at some point x in U. Then f is defined and Lipschitz in U. If at least one of (a),(b) or (c) is satisfied, then formula (3) holds. Further, if $f_t(\cdot)$ is M-P regular at x for each t, then f is M-P regular at x and equality holds in expression (3).

Proof. The Lipschitz condition on U follows from the hypotheses (i) and (ii) given above in this section. In case (b) the formula (3) holds has been shown by Birge and Qi [12], in case (a) or (c) proof is similar to the proof of Theorem 2.7.2 of [5]. The proof for equality part is identical to the corresponding part of Theorem 2.7.2 of [5] using M-P regularity and M-P subdifferential instead of regularity and Clarke subdifferential.

The advantage of Theorem 4.1 is that under moderate conditions on X and T, the right hand side of (3) is singleton almost everywhere in U. Almost everywhere means except a set of Haar measure zero, that is there exists a Borel probability measure, μ on X such that $\mu(N + x) = 0$ for all $x \in X$, and we say N is a Haar zero set.

In practical situations, for instance in control theory, a different type of integral functional occurs frequently is that X is space of functions on T. When T is countable or X is restricted to space of continuous functions on T (e.g. $X = C[0,1]$) or finite dimensional, Theorem 4.1 will apply directly. However, not generally, when X is a L^p space. We state the following Theorem for $1 \leq p \leq \infty$.

Theorem 4.2 Let (T, Δ, μ) be a σ - finite positive measure space and Y a separable Banach space, and $L^p(T, Y)$ be the space of $p-$

integrable (for $p=1$, essentially bounded)functions from T to Y. Suppose X is a closed subspace of $L^p(T,Y)$ and a family of functions $f_t: Y \to R$ such that

(i) $t \mapsto f_t(y)$ is measurable for each y in Y,

$(ii)'$ there exists $\varepsilon > 0$, a function $k \in L^q(T,R)$ (q is conjugate index to p defined as $\frac{1}{p}+\frac{1}{q}=1$, $q=1$ if $p=\infty$) such that for all $t \in T$, for all $y_1, y_2 \in x(t)+\varepsilon B_Y$, $|f_t(y_1)-f_t(y_2)| \le k(t)\|y_1-y_2\|$.

Then, for any x satisfying the conditions, $f(x)=\int_T f_t(x)\,\mu(dt)$ is Lipschitz near x and (3) holds. Further, if f_t is M-P regular at $x(t)$ for all $t \in T$, then equality holds in (3).

Proof. Similar to Theorems 2.7.3 and 2.7.5 of Clarke [5], the measurability of $t \mapsto f_t^{\Diamond}(x(t);v)$ for all $x(t)$ and v follows since the supremum over $w \in X$ in the definition of $f_t^{\Diamond}(x(t);v)$ can be replaced by supremum over countable dense subset of Y for each fixed t. We use the fact that $f_t^{\Diamond}(x(t);v)$ is sup of countable limsup of measurable functions over a countable set (see Birge and Qi [12], Theorem 5.3, have considered $Y=R^n$).

5 Pointwise Maxima-A General Formula

We have already studied functionals which are pointwise maxima of some finite indexed family of functions. Now, we study much more complex situation in which indexed set is infinite.

Let $\{f_t\}$ be a family of functions on $X, t\in T$ and T is a topological space. Suppose there is some point x in X such that f_t is locally Lipschitz at x for each $t \in T$. We define a new type of partial MP-subdifferential, in which it has also been taken care of variations in parameters t.

We denote by $\partial^{\Diamond}_{[T]} f_t(u)$ the set

$$\partial^{\Diamond} f(x) \subseteq \overline{\text{co}} \left\{ \begin{array}{r} \xi_i \in X^* \mid \xi_i \in \partial^{\Diamond} f_{t_i}(x_i), \\ x_i \to x,\ t_i \to t, t_i \in T, \\ \xi\ is\ a\ w^* - \text{cluster point of } \xi_i \end{array} \right\}.$$

Definition 5.1

The multifunction $(\tau, y) \mapsto \partial^{\Diamond} f_\tau(y)$ is said to be (weak *) closed at (t,x) provided $\partial^{\Diamond}_{[T]} f_t(u) = \partial^{\Diamond} f_t(x)$.

Clearly, in view of Lemma 2.1, if t is isolated in T, then condition certainly holds.

Lemma 5.1 We consider the following hypotheses

(i) T is sequentially compact.

(ii) There exists a neighborhood U of x such that the map $t \mapsto f_t(y)$ is upper semicontinuous for each $y \in U$.

(iii) For each $t \in T$, f_t is Lipschitz of given rank (i.e. Lipschitz constant) K on U , and $\{f_t(x) \mid t \in T\}$ is bounded. Then a function $f: X \to R$ given by

$f(y) = \max\{f_t(y) \mid t \in T\}$

is defined and finite (with the maximum defining f attained) on U , and f is Lipschitz on U of rank K.

Let $M(y) := \max\{t \in T \mid f_t(y) = f(y)\}$. Observe that $M(y)$ is nonempty and closed for each y in U . Let us denote by $P[S]$ the collection of probability radon measures on S, for any subset S of T.

Theorem 5.1 Under the hypotheses given in above lemma 5.1, suppose that either

(a) X is separable, or

(b) T is metrizable.

Then one has

$$\partial^{\Diamond} f(x) \subseteq \left\{ \int_T \partial^{\Diamond}_{[T]} f_t(x)\,\mu(dt) \mid \mu \in P[M(X)] \right\} \quad (4)$$

Further, if the multifunction $(t,x)\mapsto \partial^{\Diamond} f_t(x)$ is closed at (t,x) for each $t\in M(x)$, and if f_t is M-P regular at x for each t in $M(x)$, and equality holds in (4) with $\partial^{\Diamond}_{[T]} f_t(u)=\partial^{\Diamond} f_t(x)$.

Proof. The proof is similar to the proof of Theorem 2.8.2 of Clarke [5], except we use Michel-Penot directional derivative instead of Clarke directional derivative, and use the Mean value Theorem stated in Borwein et. al [10].

Remark 5.1 The interpretation of the set occurring on the right-hand side
expression (4) is Completely analogous to that of (1) in Theorem 2.7.2 of Clarke [5]. In particular, each element ξ of that set is an element of X, corresponding to which there is a mapping $t\mapsto\langle\xi_t,v\rangle$ from T to X^* and an element μ, of $P[M(X)]$ such that, for every v in X, $t\mapsto\langle\xi_t,v\rangle$ is μ -integrable, and $\langle\xi,v\rangle=\int_T\langle\xi_t,v\rangle\mu(dt).$

6 Proof of the multiplier Theorem

The following results are pivotal to establish the multiplier rule:

Theorem 6.1 (Ekeland's Theorem)
Let (V,d) be a complete metric space with metric d, and let $F:V\to R\cup\{+\infty\}$ be a l. s. c. function which is bounded below. If u is a point in V satisfying
$F(u)\le\inf F+\varepsilon$
for some $\varepsilon>0$, then, for every $\lambda>0$ there exists a point v in V such that
(i) $F(v)\le F(u)$,
(ii) $d(u,v)\le\lambda$,
(iii) for all $w\neq v$ in V,

$$F(w)+\frac{\varepsilon}{\lambda}d(w,v)>F(v).$$

Proposition 6.1 (see [5] Proposition(2.4.3)) Let f be Lipschitz of rank K on a set S. Let $x\in C\subset S$ and suppose that f attains a minimum over C at x. Then for any $\hat{K}\ge K$, the function $g(y)=f(y)+\hat{K}d_C(y)$ attains a minimum over S at x. If $\hat{K}>K$ and C is closed, then any other point minimizing g over S must also lie in C.

Now, we have sufficient machinery to establish the stated multiplier rule as the main theorem.

Proof. Proof of the theorem
Let

$$T=\left\{\begin{matrix} t=(\lambda,\mu,\nu)\in R\times R^m\times R^n\,| \\ \lambda\ge 0,\mu\ge 0,\ \|(\lambda,\mu,\nu)\|=1\end{matrix}\right\},$$

let any $\varepsilon>0$ be given and define $F:X\to R$ by

$$F(y)=\max_T\left\{\begin{matrix}(\lambda,\mu,\nu)\cdot \\ (f(y)-f(x)+\varepsilon,g(y),h(y))\end{matrix}\right\}.$$

Note that F is locally Lipschitz at x and that $F(x)=\varepsilon.$ We claim that $F>0$ on C, if $F(y)\le 0$ then it can be easily shown that $g_i(y)\le 0, h_j(y)=0$ (i.e. y is a feasible point for P) and $f(y)\le f(x)-\varepsilon$, a contradiction. Therefore x satisfies

$$F(u)\le\inf_C F+\varepsilon,$$

and by Theorem 6.1, there is a point u in $x+\sqrt{\varepsilon}\,\overline{B}$ such that for all

$$y\in C'(C'\cap(x+\sqrt{\varepsilon}\,\overline{B})),$$

we have

$$F(y)+\sqrt{\varepsilon}\,\|y-u\|\ge F(u).$$

If $\hat{K}$ is the Lipschitz constant then any $K>\hat{K}$ is local Lipschitz constant (ε is sufficiently small) for the function $F(y)+\sqrt{\varepsilon}\,\|y-u\|$ near the point $y=u.$
By proposition 6.1, u therefore also minimizes over some neighborhood of u,
the function

$$y\mapsto F(y)+\sqrt{\varepsilon}\,\|y-u\|+Kd_C(y),$$

$$= \max_T \{L(y,\lambda,\mu,\nu,K) \ - \lambda\ f(x) + \varepsilon\lambda\} + \sqrt{\varepsilon}\,\|y-u\|, \quad (5)$$
$$= G(y) + \sqrt{\varepsilon}\,\|y-u\|,$$

where
$G(y) = \max_T \{L(y,\lambda,\mu,\nu,K) \ - \lambda\ f(x) + \varepsilon\lambda\}$.
For ε sufficiently small, then, we have

$$0 \in \partial^{\Diamond} G(u) + \sqrt{\varepsilon}\,\overline{B}^*. \quad (6)$$

Now, using Theorem 5.1, we estimate $\partial^{\Diamond} G(u)$.
First, we claim that mapping

$$(t,y) \mapsto \partial^{\Diamond} L(y,t,K) \quad (7)$$

is closed in the sense of definition 5.1. Observe that for any pair t_1, t_2 of points in T, the function

$$y \mapsto L(y,t_1,K) - L(y,t_2,K) = (t_1 - t_2)\cdot(f,g,h)(y)$$

is Lipschitz near x of rank $K|t_1 - t_2|$, thus

$$\partial_x^{\Diamond} L(y,t_1,K) \subset \partial_x^{\Diamond} L(y,t_2,K) + K\,|t_1 - t_2|\,\overline{B}^*$$

by proposition 2.1 (v) and 2.1 (iii). The closure property, Lemma 2.1, of the M-P subdifferential implies that the map (7) is closed. Since $F(u)$ is positive, there is a Unique t_u in T at which F (and hence G) attains maximum.
Now Theorem 5.1, is applied to estimate $\partial^{\Diamond} G(u)$:

$$\partial^{\Diamond} G(u) = \int_T \partial_x^{\Diamond} L(y,t_u,K)\ \mu(dt) = \partial_x^{\Diamond} L(y,t_u,K)$$

Using the above estimation in equation (6), we get

$$0 \in \partial_x^{\Diamond} L(y,t_u,K) + \sqrt{\varepsilon}\,\overline{B}^* \quad (8)$$

Proceeding in the same manner as above for a sequence $\varepsilon_i \downarrow 0$, then the corresponding sequence $u_i \to x$ and a subsequence $t_{u_i} \to t \in T$. since the map (7) is closed; hence theorem follows from equation (8).

Remark 6.1 We can prove above multiplier rule similar to proof of Theorem 1 in [4], we use proposition 2.1 (v), Proposition 3.1 and Proposition 2.1 (iii) in stead of Propositions 8, 9 and 1 respectively in [4].

Remark 6.2 when C is convex also then $\partial^{\Diamond} d_C(x) \subseteq N_C(x)$ (normal cone in the sense of convex analysis), and multiplier rule coincides with that of given in Ioffe [3].

7 Vector Optimization Problem (VOP):

Let $f = (f_1, f_2, ..., f_p) \in R^p$ be the objective function, and then the problem (P) is called (VOP). The feasible point x is termed as an efficient (weak efficient) or Pareto optimal for the VOP, if there is no feasible point y for which $f_r(y) < f_r(x); 1 \le r \le p$
$(f_r(y) < f_r(x); \forall\ 1 \le r \le p)$.
The Lagrangian L for the VOP is given by

$$L(x,\lambda,\mu,\nu,K) = \langle \lambda, f(x)\rangle + \langle \mu, g(x)\rangle + \langle \nu, h(x)\rangle + K\|(\lambda,\mu,\nu)\|\ d_C(x)$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_p) \in R^p$.

Theorem7.1. (Multiplier rule for VOP) Assume that
$f_r, g_i, h_j (r = 1,2,.., p; i = 1,2,..m; j = 1,2,.., n.)$
are locally Lipschitz at x. If x is a weak efficient point of VOP, then for every K sufficiently large there exist

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_p) \in R_+^p, \mu_i \ge 0$$

and $\nu_j \in R$ not all zero such that

$$\mu_i\ g_i(x) = 0; i = 1,2,.., m$$

and

$$0 \in \sum_{r=1}^{p} \lambda_r\, \partial^{\Diamond} f_r(x) + \sum_{i=1}^{m} \mu_i\, \partial^{\Diamond} g_i(x) + \sum_{j=1}^{n} \nu_j\, \partial^{\Diamond} h_j(x) + K\,\|(\lambda,\mu,\nu)\|\ \partial^{\Diamond} d_C(x)$$

Proof. Proof is analogous to the proof of Theorem 1.1, we will consider ε as $p-$vector

$(\varepsilon, \varepsilon, ..., \varepsilon) \in R^p$ appearing in the definition of F .

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