Spectral analysis of a two unit deteriorating standby system with repair

WENZHI YUAN
Taiyuan Teachers College
Department of Mathematics
Taiyuan 030012
P. R. China
ywzywz123@163.com

GENQI XU
Tianjin University
Department of Mathematics
Tianjin, 300072
P. R. China
gqxu@tju.edu.cn

Abstract: In this paper, we analyze the spectra and stability of a system consisting of a working unit and repair unit, in which the working unit consists of one main unit and one standby unit, while the standby unit may deteriorate in its standby mode. Firstly, we formulate the problem into a suitable Banach space. And then we carry out a detailed spectral analysis of the system operator. Based on the spectral analysis and C_0-semigroup theory, we prove the existence of positive solution and finite expansion of the solution according to its eigenvectors. As a consequence we get that its dynamic solutions converges exponentially to the steady-state solution. Finally, we derive some reliability indices of the system.

Key–Words: C_0-semigroup theory, dynamic solution, steady-state, exponential stability, availability.

1 Introduction

With the development of the modern technology and extensive use of electronic products, the reliability problem of the reparable systems has been a hot topic. It is well-known that reliability of a system is an important concept in engineering, it takes an essential rule in the plan, design and operation strategy of various complex systems. In order to increase the reliability of a system, a repair unit is necessary for increasing the performance and reducing the downtime or the maintenance. Therefore, repairable system is not only a kind of important system discussed in reliability theory but also one of the main objects studied in reliability mathematics. Many authors have worked in this field, including system modeling (see, [1],[2],[3]) and model analysis [4],[5],[6], [7],[8] and the references therein.

We observe that reparability is not only applicable in engineering, but also applicable to various issue arisen other subject, for instance, the study of medicament[11] and human health [12], electronic-commerce, etc. Therefore, much more attention concentrate on the study of repairable system.

Different from the early study of reparable system, in which the key point emphases the reliability indices involving availability of the system, which usually were obtained by steady state, the issue is to obtain the time-dependent solution of the system govern by the partial differential equations. This is because we cannot wait for a long time in some cases, for example, the cases of [11] and [12]. The change of key point of the issue requires us to analyze completely the system including spectrum of the system operator and finite expansion of solution. From application point of view, the time we can observe the steady state of the system becomes obviously an important index, which is especial important in the investigation of human health problem or recovery. Therefore, after the mathematical modeling for the problem, our task is mainly to solve the following questions: (1) the system under consideration has a unique nonnegative time-dependent solution; (2) approximate of solution; (3) the system has a steady state, and the dynamic solution of the system converges to the steady state.

Let us recall the observation time issue. Let $\mathbf{S}$ be a reparable system and $P(t)$ be the state vector, which describe the probability in the various states. Suppose that the system has a steady state $\widehat{P}_0$. If there is a time τ_0 such that $||P(t) - \widehat{P}_0|| \leq 0.25, t \geq \tau_0$, then it is said that the steady state of $\mathbf{S}$ is observable at time τ_0. Obviously, the observable time τ_0 is a more valuable information in application. From the observation time issue we see that it is not only an issue of existence of the solution and steady state but also the quasi-exponential decay issue of the system. How to determine the decay rate of the dynamic solution, however, is hard work, which needs more detail spectral information of the operator determined by the system. In the present paper, we mainly study the spectrum of the system operator, from which we can obtain an answer for the observe time issue. In the present paper, our model under consideration after certain assumptions is the same as the one in [9] although it has different background.

Let us recall the model under consideration [9]. Suppose that a system consists of a working unit and a repair unit, the work unit consists of one main unit and one standby unit. Initially, the system is in good condition, this state is denoted by state 0. The standby unit can deteriorate in its standby mode and due to this deteriorating process, it may fail in this mode with a failure rate λ_0. Upon the failure of the standby unit in the state 0, the system goes to state 1. The failure of the operating unit in the state 0 brings the system to state 2 where the standby unit starts to work with an increased failure rate λ_1. The failure of any of the working units in state 1&2 brings the system to a completely failed state F. Common cause failure and critical human error can occur from all the three working state which cause the complete failure of the system and are denoted by C and H, respectively. A repair facility is available in the states $1, 2$ and all the completely failed states. After repair, the system goes to state 0. Repair rates from states 1 and 2 follow an exponential distribution with μ_1 and μ_2, while from states F, C and H, it follows a general distribution.

Denote by

$p_j(t)$ Probability that the system is in state j at time t $(j = 0, 1, 2, F, C, H)$;

$p_j(u, t)$ probability density w.r.t. repair time that the failed system is in state j and has an elapsed repair time lies $u(u = x, y, z; j = F, C, H)$;

$F/C/H$ system failed due to hardware failure/common cause failure/critical human error;

λ_0 constant failure rate of the standby unit in its standby mode;

λ constant failure rate of the working unit;

λ_1 constant failure rate of the standby unit in its operating mode;

$\lambda_{c_j}/\lambda_{h_j}$ constant failure rate from the state j to the state $C/H(j = 0, 1, 2)$;

μ_j constant repair rates from states j to state $0(j = 1, 2)$;

$\alpha(x)/\beta(y)/\gamma(z)$ repair rates from states $F/H/C$ to state 0.

Thus the dynamic behavior of the system is governed by the partial differential equations

$$\begin{cases} \{\frac{d}{dt} + \lambda + \lambda_0 + \lambda_{h_0} + \lambda_{c_0}\}p_0(t) \\ = \mu_1 p_1(t) + \mu_2 p_2(t) + \int_0^\infty \alpha(x) p_F(x,t)dx \\ + \int_0^\infty \beta(y) p_H(y,t)dy + \int_0^\infty \gamma(z) p_C(z,t)dz, \\ \{\frac{d}{dt} + \lambda + \mu_1 + \lambda_{h_1} + \lambda_{c_1}\}p_1(t) = \lambda_0 p_0(t), \\ \{\frac{d}{dt} + \lambda_1 + \mu_2 + \lambda_{h_2} + \lambda_{c_2}\}p_2(t) = \lambda p_0(t), \\ \{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \alpha(x)\}p_F(x,t) = 0, \\ \{\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \beta(y)\}p_H(y,t) = 0, \\ \{\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \gamma(z)\}p_C(z,t) = 0 \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} p_F(0,t) = \lambda p_1(t) + \lambda_1 p_2(t) \\ p_H(0,t) = \sum_{j=0}^{2} \lambda_{h_j} p_j(t) \\ p_C(0,t) = \sum_{j=0}^{2} \lambda_{c_j} p_j(t) \end{cases} \tag{2}$$

and the initial condition is given by

$$(p_0(0), p_1(0), p_2(0), p_F(x,0), p_C(y,0), p_H(z,0)) = (1, 0, 0, 0, 0, 0).$$

In the present paper, we shall analyze the system (1) with (2).

The rest are organized as follows. In section 2, we formulate the system (1) with (2) into a suitable Banach space, and give the system operator $\mathcal{A}$ and its dual operator $\mathcal{A}^*$. In section 3, we carry out a complete spectral analysis for the system operator $\mathcal{A}$. In section 4, we discuss the existence of positive solution and conservation property of the system, further we get the finite expansion of the solution according its eigenvectors. In section 5, we discuss the some indices of the system, and give the estimate for the observable time and availability of the system.

2 Formulation of the system

In this section, we formulate the system (1) and (2) into a suitable Banach space, and define the system $\mathcal{A}$ and find out its dual operator $\mathcal{A}^*$. In the sequel, we use natation $\mathbb{R}$ denote the real number set and $\mathbb{R}_+ = [0, \infty)$. Based on the practice meaning of the problem, we take space $\mathbb{X}$ as $\mathbb{X} = \mathbb{R}^3 \times (L^1(\mathbb{R}_+))^3$. For each $P = (p_0, p_1, p_2, p_F(x), p_H(y), p_C(z)) \in \mathbb{X}$, the norm is defined as

$$\|P\| = \sum_{i=0}^{2} |p_i| + \|p_F\|_{L^1} + |p_H\|_{L^1} + \|p_C\|_{L^1}.$$

Obviously, $\mathbb{X}$ is a Banach space.

Before we define the system operator, we make the following assumptions:

1) The general distributions $A(x) = 1 - \exp(-\int_0^x \alpha(s)ds)$, $B(y) = 1 - \exp(-\int_0^y \beta(s)ds)$, and $\Gamma(z) = 1 - \exp(-\int_0^z \gamma(s)ds)$, where $\alpha(x), \beta(y), \gamma(z)$ are nonnegative and local integrable on $[0, \infty)$, and

$$\sup_{x\geq 0} \alpha(x), \quad \sup_{y\geq 0} \beta(y), \quad \sup_{z\geq 0} \gamma(z) < \infty; \tag{3}$$

2) The functions $\alpha(x), \beta(y), \gamma(z)$ satisfy

$$\int_0^\infty \alpha(x)dx = \int_0^\infty \beta(y)dy = \int_0^\infty \gamma(z)dz = \infty. \tag{4}$$

For simplification, we set $I_0 = \lambda+\lambda_0+\lambda_{h_0}+\lambda_{c_0}$, $I_1 = \lambda + \mu_1 + \lambda_{h_1} + \lambda_{c_1}$, $I_2 = \lambda_1 + \mu_2 + \lambda_{h_2} + \lambda_{c_2}$. Now we define the operator $\mathcal{A}$ in $\mathbb{X}$ by

$$\mathcal{A}\begin{pmatrix} p_0 \\ \\ \\ \\ p_1 \\ p_2 \\ p_F(x) \\ p_H(y) \\ p_C(z) \end{pmatrix} = \begin{pmatrix} -I_0p_0 + \mu_1p_1 + \mu_2p_2 \\ +\int_0^\infty \alpha(x)p_F(x)dx \\ +\int_0^\infty \beta(y)p_H(y)dy \\ +\int_0^\infty \gamma(z)p_C(z)dz \\ \lambda_0p_0 - I_1p_1 \\ \lambda p_0 - I_2p_2 \\ -p'_F(x) - \alpha(x)p_F(x) \\ -p'_H(y) - \beta(y)p_H(y) \\ -p'_C(z) - \gamma(z)p_C(z) \end{pmatrix} \tag{5}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (p_0, p_1, p_2, p_F, p_H, p_C) \in \mathbb{X} \\ p_F(x), p_H(y), p_C(z) \in L^1(\mathbb{R}_+) \\ p'_F(x), p'_H(y), p'_C(z) \in L^1(\mathbb{R}_+) \\ p_F(0) = \lambda p_1 + \lambda_1 p_2; \\ p_H(0) = \sum_{j=0}^{2} \lambda_{h_j} p_j \\ p_C(0) = \sum_{j=0}^{2} \lambda_{c_j} p_j \end{array} \right\} \tag{6}$$

Obviously, $\mathcal{A}$ is a linear operator in $\mathbb{X}$.

With the help of above notation, we can rewrite (1) and (2) as an evolutionary equation in the Banach space $\mathbb{X}$

$$\begin{cases} \frac{dP(t)}{dt} = \mathcal{A}P(t), \quad t \geq 0; \\ P(0) = P_0 \end{cases} \tag{7}$$

where $P(t)=(p_0(t),p_1(t),p_2(t),p_F(,t),p_H(,t),p_C(,t)) \in D(\mathcal{A})$ and $P_0 = (1, 0, 0, 0, 0, 0) \in \mathbb{X}$.

Firstly we have the following result.

Theorem 1. *Let $\mathcal{A}$ be defined by (5) and (6). Then the following statements are true*

1) $\mathcal{A}$ is a closed and densely defined linear operator in $\mathbb{X}$;

2) $\mathcal{A}$ is a dissipative operator in $\mathbb{X}$.

Proof: The first assertion is a direct verification, we omit the checking detail. We only prove the second assertion.

For any real $P \in D(\mathcal{A})$, we take a real vector

$$Q = (q_0, q_1, q_2, q_F(x), q_H(y), q_C(z)),$$

where $q_0 = \|P\|\text{sign}(p_0)$, $q_1 = \|P\|\text{sign}(p_1)$, $q_2 = \|P\|\text{sign}(p_2)$, $q_F = \|P\|\text{sign}(p_F(x))$, $q_H = \|P\|\text{sign}(p_H(y))$ and $q_C = \|P\|\text{sign}(p_C(z))$. Clearly, $Q \in \mathbb{X}^*$, where $\mathbb{X}^* = \mathbb{R}^3 \times (L^\infty(\mathbb{R}_+))^3$ is the dual space of $\mathbb{X}$.

For given P and Q, we have $\langle P, Q\rangle = ||P||^2 = ||Q||^2$. Further, we have

$$\begin{aligned} & \frac{\langle \mathcal{A}P, Q\rangle}{\|P\|} \\ = \;& (-I_0p_0 + \mu_1p_1 + \mu_2p_2)\text{sign}(p_0) \\ & +\text{sign}(p_0)\int_0^\infty \alpha(x)p_F(x)dx \\ & +\text{sign}(p_0)\int_0^\infty \beta(y)p_H(y)dy \\ & +\text{sign}(p_0)\int_0^\infty \gamma(z)p_C(z)dz \\ & +(\lambda_0p_0 - I_1p_1)\text{sign}(p_1) + (\lambda p_0 - I_2p_2)\text{sign}(p_2) \\ & +\int_0^\infty (-p'_F(x) - \alpha(x)p_F(x))\text{sign}(p_F(x))dx \\ & +\int_0^\infty (-p'_H(y) - \beta(y)p_H(y))\text{sign}(p_H(y))dy \\ & +\int_0^\infty (-p'_C(z) - \gamma(z)p_C(z))\text{sign}(p_C(z))dz \\ \leq \;& -(I_0 - \lambda_0 - \lambda)|p_0| - (I_1 - \mu_1)|p_1| \\ & -(I_2-\mu_2)|p_1| + |p_F(0))| + |p_H(0)| + |p_C(0)| \end{aligned}$$

where we have used an identity

$$\int_0^\infty p'(x)\text{sign}(p(x))dx = -|p(0)|.$$

Using the boundary conditions in $D(\mathcal{A})$, we can get

$$\begin{aligned} & |p_F(0)| + |p_H(0)|] + |p_C(0)| \\ \leq \;& (\lambda_{h_0} + \lambda_{c_0})|p_0| + (\lambda + \lambda_{h_1} + \lambda_{c_1})|p_1| \\ & +(\lambda_1 + \lambda_{h_2} + \lambda_{c_2})|p_2| \\ = \;& (I_0-\lambda_0-\lambda)|p_0| + (I_1-\mu_1)|p_1| + (I_2-\mu_2)|p_1|. \end{aligned}$$

Therefore, we have $\langle \mathcal{A}P, Q\rangle \leq 0$, which means that $\mathcal{A}$ is a dissipative operator in $\mathbb{X}$. □

Since $\mathcal{A}$ is a closed and densely defined linear operator in $\mathbb{X}$, its dual operator $\mathcal{A}^*$ exists and is also a closed linear operator in $\mathbb{X}^*$. To find out the expression of $\mathcal{A}^*$, we let

$$P = (p_0, p_1, p_2, p_F(x), p_H(y), p_C(z)) \in D(\mathcal{A})$$

and

$$Q = (q_0, q_1, q_2, q_F(x), q_H(y), q_C(z)) \in D(\mathcal{A}^*).$$

Using the equality $\langle \mathcal{A}P, Q\rangle = \langle P, \mathcal{A}^*Q\rangle$, integration by part, we have

$$\begin{aligned} & \langle \mathcal{A}P, Q\rangle = \langle P, \mathcal{A}^*Q\rangle \\ = \ & [-I_0q_0+\lambda_0q_1+\lambda q_2 + \lambda_{h_0}q_H(0)+\lambda_{c_0}q_C(0)]p_0 \\ + \ & [\mu_1q_0-I_1q_1+\lambda q_F(0)+\lambda_{h_1}q_H(0)+\lambda_{c_1}q_C(0)]p_1 \\ + \ & [\mu_2q_0-I_2q_2+\lambda_1q_F(0)+\lambda_{h_2}q_H(0)+\lambda_{c_2}q_C(0)]p_2 \\ + \ & \int_0^\infty [q_F'(x)-\alpha(x)q_F(x)+q_0\alpha(x)]p_F(x)dx \\ + \ & \int_0^\infty [q_H'(x)-\beta(y)q_H(y)+q_0\beta(y)]p_H(y)dy \\ + \ & \int_0^\infty [q_C'(z)-\gamma(z)q_C(z)+q_0\gamma(z)]p_C(z)dz \end{aligned}$$

.

Therefore, we get the expression of $\mathcal{A}^*$

$$\mathcal{A}^*\begin{pmatrix} q_0 \\ \\ q_1 \\ \\ q_2 \\ \\ q_F(x) \\ q_H(y) \\ q_C(z)\end{pmatrix} = \begin{pmatrix} -I_0q_0 + \lambda_0q_1 + \lambda q_2 \\ +\lambda_{h_0}q_H(0) + \lambda_{c_0}q_C(0) \\ \mu_1q_0 - I_1q_1 + \lambda q_F(0) \\ +\lambda_{h_1}q_H(0) + \lambda_{c_1}q_C(0) \\ \mu_2q_0 - I_2q_2 + \lambda_1q_F(0) \\ +\lambda_{h_2}q_H(0) + \lambda_{c_2}q_C(0) \\ q_F'(x)-\alpha(x)q_F(x)+q_0\alpha(x) \\ q_H'(x)-\beta(y)q_H(y)+q_0\beta(y) \\ q_C'(z)-\gamma(z)q_C(z)+q_0\gamma(z)\end{pmatrix} \quad (8)$$

with domain

$$D(\mathcal{A}^*) = \left\{\begin{array}{l} (q_0, q_1, q_2, q_F(x), q_H(y), q_C(z)) \\ \in \mathbb{R}^3 \times (L^\infty(\mathbb{R}_+))^3 = \mathbb{X}^* \\ q_F(x), q_H(y), q_C(z) \\ \text{are absolutely continuous, and} \\ q_F'(x), q_H'(y), q_C'(z) \in L^\infty(\mathbb{R}_+) \end{array}\right\} \quad (9)$$

3 Spectral analysis of $\mathcal{A}$

In this section we shall carry out a complete spectral analysis of $\mathcal{A}$. In what follows we always regard $\mathbb{X}$ as a complex Banach space.

3.1 Spectral analysis

Let $s \in \mathbb{C}$. For any $F \in \mathbb{X}$ fixed, we consider the resolvent equation $[sI - \mathcal{A}]P = F$ where $P = (p_0, p_1, p_2, p_F(x), p_H(y), p_C(z)) \in D(\mathcal{A})$. That is

$$\left\{\begin{array}{l} (s+I_0)p_0 - \sum\limits_{j=1}^2 \mu_jp_j - \int_0^\infty \alpha(x)p_F(x)dx \\ -\int_0^\infty \beta(y)p_H(y)dy - \int_0^\infty \gamma(z)p_C(z)dz = f_0, \\ -\lambda_0p_0 + (s + I_1)p_1 = f_1, \\ -\lambda p_0 + (s + I_2)p_2 = f_2, \\ p_F'(x) + (s + \alpha(x))p_F(x) = f_F(x), \\ p_H'(y) + (s + \beta(y))p_H(y) = f_H(y), \\ p_C'(z) + (s + \gamma(z))p_C(z) = f_C(z), \end{array}\right. \quad (10)$$

and boundary conditions

$$\left\{\begin{array}{l} p_F(0) = \lambda p_1 + \lambda_1 p_2, \\ p_H(0) = \sum\limits_{j=0}^2 \lambda_{h_j}p_j, \\ p_C(0) = \sum\limits_{j=0}^2 \lambda_{c_j}p_j. \end{array}\right. \quad (11)$$

Solving the differential equations in (10) we get the formal solution

$$\left\{\begin{array}{rcl} p_F(x) &=& p_F(0)e^{-\int_0^x (s+\alpha(\tau))d\tau} \\ && + \int_0^x e^{-\int_r^x (s+\alpha(\tau))d\tau} f_F(r)dr, \\ p_H(y) &=& p_H(0)e^{-\int_0^y (s+\beta(\tau))d\tau} \\ && + \int_0^y e^{-\int_r^y (s+\beta(\tau))d\tau} f_H(r)dr, \\ p_C(z) &=& p_C(0)e^{-\int_0^z (s+\gamma(\tau))d\tau} \\ && + \int_0^z e^{-\int_r^z (s+\gamma(\tau))d\tau} f_C(r)dr. \end{array}\right. \quad (12)$$

In order that $p_F(x), p_H(y), p_C(z) \in L^1(\mathbb{R}_+)$, it must hold that

$$\begin{array}{l} e^{-\int_0^x (s+\alpha(\tau))d\tau}, e^{-\int_0^y (s+\beta(\tau))d\tau}, e^{-\int_0^z (s+\gamma(\tau))d\tau}, \\ \int_0^x e^{-\int_r^x (s+\alpha(\tau))d\tau} f_F(r)dr, \int_0^y e^{-\int_r^y (s+\beta(\tau))d\tau} f_H(r)dr, \\ \int_0^z e^{-\int_r^z (s+\gamma(\tau))d\tau} f_C(r)dr \in L^1(\mathbb{R}_+). \end{array}$$

These imply that s must satisfy conditions

$$\begin{array}{l} \sup_{r\ge0} \int_r^\infty e^{-\int_r^x (\Re s+\alpha(\tau))d\tau} dx < \infty, \\ \sup_{r\ge0} \int_r^\infty e^{-\int_r^y (\Re s+\beta(\tau))d\tau} dy < \infty, \\ \sup_{r\ge0} \int_r^\infty e^{-\int_r^z (\Re s+\gamma(\tau))d\tau} dz < \infty. \end{array}$$

Therefore, we define non-negative real numbers $\widehat{\mu}_\alpha$, $\widehat{\mu}_\beta$ and $\widehat{\mu}_\gamma$ as follows

$$\begin{array}{l} \widehat{\mu}_\alpha = \sup\{\eta \ge 0 \mid \sup\limits_{r\ge0} \int_0^\infty e^{\eta x - \int_0^x \alpha(\tau+r)d\tau} dx < \infty\}, \\ \widehat{\mu}_\beta = \sup\{\eta \ge 0 \mid \sup\limits_{r\ge0} \int_0^\infty e^{\eta y - \int_0^y \beta(\tau+r)d\tau} dy < \infty\}, \\ \widehat{\mu}_\gamma = \sup\{\eta \ge 0 \mid \sup\limits_{r\ge0} \int_0^\infty e^{\eta z - \int_0^z \gamma(\tau+r)d\tau} dz < \infty\},. \end{array}$$

Obviously, when $\eta < \widehat{\mu}_\alpha$, then the integral for $\forall r \geq 0$,

$$\int_r^\infty e^{-\int_r^x(\alpha(\tau)-\eta)d\tau}dx=\int_0^\infty e^{-\int_0^x(\alpha(\tau+r)-\eta)d\tau}dx<\infty,$$

while for $\eta > \widehat{\mu}_\alpha$, it must be

$$\int_r^\infty e^{-\int_r^x(\alpha(\tau)-\eta)d\tau}dx=\infty.$$

For the other integrals, similar results hold true.

Note that real numbers $\widehat{\mu}_\alpha, \widehat{\mu}_\beta, \widehat{\mu}_\gamma$ are the measure of essential repair rate of the system. Set

$$\widehat{\mu}=\min\{\widehat{\mu}_\alpha, \widehat{\mu}_\beta, \widehat{\mu}_\gamma\}. \tag{13}$$

Obviously, when $\Re s+\widehat{\mu} < 0$, at least one of $p_F(x), p_H(y), p_C(z)$ given in (12) is not in $L^1(\mathbb{R}_+)$. Therefore, $\{s \in \mathbb{C} \mid \Re s+\widehat{\mu}<0\} \subset \sigma(\mathcal{A})$.

Without loss of generality we can assume that the functions $e^{-\int_0^x(\alpha(\tau+r)-\widehat{\mu}_\alpha)d\tau}$, $e^{-\int_0^y(\beta(\tau+r)-\widehat{\mu}_\beta)d\tau}$ and $e^{-\int_0^z(\gamma(\tau+r)-\widehat{\mu}_\gamma)d\tau}$ are uniformly bounded in $(x,r),(y,r)$ and (z,r), respectively. Set

$$\begin{aligned}N_\alpha&=\sup_{r,x\geq 0}e^{-\int_0^x(\alpha(\tau+r)-\widehat{\mu}_\alpha)d\tau},\\ N_\beta&=\sup_{r,y\geq 0}e^{-\int_0^y(\beta(\tau+r)-\widehat{\mu}_\beta)d\tau},\\ N_\gamma&=\sup_{r,z\geq 0}e^{-\int_0^x(\gamma(\tau+r)-\widehat{\mu}_\gamma)d\tau}.\end{aligned}$$

For $\Re s+\widehat{\mu}>0$, we have the following estimates

$$\begin{aligned}&\int_0^\infty|p_F(x)|dx\\ \leq\ & |p_F(0)|\int_0^\infty e^{-\int_0^x(\Re s+\alpha(\tau))d\tau}dx\\ &+\int_0^\infty|f_F(r)|dr\int_r^\infty e^{-\int_r^x(\Re s+\alpha(\tau))d\tau}dx\\ \leq\ & |p_F(0)|\frac{N_\alpha}{\Re s+\widehat{\mu}_\alpha}+\frac{N_\alpha}{\Re s+\widehat{\mu}_\alpha}||f_F||_{L^1},\end{aligned}$$

$$\begin{aligned}&\int_0^\infty|p_H(y)|dy\\ \leq\ & |p_H(0)|\int_0^\infty e^{-\int_0^y(\Re s+\beta(\tau))d\tau}dy\\ &+\int_0^\infty|f_H(r)|dr\int_r^\infty e^{-\int_r^y(\Re s+\beta(\tau))d\tau}dy\\ \leq\ & |p_H(0)|\frac{N_\beta}{\Re s+\widehat{\mu}_\beta}+\frac{N_\beta}{\Re s+\widehat{\mu}_\beta}||f_H||_{L^1},\end{aligned}$$

$$\begin{aligned}&\int_0^\infty|p_C(z)|dz\\ \leq\ & |p_C(0)|\int_0^\infty e^{-\int_0^z(\Re s+\gamma(\tau))d\tau}dz\\ &+\int_0^\infty|f_C(r)|dr\int_r^\infty e^{-\int_r^z(\Re s+\gamma(\tau))d\tau}dz\\ \leq\ & |p_C(0)|\frac{N_\gamma}{\Re s+\widehat{\mu}_\gamma}+\frac{N_\gamma}{\Re s+\widehat{\mu}_\gamma}||f_C||_{L^1}.\end{aligned}$$

So we have $p_F(x), p_H(y), p_C(z) \in L^1(\mathbb{R}_+)$. Note that these functions are the formal solution of the differential equations in (10). Substituting them into the first equation in (10) and the boundary conditions (11) lead to algebraic equations with unknown variants $p_0, p_1, p_2, p_F(0), p_H(0)$ and $p_C(0)$

$$\begin{cases}(s+I_0)p_0-\mu_1p_1-\mu_2p_2-p_F(0)(1-sG_\alpha(s))\\ -p_H(0)(1-sG_\beta(s))-p_C(0)(1-sG_\gamma(s))=\widehat{f_0},\\ -\lambda_0p_0+(s+I_1)p_1=f_1,\\ -\lambda p_0+(s+I_2)p_2=f_2,\\ -\lambda p_1-\lambda_1p_2+p_F(0)=0,\\ -\lambda_{h_0}p_0-\lambda_{h_1}p_1-\lambda_{h_2}p_2+p_H(0)=0,\\ -\lambda_{c_0}p_0-\lambda_{c_1}p_1-\lambda_{c_2}p_2+p_C(0)=0.\end{cases} \tag{14}$$

Eliminating $p_F(0)$, $p_C(0)$ and $p_H(0)$ from above equations yield

$$\begin{cases}[s+\lambda+\lambda_0+s(\lambda_{h_0}G_\beta(s)+\lambda_{c_0}G_\gamma(s))]p_0\\ +[s(\lambda G_\alpha(s)+\lambda_{h_1}G_\beta(s)+\lambda_{c_1}G_\gamma(s))-I_1]p_1\\ +[s(\lambda_1G_\alpha(s)+\lambda_{h_2}G_\beta(s)+\lambda_{c_2}G_\gamma(s))-I_2]p_2=\widehat{f_0},\\ -\lambda_0p_0+(s+I_1)p_1=f_1,\\ -\lambda p_0+(s+I_2)p_2=f_2\end{cases} \tag{15}$$

where

$$\begin{cases}G_\alpha(s)=\int_0^\infty e^{-\int_0^x(s+\alpha(\tau))d\tau}dx,\\ G_\beta(s)=\int_0^\infty e^{-\int_0^y(s+\beta(\tau))d\tau}dy,\\ G_\gamma(s)=\int_0^\infty e^{-\int_0^z(s+\gamma(\tau))d\tau}dz.\end{cases} \tag{16}$$

and the inhomogeneous term $\widehat{f_0}$ is

$$\widehat{f_0}=f_0+\mathcal{F}_\alpha(s)+\mathcal{F}_\beta(s)+\mathcal{F}_\gamma(s)$$

where

$$\begin{aligned}\mathcal{F}_\alpha(s)&=\int_0^\infty\alpha(x)dx\int_0^x f_F(r)e^{-\int_r^x[s+\alpha(\tau)]d\tau}dr,\\ \mathcal{F}_\beta(s)&=\int_0^\infty\beta(y)dy\int_0^y f_H(r)e^{-\int_r^y[s+\beta(\tau)]d\tau}dr,\\ \mathcal{F}_\gamma(s)&=\int_0^\infty\gamma(z)dz\int_0^z f_C(r)e^{-\int_r^z[s+\gamma(\tau)]d\tau}dr.\end{aligned}$$

A direct calculation gives the determinant of the coefficient matrix of (15)

$$\begin{aligned}D(s)=s\{&s^2+(I_1+I_2+\lambda+\lambda_0)s\\ &+(I_1I_2+\lambda I_1+\lambda_0I_2)\\ &+G_\alpha(s)[\lambda(\lambda_0+\lambda_1)s+\lambda(\lambda_0I_2+\lambda_1I_1)]\\ &+G_\beta(s)[\lambda_{h_0}s^2+(\lambda_{h_0}I_1+\lambda_{h_0}I_2+\lambda_0\lambda_{h_1}+\lambda\lambda_{h_2})s\\ &+\lambda_0\lambda_{h_1}I_2+\lambda\lambda_{h_2}I_1+\lambda_{h_0}I_1I_2]\\ &+G_\gamma(s)[\lambda_{c_0}s^2+(\lambda_{c_0}I_1+\lambda_{c_0}I_2+\lambda_0\lambda_{c_1}+\lambda\lambda_{c_2})s\\ &+\lambda_0\lambda_{c_1}I_2+\lambda\lambda_{c_2}I_1+\lambda_{c_0}I_1I_2]\}.\end{aligned} \tag{17}$$

If $s_1 \in \mathbb{C}$ such that $D(s_1) \neq 0$, solving the algebraic equations (15) we can get

$$\begin{cases} p_0^{(s_1)} = \frac{1}{D(s_1)}[d_{11}(s_1)\widehat{f}_0 + d_{21}(s_1)f_1 + d_{31}(s_1)f_2], \\ p_1^{(s_1)} = \frac{1}{D(s_1)}[d_{12}(s_1)\widehat{f}_0 + d_{22}(s_1)f_1 + d_{32}(s)f_2], \\ p_2^{(s_1)} = \frac{1}{D(s_1)}[d_{13}(s_1)\widehat{f}_0 + d_{23}(s_1)f_1 + d_{33}(s_1)f_2] \end{cases} \tag{18}$$

where $d_{ij}(s_1)(i,j=1,2,3)$ are the algebraic cofactor of $D(s_1)$, they are given by

$$\begin{cases} d_{11}(s_1) = (s_1+I_1)(s_1+I_2), \\ d_{21}(s_1) = -(s_1+I_2) \\ \times[s_1(\lambda G_\alpha(s_1) + \lambda_{h_1}G_\beta(s_1) + \lambda_{c_1}G_\gamma(s_1)) - I_1], \\ d_{31}(s_1) = -(s_1+I_1) \\ \times[s_1(\lambda_1 G_\alpha(s_1) + \lambda_{h_2}G_\beta(s_1) + \lambda_{c_2}G_\gamma(s_1)) - I_2], \\ d_{12}(s_1) = \lambda_0(s_1+I_2), \\ d_{22}(s_1) = (s_1+I_2) \\ \times[s_1+\lambda+\lambda_0+s_1(\lambda_{h_0}G_\beta(s_1)+\lambda_{c_0}G_\gamma(s_1))] \\ +\lambda[s_1(\lambda_1 G_\alpha(s_1) + \lambda_{h_2}G_\beta(s_1)+\lambda_{c_2}G_\gamma(s_1))-I_2], \\ d_{32}(s_1) = -\lambda_0 \\ \times[s_1(\lambda_1 G_\alpha(s_1) + \lambda_{h_2}G_\beta(s_1) + \lambda_{c_2}G_\gamma(s_1))-I_2], \\ d_{13}(s_1) = \lambda(s_1+I_1), \\ d_{23}(s_1) = -\lambda \\ \times[s_1(\lambda G_\alpha(s_1) + \lambda_{h_1}G_\beta(s_1) + \lambda_{c_1}G_\gamma(s_1)) - I_1], \\ d_{33}(s_1) = (s_1+I_1) \\ \times[s_1+\lambda+\lambda_0+s_1(\lambda_{h_0}G_\beta(s_1)+\lambda_{c_0}G_\gamma(s_1))] \\ +\lambda_0[s_1(\lambda G_\alpha(s_1)+\lambda_{h_1}G_\beta(s_1)+\lambda_{c_1}G_\gamma(s_1))-I_1]. \end{cases} \tag{19}$$

From (11) we can get

$$\begin{cases} p_F^{(s_1)}(0) = \frac{\lambda}{D(s_1)}[d_{12}(s_1)\widehat{f}_0+d_{22}(s_1)f_1+d_{32}(s_1)f_2] \\ +\frac{\lambda_1}{D(s_1)}[d_{13}(s_1)\widehat{f}_0 + d_{23}(s_1)f_1 + d_{33}(s_1)f_2], \\ p_H^{(s_1)}(0) = \frac{\lambda_{h_0}}{D(s_1)}[d_{11}(s_1)\widehat{f}_0+d_{21}(s_1)f_1+d_{31}(s_1)f_2] \\ +\frac{\lambda_{h_1}}{D(s_1)}[d_{12}(s_1)\widehat{f}_0 + d_{22}(s_1)f_1 + d_{32}(s_1)f_2] \\ +\frac{\lambda_{h_2}}{D(s_1)}[d_{13}(s_1)\widehat{f}_0 + d_{23}(s_1)f_1 + d_{33}(s_1)f_2], \\ p_C^{(s_1)}(0) = \frac{\lambda_{c_0}}{D(s_1)}[d_{11}(s_1)\widehat{f}_0+d_{21}(s_1)f_1+d_{31}(s_1)f_2] \\ +\lambda_{c_1}\frac{1}{D(s_1)}[d_{12}(s_1)\widehat{f}_0 + d_{22}(s_1)f_1 + d_{32}(s_1)f_2] \\ +\lambda_{c_2}\frac{1}{D(s_1)}[d_{13}(s_1)\widehat{f}_0 + d_{23}(s_1)f_1 + d_{33}(s_1)f_2]. \end{cases} \tag{20}$$

According to (12) we have

$$\begin{cases} p_F^{(s_1)}(x) = p_F^{(s_1)}(0)e^{-\int_0^x(s_1+\alpha(\tau))d\tau} \\ +\int_0^x e^{-\int_r^x(s_1+\alpha(\tau))d\tau}f_F(r)dr, \\ p_H^{(s_1)}(y) = p_H^{(s_1)}(0)e^{-\int_0^y(s_1+\beta(\tau))d\tau} \\ +\int_0^y e^{-\int_r^y(s_1+\beta(\tau))d\tau}f_H(r)dr, \\ p_C^{(s_1)}(z) = p_C^{(s_1)}(0)e^{-\int_0^z(s_1+\gamma(\tau))d\tau} \\ +\int_0^z e^{-\int_r^z(s_1+\gamma(\tau))d\tau}f_C(r)dr. \end{cases} \tag{21}$$

Thus we obtain unique a solution of (10) and (11) in $\mathbb{X}$ whose entries are determined by (18) and (21). So $P = (p_0^{(s_1)}, p_1^{(s_1)}, p_2^{(s_1)}, p_F^{(s_1)}(x), p_H^{(s_1)}(y), p_C^{(s_1)}(z)) \in D(\mathcal{A})$ and $(s_1I - \mathcal{A})P = F$. Therefore, $s_1 \in \rho(\mathcal{A})$.

For $s \in \mathbb{C}$ with $\Re s + \widehat{\mu} > 0$, the functions $G_\alpha(s)$, $G_\beta(s)$ and $G_\gamma(s)$ defined by (16) have meaning. If $s_0 \in \mathbb{C}$ with $\Re s_0 + \widehat{\mu} > 0$ such that $D(s_0) = 0$, the homogeneous algebraic equations for $s = s_0$

$$\begin{cases} (s+I_0)p_0 - \mu_1 p_1 - \mu_2 p_2 - p_F(0)(1-sG_\alpha(s)) \\ -p_H(0)(1-sG_\beta(s)) - p_C(0)(1-sG_\gamma(s)) = 0, \\ -\lambda_0 p_0 + (s+I_1)p_1 = 0, \\ -\lambda p_0 + (s+I_2)p_2 = 0, \\ -\lambda p_1 - \lambda_1 p_2 + p_F(0) = 0, \\ -\lambda_{h_0}p_0 - \lambda_{h_1}p_1 - \lambda_{h_2}p_2 + p_H(0) = 0, \\ -\lambda_{c_0}p_0 - \lambda_{c_1}p_1 - \lambda_{c_2}p_2 + p_C(0) = 0 \end{cases} \tag{22}$$

have a non-zero solution of the form

$$\begin{cases} p_0^{(s_0)} = (s_0+I_1)(s_0+I_2), \\ p_1^{(s_0)} = \lambda_0(s_0+I_2), \\ p_2^{(s_0)} = \lambda(s_0+I_1), \\ p_F^{(s_0)}(0) = \lambda\lambda_0(s_0+I_2) + \lambda_1\lambda(s_0+I_1), \\ p_H^{(s_0)}(0) = \lambda_{h_0}(s_0+I_1)(s_0+I_2) \\ +\lambda_{h_1}\lambda_0(s_0+I_2) + \lambda_{h_2}\lambda(s_0+I_1), \\ p_C^{(s_0)}(0) = \lambda_{c_0}(s_0+I_1)(s_0+I_2) \\ +\lambda_{c_1}\lambda_0(s_0+I_2) + \lambda_{c_2}\lambda(s_0+I_1). \end{cases} \tag{23}$$

Using (12) with $f_F(x) = f_H(y) = f_C(z) = 0$ and (23) we can show that the functions

$$\begin{cases} p_0^{(s_0)} = (s_0+I_1)(s_0+I_2) \\ p_1^{(s_0)} = \lambda_0(s_0+I_2) \\ p_2^{(s_0)} = \lambda(s_0+I_1) \\ p_F^{(s_0)}(x) = \\ = [\lambda\lambda_0(s_0+I_2) + \lambda_1\lambda(s_0+I_1)]e^{-\int_0^x(s_0+\alpha(\tau))d\tau} \\ p_H^{(s_0)}(y) = [\lambda_{h_0}(s_0+I_1)(s_0+I_2)+\lambda_{h_1}\lambda_0(s_0+I_2) \\ +\lambda_{h_2}\lambda(s_0+I_1)]e^{-\int_0^y(s_0+\beta(\tau))d\tau} \\ p_C^{(s_0)}(x) = [\lambda_{c_0}(s_0+I_1)(s_0+I_2)+\lambda_{c_1}\lambda_0(s_0+I_2) \\ +\lambda_{c_2}\lambda(s_0+I_1)]e^{-\int_0^z(s_0+\gamma(\tau))d\tau} \end{cases} \tag{24}$$

satisfy the homogeneous equations (10) with $f_0 = f_1 = f_2 = f_F(x) = f_H(y) = f_C(z) = 0$ and (11) and $p_F^{(s_0)}(x), p_H^{(s_0)}(y), p_C^{(s_0)}(z) \in L^1(\mathbb{R}_+)$ for $\Re z + \widehat{\mu} > 0$. Set

$$P^{(s_0)} = (p_0^{(s_0)}, p_1^{(s_0)}, p_2^{(s_0)}, p_F^{(s_0)}(x), p_H^{(s_0)}(y), p_C^{(s_0)}(z)),$$

we have $P^{(s_0)} \in D(\mathcal{A})$ and $\mathcal{A}P^{(s_0)} = s_0 P^{(s_0)}$. So s_0 is an eigenvalue of $\mathcal{A}$.

Summarizing the discussion above, we have proved the following result.

Theorem 2. *Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before, and $\widehat{\mu}$ be defined by (13). Then the following assertions are true:*

1) The half-plane $\{s \in \mathbb{C} \mid \Re s + \widehat{\mu} < 0\}$ are in the spectrum of $\mathcal{A}$;

2) The set $\{s \in \mathbb{C} \mid \Re s + \widehat{\mu} > 0, D(s) \neq 0\}$ is in the resolvent set of $\mathcal{A}$;

3) The set $\{s \in \mathbb{C} \mid \Re s + \widehat{\mu} > 0, D(s) = 0\}$ consists of all eigenvalues of $\mathcal{A}$;

4) The spectrum $\sigma(\mathcal{A})$ distributes symmetrically with respect to the real axis.

Note that when $s_1 \in \mathbb{C}$ with $\Re s_1 + \widehat{\mu} > 0$ and $D(s_1) \neq 0$, we have $s_1 \in \rho(\mathcal{A})$. So the solution of (10) and (11) is given by $P = R(s_1, \mathcal{A})F$. In this case, according to the previous calculation, we have norm estimate

$$\begin{aligned} ||P||_{\mathbb{X}} &= \sum_{i=0}^{2} |p_i^{(s_1)}| + \int_0^\infty |p_F^{(s_1)}(x)|dx \\ &\quad + \int_0^\infty |p_H^{(s_1)}(y)|dy + \int_0^\infty |p_C^{(s_1)}(z)|dz \\ &\le \sum_{i=0}^{2} |p_i^{(s_1)}| \\ &+ N\frac{[|p_F^{(s_1)}(0)| + |p_H^{(s_1)}(0)| + |p_C^{(s_1)}(0)|]}{\Re s_1 + \widehat{\mu}} \\ &+ N\frac{[||f_F||_{L^1} + ||f_H||_{L^1} + ||f_C||_{L^1}]}{\Re s_1 + \widehat{\mu}} \end{aligned}$$

where $N = \max\{N_\alpha, N_\beta, N_\gamma\}$.

We estimate the term

$$\begin{aligned} &\sum_{i=0}^{2} |p_i^{(s_1)}| + N\frac{[|p_F^{(s_1)}(0)| + |p_H^{(s_1)}(0)| + |p_C^{(s_1)}(0)|]}{\Re s_1 + \widehat{\mu}} \\ \le\ & (1 + \frac{N(\lambda_{h_0} + \lambda_{c_0})}{\Re s_1 + \widehat{\mu}})|p_0^{(s_1)}| \\ & +(1 + \frac{N(\lambda + \lambda_{h_1} + \lambda_{c_1})}{\Re s_1 + \widehat{\mu}})|p_1^{(s_1)}| \\ & +(1 + \frac{N(\lambda_1 + \lambda_{h_2} + \lambda_{c_2})}{\Re s_1 + \widehat{\mu}})|p_2^{(s_1)}| \\ \le\ & N_1 \sum_{i=0}^{2} |p_i^{(s_1)}| \end{aligned}$$

where N_1 is the maximum value of $(1 + \frac{N(\lambda_{h_0}+\lambda_{c_0})}{\Re s_1 + \widehat{\mu}})$, $(1 + \frac{N(\lambda+\lambda_{h_1}+\lambda_{c_1})}{\Re s_1 + \widehat{\mu}})$ and $(1 + \frac{N(\lambda_1+\lambda_{h_2}+\lambda_{c_2})}{\Re s_1 + \widehat{\mu}})$. According to (18) it holds

$$\sum_{i=0}^{2} |p_i^{(s_1)}| \le \frac{\max\limits_{1\le i\le 3} \sum\limits_{j=1}^{3} |d_{ij}(s_1)|(|\widehat{f}_0| + |f_1| + |f_2|)}{|D(s_1)|},$$

while

$$\begin{aligned} & |\widehat{f}_0| + |f_1| + |f_2| \\ =\ & |f_0| + |f_1| + |f_2| + |\mathcal{F}_\alpha(s_1)| \\ & +|\mathcal{F}_\beta(s_1)| + |\mathcal{F}_\gamma(s_1)| \\ \le\ & |f_0| + |f_1| + |f_2| + ||f_F||_{L^1}[1 + \frac{N_\alpha|\Re s_1|}{\Re s_1 + \widehat{\mu}_\alpha}] \\ & +||f_H||_{L^1}[1 + \frac{N_\beta|\Re s_1|}{\Re s_1 + \widehat{\mu}_\beta}] \\ & +||f_C||_{L^1}[1 + \frac{N_\gamma|\Re s_1|}{\Re s_1 + \widehat{\mu}_\gamma}] \\ \le\ & [1 + \frac{N|\Re s_1|}{\Re s_1 + \widehat{\mu}}]||F||_{\mathbb{X}}, \end{aligned}$$

so we have

$$\begin{aligned} & \sum_{i=0}^{2} |p_i^{(s_1)}| \\ \le\ & \frac{\max\limits_{1\le i\le 3} \sum\limits_{j=1}^{3} |d_{ij}(s_1)|}{|D(s_1)|} \left[1 + \frac{N|\Re s_1|}{\Re s_1 + \widehat{\mu}}\right] ||F||_{\mathbb{X}}. \end{aligned}$$

Using (19) we can get that there is a positive constant M such that

$$\max_{1\le i\le 3} \sum_{j=1}^{3} |d_{ij}(s_1)| \le M\left(|s_1|^3 + \frac{N}{(\Re s_1 + \widehat{\mu})}\right)$$

for $\forall \Re s_1 + \widehat{\mu} > 0$. Therefore, we have

$$\begin{aligned} & ||P||_{\mathbb{X}} \le \sum_{i=0}^{2} |p_i^{(s_1)}| \\ & +\frac{N}{\Re s_1 + \widehat{\mu}}\left[|p_F^{(s_1)}(0)| + |p_H^{(s_1)}(0)| + p_C^{(s_1)}(0)\right] \\ & +\frac{N}{\Re s_1 + \widehat{\mu}}[||f_F||_{L^1} + ||f_H||_{L^1} + ||f_C||_{L^1}] \\ \le\ & N_1 \sum_{i=0}^{2} |p_i^{(s_1)}| + N\frac{||f_F||_{L^1} + ||f_H||_{L^1} + ||f_C||_{L^1}}{\Re s_1 + \widehat{\mu}} \\ \le\ & N_1 \frac{M}{|D(s_1)|}\left(|s_1|^3 + \frac{N}{(\Re s_1 + \widehat{\mu})}\right) \\ & \times\left[1 + \frac{N|\Re s_1|}{\Re s_1 + \widehat{\mu}}\right] ||F||_{\mathbb{X}} + \frac{N}{\Re s_1 + \widehat{\mu}}||F||_{\mathbb{X}} \\ \le\ & H(s_1)||F||_{\mathbb{X}} \end{aligned}$$

where $H(s_1) = \frac{MN_1}{|D(s_1)|}\left(|s_1|^3 + \frac{N}{(\Re s_1 + \widehat{\mu})}\right)\left[1 + \frac{N|\Re s_1|}{\Re s_1 + \widehat{\mu}}\right] + \frac{N}{\Re s_1 + \widehat{\mu}}$.

Since $D(s)$ is analysis in the half-plane $\Re s + \widehat{\mu} > 0$, we have $\lim\limits_{\Im s \to \infty} \frac{D(s)}{s^3 + I_0 I_1 I_2} = 1$, the limit is uniformly in the region $\Re s + \widehat{\mu} \ge \delta > 0$. So the term $\frac{\left(|s_1|^3+\frac{N}{(\Re s_1+\widehat{\mu})}\right)}{|D(s_1)|}$ is bounded as $|\Im s_1| \to \infty$ with $\Re s_1 + \widehat{\mu} \geq \delta > 0$. We can define the number

$$M(\Re s_1) = \frac{|D(s_1)|}{|s_1|^3 + I_0 I_1 I_2}(\Re s_1 + \widehat{\mu})H(s_1).$$

Obviously, when $\Re s_1 + \widehat{\mu} \geq \delta > 0$, $M(\Re s_1)$ is bounded uniformly. In addition, $\mathcal{A}$ is a dissipative operator in $\mathbb{X}$, we also have $||R(s_1, A)|| \leq \frac{1}{\Re s_1}$ as $\Re s_1 > 0$. So far we have proved the following result.

Theorem 3. *Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before, and let $D(s)$ be defined by (17). Then for any $s \in \{s \in \mathbb{C} \mid \Re s + \widehat{\mu} > 0, D(s) \neq 0\}$, there exists a nonnegative function $M(\Re s)$ such that*

$$||R(s, A)|| \leq \frac{(|s|^3 + I_0 I_1 I_2)M(\Re s)}{|D(s)|(\Re s + \widehat{\mu})}.$$

In particular, when $\Re s > 0$, it holds that $||R(s, A)|| \leq \frac{1}{\Re s}$.

As a consequence of Theorem 3, we have the following corollary thank to the semigroup theory (see, [13]).

Corollary 4. *Let $\mathcal{A}$ be defined by (5)-(6), and let the conditions (3) and (4) hold. Then $\mathcal{A}$ generates a C_0-semigroup on $\mathbb{X}$ of contraction. Hence the system (7) is well-posed in $\mathbb{X}$.*

3.2 Eigenvalues of $\mathcal{A}$ and their distribution

From Theorem 2 we see that $s \in \mathbb{C}$ with $\Re s + \widehat{\mu} > 0$ is an eigenvalue of $\mathcal{A}$ if and only if $D(s) = 0$. Since $\mathcal{A}$ is a dissipative operator, we have $\sigma(\mathcal{A}) \subset \{s \in \mathbb{C} \mid \Re s \leq 0\}$. Therefore, we only need to discuss zero of $D(s)$ in the region $\Re s + \widehat{\mu} > 0$ and $\Re s \leq 0$. In this subsection, we shall discuss the existence of eigenvalues of $\mathcal{A}$.

3.2.1 Eigenvalue 0

From (17) we see that $s = 0$ is a zero of $D(s)$, denote it $\gamma_0 = 0$. The functions defined by (24) with $s_0 = 0$ are the formal solution to the eigenvalue problem of $\mathcal{A}$. If $\widehat{\mu} > 0$, they are in $L^1(\mathbb{R}_+)$ and hence $s_0 = 0$ is always an eigenvalue of $\mathcal{A}$. If $\widehat{\mu} = 0$, we have the following result.

Theorem 5. *Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before. If $\alpha(x)$, $\beta(y)$ and $\gamma(z)$ satisfy the condition*

$$e^{-\int_0^x \alpha(\tau)d\tau}, e^{-\int_0^y \beta(\tau)d\tau}, e^{-\int_0^z \gamma(\tau)d\tau} \in L^1(\mathbb{R}_+). \quad (25)$$

then $\gamma_0 = 0$ is a simple eigenvalue of $\mathcal{A}$, and corresponding an eigenvector is

$$\widehat{P}_0 = \frac{1}{Z}(p_0^{(0)}, p_1^{(0)}, p_2^{(0)}, p_F^{(0)}(x), p_H^{(0)}(y), p_C^{(0)}(z)) \quad (26)$$

where

$$\begin{cases} p_0^{(0)} = I_1 I_2, \\ p_1^{(0)} = \lambda_0 I_2, \\ p_2^{(0)} = \lambda I_1, \\ p_F^{(0)}(x) = [\lambda\lambda_0 I_2 + \lambda_1 \lambda I_1]e^{-\int_0^x \alpha(\tau)d\tau}, \\ p_H^{(0)}(y) = [\lambda_{h_0} I_1 I_2 + \lambda_{h_1}\lambda_0 I_2 + \lambda_{h_2}\lambda I_1]e^{-\int_0^y \beta(\tau)d\tau}, \\ p_C^{(0)}(z) = [\lambda_{c_0} I_1 I_2 + \lambda_{c_1}\lambda_0 I_2 + \lambda_{c_2}\lambda I_1]e^{-\int_0^z \gamma(\tau)d\tau} \end{cases} \quad (27)$$

and

$$\begin{aligned} Z \;=\;& I_1 I_2 + \lambda_0 I_2 + \lambda I_1 + [\lambda\lambda_0 I_2 + \lambda_1\lambda I_1]G_\alpha(0) \\ &+ [\lambda_{h_0} I_1 I_2 + \lambda_{h_1}\lambda_0 I_2 + \lambda_{h_2}\lambda I_1]G_\beta(0) \\ &+ [\lambda_{c_0} I_1 I_2 + \lambda_{c_1}\lambda_0 I_2 + \lambda_{c_2}\lambda I_1]G_\gamma(0) \end{aligned} \quad (28)$$

In particular, the entries of $\widehat{P}_0$ are positive and $||\widehat{P}_0||_{\mathbb{X}} = 1$.

Proof: We verify all assertions by three steps.

Step 1: $\gamma_0 = 0$ is an eigenvalue of $\mathcal{A}$.

Under the condition (25) the functions defined in (27) are in $L^1(\mathbb{R}_+)$. Hence

$$\widehat{P}_0 = \frac{1}{Z}(p_0^{(0)}, p_1^{(0)}, p_2^{(0)}, p_F^{(0)}(x), p_H^{(0)}(y), p_C^{(0)}(z))$$

is element of $D(\mathcal{A})$ whose entry are determined by (27) and $\mathcal{A}\widehat{P}_0 = 0$. Therefore, 0 is an eigenvalue of $\mathcal{A}$ and $\widehat{P}_0$ is an eigenvector. Obviously, the $p_0^{(0)}, p_1^{(0)}, p_2^{(0)}, p_F^{(0)}(x), p_H^{(0)}(y), p_C^{(0)}(z)$ defined in (27) are positive. Therefore, $\widehat{P}_0$ is a positive vector in $\mathbb{X}$. In particular, the eigen-subspace corresponding to γ_0 is one-dimensional.

Step 2. 0 is an eigenvalue of $\mathcal{A}^*$ and $Q_0 = (1, 1, 1, 1, 1, 1) \in D(\mathcal{A}^*)$ is a corresponding eigenfunction.

From the expression of $\mathcal{A}^*$ in (8)–(9) we see that $Q_0 = (1, 1, 1, 1, 1, 1) \in D(\mathcal{A}^*)$ and $\mathcal{A}^* Q = 0$. So Q is an eigenvector of $\mathcal{A}^*$ corresponding of to 0.

Step 3: 0 is a simple eigenvalue of $\mathcal{A}$.

Let $\widehat{P}_0 = \frac{1}{Z}(p_0^{(0)}, p_1^{(0)}, p_2^{(0)}, p_F^{(0)}(x), p_H^{(0)}(y), p_C^{(0)}(z))$ whose entry is determined by (27) and $Q_0 = (1, 1, 1, 1, 1, 1)$. Then we have

$$\begin{aligned} \langle \widehat{P}_0, Q_0 \rangle \;&=\; \tfrac{1}{Z}\sum_{i=0}^{2} p_i^{(0)} + \tfrac{1}{Z}\int_0^\infty p_F^{(0)}(x)dx \\ &\quad + \tfrac{1}{Z}\int_0^\infty p_H^{(0)}(y)dy + \tfrac{1}{Z}\int_0^\infty p_C^{(0)}(z)dz \\ &=\; ||\widehat{P}_0||_{\mathbb{X}} = 1 \end{aligned}$$

Therefore, 0 is a simple eigenvalue of $\mathcal{A}$. □

Theorem 6. *If the functions $\alpha(x)$, $\beta(y)$ and $\gamma(z)$ satisfy conditions*

$$\begin{cases} \sup\limits_{r\in\mathbb{R}_+} \int_r^\infty e^{-\int_r^x \alpha(\tau)ds} d\tau < \infty, \\ \sup\limits_{r\in\mathbb{R}_+} \int_r^\infty e^{-\int_r^y \beta(\tau)d\tau} dy < \infty, \\ \sup\limits_{r\in\mathbb{R}_+} \int_r^\infty e^{-\int_r^z \gamma(\tau)d\tau} dz < \infty, \end{cases} \quad (29)$$

then $i\mathbb{R}\backslash\{0\} \subset \rho(A)$.

Proof If (29) hold, then (25) also hold. In this case, 0 always is an eigenvalue of $\mathcal{A}$. So we only need to prove that the condition (29) implies that $i\mathbb{R}\backslash\{0\} \subset \rho(\mathcal{A})$.

For any $b \in \mathbb{R}, b \neq 0$, the matrix $\Delta(ib)$ of coefficients of (22) is

$$\begin{pmatrix} ib+I_0 & -\mu_1 & -\mu_2 & -f_\alpha(ib) & -f_\beta(ib) & -f_\gamma(ib) \\ -\lambda_0 & ib+I_1 & 0 & 0 & 0 & 0 \\ -\lambda & 0 & ib+I_2 & 0 & 0 & 0 \\ 0 & -\lambda & -\lambda_1 & 1 & 0 & 0 \\ -\lambda_{h_0} & -\lambda_{h_1} & -\lambda_{h_2} & 0 & 1 & 0 \\ -\lambda_{c_0} & -\lambda_{c_1} & -\lambda_{c_2} & 0 & 0 & 1 \end{pmatrix} \quad (30)$$

where $f_\alpha(s) = 1 - sG_\alpha(s)$, $f_\beta(s) = 1 - sG_\beta(s)$ and $f_\gamma(s) = 1 - sG_\gamma(s)$. Since

$$I_0 = \lambda_0 + \lambda + \lambda_{h_0} + \lambda_{c_0} < \sqrt{I_0^2 + b^2} = |I_0 + ib|,$$

$$I_1 = \lambda + \mu_1 + \lambda_{h_1} + \lambda_{c_1} < \sqrt{I_1^2 + b^2} = |I_1 + ib|,$$

$$I_2 = \lambda_1 + \mu_2 + \lambda_{h_2} + \lambda_{c_2} < \sqrt{I_2^2 + b^2} = |I_2 + ib|,$$

$|f_\alpha(ib)| < 1$, $|f_\beta(ib)| < 1$ and $|f_\gamma(ib)| < 1$, $\Delta_1(ib)$ is a strictly diagonal-dominant matrix about column, which implies $D(ib) = \det \Delta_1(ib) \neq 0, \forall b \neq 0, b \in \mathbb{R}$. Therefore, there is not the eigenvalue of $\mathcal{A}$ on the imaginary axis besides $s = 0$. In this case, the equations (15) has uniquely a solution $p_0^{(s_1)}, p_1^{(s_1)}, p_2^{(s_1)}$ satisfaction (18) with $s_1 = ib$. Under the conditions (29), we can verify that the functions defined by (21) with $s_1 = ib(b \neq 0)$ are in $L^1(\mathbb{R}_+)$, so $(p_0^{(s_1)}, p_1^{(s_1)}, p_2^{(s_1)}, p_F^{(s_1)}(x), p_H^{(s_1)}(y), p_C^{(s_1)}(z)) \in D(\mathcal{A})$ is unique a solution of resolvent equation $(ibI - \mathcal{A})P = F$. Therefore, $ib \in \rho(\mathcal{A})$, that is $i\mathbb{R}\backslash\{0\} \subset \rho(\mathcal{A})$. □

The condition (29) is necessary for $i\mathbb{R}\backslash\{0\} \subset \rho(\mathcal{A})$. If one of them fails, then the imaginary axis will be in $\sigma(\mathcal{A})$. Let us consider the following example.

Example 3.1. *Let functions $\alpha(x)$, $\beta(y)$ and $\gamma(z)$ be given by*

$$\alpha(x) = (\alpha+1)\frac{[\ln(1+x)]^\alpha}{1+x},$$

$$\beta(y) = (\beta+1)\frac{[\ln(1+y)]^\beta}{1+y},$$

$$\gamma(x) = (\gamma+1)\frac{[\ln(1+z)]^\gamma}{1+z}.$$

Their distributions are $A(x) = 1 - \exp(-[\ln(1+x)]^{\alpha+1})$, $B(y) = 1 - \exp(-[\ln(1+y)]^{\beta+1})$ and $\Gamma(z) = 1 - \exp(-[\ln(1+z)]^{\gamma+1})$, respectively.

Obviously, it holds that, for each $r > 0$,

$$\exp([\ln(1+r)]^{\alpha+1}) \int_r^\infty \exp(-[\ln(1+x)]^{\alpha+1}) dx < \infty$$

and

$$\lim_{r\to\infty} \exp([\ln(1+r)]^{\alpha+1}) \int_r^\infty \exp(-[\ln(1+x)]^{\alpha+1}) dx = \lim_{r\to\infty} \frac{1+r}{(\alpha+1)[\ln(1+r)]^\alpha} = \infty.$$

For for any $f \in L^1(\mathbb{R}_+)$, $f(x) \geq 0$,

$$\begin{aligned} &\int_0^\infty \left| \int_0^x e^{-\int_r^x (ib+\alpha(s))ds} f(r) e^{-ibr} dr \right| dx \\ =& \int_0^\infty \left| \int_0^x e^{-\int_r^x \alpha(s))ds} f(r) dr \right| dx \\ =& \int_0^\infty \int_0^x e^{-\int_r^x \alpha(s))ds} f(r) dr dx \\ =& \int_0^\infty f(r) dr \int_r^\infty e^{-\int_r^x \alpha(s))ds} dx \\ =& \int_0^\infty f(r) \exp([\ln(1+r)]^{\alpha+1}) dr \\ & \int_r^\infty \exp(-[\ln(1+x)]^{\alpha+1}) dx \end{aligned}$$

The uniformly bounded principle asserts that there is at least one $f_0 \in L^1(\mathbb{R}+)$ such that $\int_0^\infty f_0(r) \exp([\ln(1+r)]^{\alpha+1}) dr \int_r^\infty \exp(-[\ln(1+x)]^{\alpha+1}) dx = \infty$. Therefore, $i\mathbb{R} \subset \sigma(\mathcal{A})$.

3.3 Other eigenvalues

Theorem 7. *Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before. Suppose that $\widehat{\mu} = min\{\widehat{\mu}_\alpha, \widehat{\mu}_\beta, \widehat{\mu}_\gamma\} > 0$. Then $\sigma(\mathcal{A})$ has the following properties*

(1). $\forall\delta > 0$, there are at most finitely many eigenvalues of $\mathcal{A}$ in the region $\{s \in \mathbb{C} \mid \Re s + \widehat{\mu} \geq \delta\}$;

(2). There exists a constant $\omega_1 > 0$ such that the region $\{s \in \mathbb{C} \mid \Re s > -\omega_1\}$ has only one eigenvalue $\gamma_0 = 0$.

Proof In the half-plane $\Re s + \widehat{\mu} > 0$ we have proved that there are eigenvalues of $\mathcal{A}$ only, and s is an eigenvalue of $\mathcal{A}$ if and only if $D(s) = 0$. When $\widehat{\mu} > 0$ there is only one zero of $D(s)$ on the imaginary axis according to Theorem 6. We consider the zeros of

$D(s)$ in the region $-\widehat{\mu}+\delta \le \Re s < 0$. Observing that the functions $f_\alpha(s) = \int_0^\infty \alpha(x)e^{-\int_0^x (s+\alpha(r))dr}dx$, $f_\beta(s) = \int_0^\infty \beta(y)e^{-\int_0^y (s+\beta(r))dr}dy$ and $f_\gamma(s) = \int_0^\infty \gamma(z)e^{-\int_0^z (s+\gamma(r))dr}dz$ are analysis in the region. The Riemann Lemma asserts that

$$\lim_{\Im s\to\infty} f_\alpha(s) = \lim_{\Im s\to\infty} f_\beta(s) = \lim_{\Im s\to\infty} f_\gamma(s) = 0.$$

Therefore, $\lim\limits_{\Im s\to\infty} \frac{D(s)}{s^3} = 1$ is uniformly in the region $-\widehat{\mu}+\delta \le \Re s \le 0$. So $D(s)$ has at most finite number of zeros in $-\widehat{\mu}+\delta \le \Re s \le 0$. So $\mathcal{A}$ has at most finite number eigenvalue in $-\widehat{\mu}+\delta \le \Re s \le 0$.

Set $H_1(s) = \frac{D(s)}{s}$. So $s \ne 0$ is a zero of $D(s)$ if and only if it is that of $H_1(s)$. Since $\overline{H_1(s)} = H_1(\overline{s})$, its zeros are symmetrically with respect to the real axis. Note that $\widehat{\mu} > 0$ implies $H_1(ib) \ne 0, b \in \mathbb{R}$. Let the zeros of $H_1(s)$ in the region $-\widehat{\mu}+\delta \le \Re s \le 0$ be $z_k, k = 1, 2, \cdots, m$. We can set

$$\omega_1 = \min_{1\le k\le m} |\Re z_k|.$$

There is no zero of $H_1(s)$ as $\Re s > -\omega_1$. Hence there is only one eigenvalue $\gamma_0 = 0$ of $\mathcal{A}$ in the region $\{s \in \mathbb{C} \mid \Re s > -\omega_1\}$. □

Now let us estimate the real part of zeros of $H_1(s)$. For any $s \in \mathbb{C}$ with $-\widehat{\mu} < \Re s \le 0$, we have

$$\begin{aligned}
&H_1(s) = s^2 + (I_1+I_2+\lambda+\lambda_0)s\\
&+(I_1I_2+\lambda I_1+\lambda_0 I_2)\\
&+G_\alpha(s)[\lambda(\lambda_0+\lambda_1)s+\lambda(\lambda_0 I_2+\lambda_1 I_1)]\\
&+G_\beta(s)[\lambda_{h_0}s^2+(\lambda_{h_0}I_1+\lambda_{h_0}I_2+\lambda_0\lambda_{h_1}+\lambda\lambda_{h_2})s\\
&\quad+\lambda_0\lambda_{h_1}I_2+\lambda\lambda_{h_2}I_1+\lambda_{h_0}I_1I_2]\\
&+G_\gamma(s)[\lambda_{c_0}s^2+(\lambda_{c_0}I_1+\lambda_{c_0}I_2+\lambda_0\lambda_{c_1}+\lambda\lambda_{c_2})s\\
&\quad+\lambda_0\lambda_{c_1}I_2+\lambda\lambda_{c_2}I_1+\lambda_{c_0}I_1I_2]\\
&= h_0(s)s^2+[(I_1+I_2)h_0(s)+\lambda_0h_1(s)+\lambda h_2(s)]s\\
&+I_1I_2h_0(s)+\lambda_0I_2h_1(s)+\lambda I_1h_2(s)
\end{aligned}$$

where

$$h_0(s) = 1+\lambda_{h_0}G_\beta(s)+\lambda_{c_0}G_\gamma(s),$$

$$h_1(s) = 1+\lambda G_\alpha(s)+\lambda_{h_1}G_\beta(s)+\lambda_{c_1}G_\gamma(s),$$

$$h_2(s) = 1+\lambda_1 G_\alpha(s)+\lambda_{h_2}G_\beta(s)+\lambda_{c_2}G_\gamma(s).$$

If $h_0(s) \ne 0$, we define functions $f_j(s), j = 1, 2$ by

$$f_1(s) = \frac{-[(I_1+I_2)h_0(s)+\lambda_0h_1(s)+\lambda h_2(s)]+\sqrt{\Delta_1(s)}}{2h_0(s)} \tag{31}$$

and

$$f_2(s) = \frac{-[(I_1+I_2)h_0(s)+\lambda_0h_1(s)+\lambda h_2(s)]-\sqrt{\Delta_1(s)}}{2h_0(s)} \tag{32}$$

where

$$\begin{aligned}
&\triangle_1(s) = [(I_1+I_2)h_0(s)+\lambda_0h_1(s)+\lambda h_2(s)]^2\\
&-4h_0(s)[I_1I_2h_0(s)+\lambda_0I_2h_1(s)+\lambda I_1h_2(s)].
\end{aligned}$$

Clearly, $H_1(s) = h_0(s)(s-f_1(s))(s-f_2(s))$ and $\Re f_2(s) \le \Re f_1(s)$. Therefore, $s \in \mathbb{C}$ such that $H_1(s) = 0$ if and only if $h_0(s) \ne 0$ and

$$s = f_1(s), \quad \text{or} \quad s = f_2(s).$$

If $h_0(s) = 0$, then s is a zero of $H_1(s)$ if and only if it satisfies $\lambda_0h_1(s)s+\lambda h_2(s)s+\lambda_0I_2h_1(s)+\lambda I_1h_2(s) = 0$.

We now calculate the function $\Delta_1(s)$

$$\begin{aligned}
&\triangle_1(s)\\
=\;& [(I_1+I_2)h_0(s)+\lambda_0h_1(s)+\lambda h_2(s)]^2\\
-\;& 4h_0(s)[I_1I_2h_0(s)+\lambda_0I_2h_1(s)+\lambda I_1h_2(s)]\\
=\;& (I_1-I_2)^2h_0^2(s)+[\lambda_0h_1(s)+\lambda h_2(s)]^2\\
+\;& 2(I_1-I_2)\lambda_0h_0(s)h_1(s)-2(I_1-I_2)\lambda h_0(s)h_2(s)\\
=\;& [(I_1-I_2)h_0(s)+\lambda_0h_1(s)+\lambda h_2(s)]^2\\
&-4(I_1-I_2)\lambda h_0(s)h_2(s).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&|\sqrt{\Delta_1(s)}| \le |I_1-I_2||h_0(s)|+\lambda_0|h_1(s)|+\lambda|h_2(s)|\\
&\le |I_1-I_2|h_0(\Re s)+\lambda_0h_1(\Re s)+\lambda h_2(\Re s)
\end{aligned}$$

and

$$\begin{aligned}
\Re f_1(s) \;\le\;& -\frac{(I_1+I_2)}{2} - \Re\left(\frac{\lambda_0h_1(s)+\lambda h_2(s)}{2h_0(s)}\right)\\
&+\frac{|I_1-I_2|h_0(\Re s)+\lambda_0h_1(\Re s)+\lambda h_2(\Re s)}{2|h_0(s)|}\\
\le\;& -\min\{I_1,I_2\} - \Re\left(\frac{\lambda_0h_1(s)+\lambda h_2(s)}{2h_0(s)}\right)\\
&+\frac{\lambda_0h_1(\Re s)+\lambda h_2(\Re s)}{2h_0(\Re s)}
\end{aligned}$$

In particular, $h_0(r)$, $h_1(r)$ and $h_2(r)$ are the real functions for $r \in (-\widehat{\mu}, 0)$, we have $\Re f_1(r) \le -\min\{I_1, I_2\}$.

Note that the real functions $G_\alpha(r)$, $G_\beta(r)$ and $G_\gamma(r)$ are nonnegative decreasing functions in $-\widehat{\mu} < r \le 0$, we assume without loss of generality that

$$\lim_{r\to-\widehat{\mu}} \int_0^\infty e^{-\int_0^x (r+\alpha(\tau))d\tau}dx = \infty,$$

$$\lim_{r\to-\widehat{\mu}} \int_0^\infty e^{-\int_0^y (r+\beta(\tau))d\tau}dy = G_\beta(-\widehat{\mu}) < +\infty,$$

$$\lim_{r\to-\widehat{\mu}} \int_0^\infty e^{-\int_0^z (r+\gamma(\tau))d\tau}dz = G_\gamma(-\widehat{\mu}) < +\infty.$$

Thus we have $\lim_{r \to -\widehat{\mu}} H_1(r)$

$$= \begin{cases} -\infty, & -(\lambda_0+\lambda)\widehat{\mu} + (\lambda_0 I_2 + \lambda I_1) < 0 \\ +\infty, & -(\lambda_0+\lambda)\widehat{\mu} + (\lambda_0 I_2 + \lambda I_1) > 0. \end{cases}$$

Obviously, when $-(\lambda_0+\lambda)\widehat{\mu} + (\lambda_0 I_2 + \lambda I_1) > 0$, there is no real zero of $H_1(r)$; when $-(\lambda_0+\lambda)\widehat{\mu} + (\lambda_0 I_2 + \lambda I_1) < 0$, there is at most two negative real zero of $H_1(r)$, whose zeros are given by

$$r_{1,2} = \frac{-[(I_1+I_2)h_0(r)+\lambda_0 h_1(r)+\lambda h_2(r)] \pm \sqrt{\Delta_1(r)}}{2h_0(r)}$$

where $\Delta_1(r) \geq 0$. When $I_1 = I_2$, we have $r_1 = -I_1$ and

$$r_2 = -I_1 - \frac{[\lambda_0 h_1(r) + \lambda h_2(r)]}{h_0(r)}$$

3.4 Special case

In this subsection we discuss the special case that $\alpha(x)$, $\beta(y)$ and $\gamma(z)$ are the constant functions. In this case, $\widehat{\mu} = \min\{\alpha, \beta, \gamma\}$ and $G_\alpha(s) = (s+\alpha)^{-1}$,$G_\beta(s) = (s+\beta)^{-1}$ and $G_\gamma(s) = (s+\gamma)^{-1}$. Hence

$$H_1(s) = \frac{H_2(s)}{(s+\alpha)(s+\beta)(s+\gamma)}$$

and

$$H_2(s) = s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

where $a_j, j = 0,1,2,3,4$ are real coefficients that are determined via $H_1(s)$. Clearly, $H_2(s)$ has five zeros.

4 Analysis of Stability

4.1 Existence of positive solutions and conservation

In this subsection, we shall prove the existence of positive solutions to (1) and (2) since it is a practice problem. We complete the proof by showing $\mathcal{A}$ generates positive semigroup on $\mathbb{X}$. Firstly we recall some notion.

Definition 8. *Let $\mathbb{X} = \mathbb{R}^3 \times (L^1(\mathbb{R}_+))^3$ be the real Banach space, and let*

$$\mathbb{X}_+ = \{(p_0, p_1, p_2, p_F(x), p_H(y), p_C(z)) \in \mathbb{X} | p_k \geq 0\}.$$

The set $\mathbb{X}_+$ is called a positive cone in $\mathbb{X}$.

A bounded linear operator T is said to be a positive operator if $T\mathbb{X}_+ \subset \mathbb{X}_+$. A positive operator T is said to be positive conservation if for any $P \in \mathbb{X}_+$, $||TP|| = ||P||$.

A linear operator $\mathcal{L} : D(\mathcal{L}) \subset \mathbb{X} \to \mathbb{X}$ is said to be dispersive if for any $P \in D(\mathcal{L})$ there exists a $\Phi \in \mathbb{X}_+^*$ with $||\Phi||_{\mathbb{X}^*} \leq 1$ such that $\langle P, \Phi\rangle = ||P_+||$ and $\langle \mathcal{L}P, \Phi\rangle \leq 0$, where $\mathbb{X}_+^*$ is the dual positive cone in $\mathbb{X}^*$

Note that $(\mathbb{X}, \mathbb{X}_+)$ is a Banach lattice. In particular, $\mathbb{X}^* = \mathbb{R}^3 \times (L^1(\mathbb{R}_+))^3$ and $\mathbb{X}_+^* = \mathbb{R}_+^3 \times (L_+^\infty(\mathbb{R}_+))^3$ where $L_+^\infty(\mathbb{R}_+)$ consists of all nonnegative functions in $L^\infty(\mathbb{R}_+)$. So we have the following result.

Theorem 9. *Let space $\mathbb{X}$ and operator $\mathcal{A}$ be defined as before. Then $\mathcal{A}$ generates a positive C_0-semigroup of contractions on $\mathbb{X}$.*

Proof Let $\mathcal{A}$ be defined by (5) and (6). According to theory of the positive semigroup (see, [14]), $\mathcal{A}$ generates a positive C_0-semigroup of contractions if and only if $\mathcal{A}$ is a dispersive and $\mathcal{R}(I - \mathcal{A}) = \mathbb{X}$. Since Theorem 2 has asserted that $\mathcal{R}(I - \mathcal{A}) = \mathbb{X}$, we only need to prove $\mathcal{A}$ is a dispersive operator.

For each $P \in D(\mathcal{A})$, we defined a vector

$$\Phi = (\mathrm{sign}_+(p_0), \mathrm{sign}_+(p_1), \mathrm{sign}_+(p_2), \mathrm{sign}_+(p_F(x)), \mathrm{sign}_+(p_H(y)), \mathrm{sign}_+(p_C(z)))$$

where $P = (p_0, p_1, p_2, p_F(x), p_H(y), p_C(z))$ and

$$\mathrm{sign}_+(p_k(x)) = \begin{cases} 1, & p_k(x) > 0 \\ 0, & p_k(x) \leq 0 \end{cases}$$

Clearly, $\Phi \in \mathbb{X}_+^*$ and $||\Phi||_{\mathbb{X}^*} \leq 1$. Further we have

$$\begin{aligned} \langle P, \Phi\rangle &= p_0^+ + p_1^+ + p_2^+ + \int_0^\infty p_F^+(x)dx \\ &\quad + \int_0^\infty p_H^+(y)dy + \int_0^\infty p_C^+(z)dz \\ &= ||P_+|| \end{aligned}$$

and

$$\begin{aligned} &\langle \mathcal{A}P, \Phi\rangle \\ =\ & (-I_0 p_0 + \mu_1 p_1 + \mu_2 p_2)\mathrm{sign}_+(p_0) \\ & +\mathrm{sign}_+(p_0)\int_0^\infty \alpha(x)p_F(x)dx \\ & +\mathrm{sign}_+(p_0)\int_0^\infty \beta(y)p_H(y)dy \\ & +\mathrm{sign}_+(p_0)\int_0^\infty \gamma(z)p_C(z)dz \\ +\ & (\lambda_0 p_0 - I_1 p_1)\mathrm{sign}_+(p_1) + (\lambda p_0 - I_2 p_2)\mathrm{sign}_+(p_2) \\ & - \int_0^\infty (p_F'(x) + \alpha(x)p_F(x))\mathrm{sign}_+(p_F(x))dx \\ & - \int_0^\infty (p_H'(y) + \beta(y)p_H(y))\mathrm{sign}_+(p_H(y))dy \\ & - \int_0^\infty (p_C'(z) + \gamma(z)p_C(z))\mathrm{sign}_+(p_C(z))dz \end{aligned}$$

$$\begin{aligned} &\le -(I_0-\lambda_0-\lambda)p_0^+-(I_1-\mu_1)p_1^+ \\ &\quad -(I_2-\mu_2)p_1^++p_F^+(0)+p_H^+(0)+p_C^+(0) \end{aligned}$$

where we have used the equalities

$$\int_0^\infty p'(x)\text{sign}_+(p(x))dx = -p^+(0).$$

Using the boundary conditions in $D(\mathcal{A})$, we get

$$\begin{aligned} & p_F^+(0)+p_H^+(0)+p_C^+(0) \\ \le\ & (\lambda_{h_0}+\lambda_{c_0})p_0^+ + (\lambda+\lambda_{h_1}+\lambda_{c_1})p_1^+ \\ & +(\lambda_1+\lambda_{h_2}+\lambda_{c_2})p_2^+ \\ =\ & (I_0-\lambda_0-\lambda)p_0^+ + (I_1-\mu_1)p_1^+ \\ & +(I_2-\mu_2)p_1^+. \end{aligned}$$

Therefore, we have $\langle \mathcal{A}P, \Phi\rangle \le 0$ that means that $\mathcal{A}$ is a dispersive operator in $\mathbb{X}$. Since $\mathcal{A}$ generates a C_0 semigroup, so the semigroup is a positive semigroup on $\mathbb{X}$. □

Theorem 10. *Let $T(t)$ be the positive semigroup generated by $\mathcal{A}$. Then $T(t)$ is the positive conservation, i.e., $\|T(t)P_0\| = \|P_0\|, t \ge 0$ provided that $P_0 \in D(\mathcal{A}) \bigcap \mathbb{X}_+$.*

Proof Let $P_0 \in D(\mathcal{A})$ and $P_0 > 0$. Since $T(t)$ is a positive C_0-semigroup, $T(t)P_0 \in D(\mathcal{A}) \cap \mathbb{X}_+$ is a classical solution of system (7). Set $P(t) = T(t)P_0$ and

$$P(t) = (p_0(t), p_1(t), p_2(t), p_F(x,t), p_H(y,t), p_C(z,t)).$$

Thus $P(t)$ satisfy equations (1) and (2) and has norm

$$\begin{aligned} \|P(t)\| &= p_0(t)+p_1(t)+p_2(t) \\ &\quad + \int_0^\infty p_F(x,t)dx + \int_0^\infty p_H(y,t)dy \\ &\quad + \int_0^\infty p_C(z,t)dz. \end{aligned}$$

Using the partial differential equations (1) and boundary conditions (2), we have

$$\begin{aligned} \frac{d}{dt}\|P(t)\| &= \frac{dp_0(t)}{dt} + \frac{dp_1(t)}{dt} + \frac{dp_2(t)}{dt} \\ &\quad + \frac{d}{dt}\int_0^\infty p_F(x,t)dx + \frac{d}{dt}\int_0^\infty p_H(y,t)dy \\ &\quad + \frac{d}{dt}\int_0^\infty p_C(z,t)dz = 0, \end{aligned}$$

this means that $||P(t)||$ is constant in t. Therefore, $\|P(t)\| = \|P(0)\| = \|P_0\|$. By the density of $D(\mathcal{A})$ in $\mathbb{X}$ and continuity of $T(t)$, the relation also holds on $\mathbb{X}_+$. The proof is then complete. □

Theorems 9 and 10 together the semigroup theory yield the following result.

Corollary 11. *The differential equations (1) and (2) have uniquely a positive solution for initial data $P_0 = (1,0,0,0,0,0) \in \mathbb{X}$.*

4.2 Finite expansion of solution

From now on we suppose that $\widehat{\mu} > 0$. According to the result of Theorem 7, for any small $\delta > 0$, the region $\{s \in \mathbb{C} \mid -\widehat{\mu}+\delta \le \Re s \le 0\}$ has only finitely many number of eigenvalues of $\mathcal{A}$. Without loss of generality we assume that $\Gamma_1 : \Re s = -\widehat{\mu}+\delta \in \rho(\mathcal{A})$.

Let $T(t)$ be the C_0 semigroup generated by $\mathcal{A}$. According to theory of linear operator semigroup, for $\alpha > 0$, it holds that

$$T(t)P = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}R(s,\mathcal{A})Pds, \quad P \in \mathbb{X}.$$

Now let

$$S_M(t)P = \frac{1}{2\pi i}\int_{-\widehat{\mu}+\delta-iM}^{-\widehat{\mu}+\delta+iM} e^{st}R(s,\mathcal{A})Pds.$$

For sufficient large M we have

$$\begin{aligned} & \frac{1}{2\pi i}\int_{\alpha-iM}^{\alpha+iM} e^{st}R(s,\mathcal{A})Pds \\ =\ & \sum_{s_i\in\sigma_p(\mathcal{A}),\Re s_i \ge -\widehat{\mu}+\delta} \frac{1}{2\pi i}\int_{|s-s_i|=\varepsilon} e^{st}R(s,\mathcal{A})Pds \\ & + \frac{1}{2\pi i}\int_{-\widehat{\mu}+\delta}^{\alpha} e^{(iM+\gamma)t}R(iM+\gamma,\mathcal{A})Pd\gamma \\ & - \frac{1}{2\pi i}\int_{-\widehat{\mu}+\delta}^{\alpha} e^{(-iM+\gamma)t}R(-iM+\gamma,\mathcal{A})Pd\gamma \\ & + S_M(t)P \end{aligned}$$

Using the estimates in Theorem 3, we can get that

$$\lim_{M\to\infty}\frac{1}{2\pi i}\int_{-\widehat{\mu}+\delta}^{\alpha} e^{(iM+\gamma)t}R(iM+\gamma,\mathcal{A})Pd\gamma = 0$$

and

$$\lim_{M\to\infty}\frac{1}{2\pi i}\int_{-\widehat{\mu}+\delta}^{\alpha} e^{(-iM+\gamma)t}R(-iM+\gamma,\mathcal{A})Pd\gamma = 0.$$

Therefore, we have

$$T(t)P = \sum_{s_i\in\sigma_p(\mathcal{A}),\Re s_i \ge -\widehat{\mu}+\delta} T(t)E(s_i,A)P + S(t)P$$

where

$$\begin{aligned} S(t)P &= \lim_{M\to\infty} S_M(t)P \\ &= \frac{1}{2\pi i}\int_{-\widehat{\mu}+\delta-i\infty}^{-\widehat{\mu}+\delta+i\infty} e^{st}R(s,\mathcal{A})Pds. \end{aligned}$$

Obviously this presentation implies that there exists a constant $C > 0$ such that

$$||S(t)P|| \leq Ce^{(-\widehat{\mu}+\delta)t}||P||, \quad \forall P \in \mathbb{X}.$$

Therefore, we have the following result.

Theorem 12. *Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before and $T(t)$ be the semigroup generated by $\mathcal{A}$. Suppose that, for $\delta > 0$ small enough, the eigenvalues of $\mathcal{A}$ in the half-plane $\Re s \geq -\widehat{\mu} + \delta$ are given by $\gamma_0, s_1, \overline{s_1}, s_2, \overline{s_2}, \cdots, s_m, \overline{s_m}$ with $\Re s_{j+1} < \Re s_j$. Then we have the finite expansion of the semigroup $T(t)$*

$$T(t)P = \langle P, Q\rangle \widehat{P}_0 + \sum_{j=1}^{m} T(t)[E(s_j, A) + E(\overline{s_j}, A)]P + S(t)P \tag{33}$$

where $\widehat{P}_0 = \frac{1}{Z}(p_0^0, p_1^0, p_2^0, p_F^0(x), p_H^0(y), p_C^0(z))$ whose entries are determined by (28), and $Q = (1, 1, 1, 1, 1, 1)$.

From Theorem 5 we see that $\widehat{P}_0$ is the steady-state solution with $||\widehat{P}_0|| = 1$. Due to $\Re s_j < 0$ we see from Theorem 12 that for any $P \in \mathbb{X}, \lim_{t\to\infty} T(t)P = \langle P, Q\rangle \widehat{P}_0$. In particular, we have the following estimate for its convergence.

Corollary 13. *Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before, and let $T(t)$ be the semigroup generated by $\mathcal{A}$. Suppose that $\widehat{\mu} > 0$ and $0 < \omega_1 < |\Re s_1|$. Then for any initial $P(0)$, we have*

$$\|P(t) - \langle P(0), Q\rangle \widehat{P}_0\| \leq 2e^{-\omega_1 t}\|P(0)\|, \forall t \geq 0 \tag{34}$$

where $P(t) = T(t)P(0)$.

Proof Since the Riesz spectral project corresponding to γ_0 is given by

$$E(\gamma_0, \mathcal{A})F = \frac{1}{2\pi i}\int_{|s|=\varepsilon} \mathcal{R}(s, \mathcal{A})F ds = \langle F, Q\rangle \widehat{P}_0$$

for $\forall F \in \mathbb{X}$. This leads to $\|E(\gamma_0, \mathcal{A})\| = \|Q\|\|\widehat{P}_0\| = 1$. Since $T(t)$ is a semigroup in the subspace $(I - E(\gamma_0, \mathcal{A}))\mathbb{X}$, we have

$$\begin{aligned} & \|P(t) - \langle P(0), Q\rangle \widehat{P}_0\| \\ = & \|T(t)(I - E(\gamma_0, \mathcal{A}))P(0)\| \\ \leq & 2e^{-\omega_1 t}\|P(0)\|. \end{aligned}$$

The desired result follows. □

Remark 14. *In Corollary 11, usually we have $-\omega_1 \neq \Re s_1$. If s_1 is an eigenvalue of $\mathcal{A}$ without the send order root vector, then we can take $-\omega_1 = \Re s_1$.*

5 Some indices of the system

5.1 The observable time

Quasi-exponential decaying of the system means that one can see the steady state of system in a relatively short period. For the system under consideration, the dynamic solution of system is given by $P(t) = T(t)P(0)$

$$= (p_0(t), p_1(t), p_2(t), p_F(x,t), p_H(y,t), p_C(z,t)).$$

with initial date $P(0) = (1, 0, 0, 0, 0, 0)$ and the steady-state of system is $\langle P(0), Q\rangle \widehat{P}_0 = \widehat{P}_0$ where

$$\widehat{P}_0 = \frac{1}{Z}(p_0^{(0)}, p_1^{(0)}, p_2^{(0)}, p_F^{(0)}(x), p_H^{(0)}(y), p_C^{(0)}(z)),$$

and

$$\begin{cases} p_0^{(0)} = I_1 I_2, \\ p_1^{(0)} = \lambda_0 I_2, \\ p_2^{(0)} = \lambda I_1, \\ p_F^{(0)}(x) = [\lambda\lambda_0 I_2 + \lambda_1 \lambda I_1]e^{-\int_0^x \alpha(\tau)d\tau}, \\ p_H^{(0)}(y) = [\lambda_{h_0} I_1 I_2 + \lambda_{h_1}\lambda_0 I_2 + \lambda_{h_2}\lambda I_1]e^{-\int_0^y \beta(\tau)d\tau}, \\ p_C^{(0)}(z) = [\lambda_{c_0} I_1 I_2 + \lambda_{c_1}\lambda_0 I_2 + \lambda_{c_2}\lambda I_1]e^{-\int_0^z \gamma(\tau)d\tau} \end{cases} \tag{35}$$

and

$$\begin{aligned} Z \;=\; & I_1 I_2 + \lambda_0 I_2 + \lambda I_1 + [\lambda\lambda_0 I_2 + \lambda_1 \lambda I_1]G_\alpha(0) \\ & + [\lambda_{h_0} I_1 I_2 + \lambda_{h_1}\lambda_0 I_2 + \lambda_{h_2}\lambda I_1]G_\beta(0) \\ & + [\lambda_{c_0} I_1 I_2 + \lambda_{c_1}\lambda_0 I_2 + \lambda_{c_2}\lambda I_1]G_\gamma(0) \end{aligned} \tag{36}$$

For a system S, whose dynamic solution is $P(t)$ with initial data $||P_0|| = 1$ and the steady state is $\widehat{P}_0$, if there is a time τ_0 such that when $t > \tau_0$, it holds that $||P(t) - \widehat{P}_0|| \leq 0.25$, then we say that we can see the steady state of the system at τ_0.

According to Corollary 13, we have estimate $\|P(t) - \widehat{P}_0\| \leq 2e^{-\omega_1 t}$. Obviously, for $\tau_0 = \frac{3\ln 2}{\omega_1}$, when $t > \tau_0$, we have $\|P(t) - \widehat{P}_0\| \leq 0.25$. Therefore, we can see the steady-state at $\tau_0 = \frac{3\ln 2}{\omega_1}$.

5.2 The estimation of availability of the system

The instantaneous availability of the system is the probability of the system in work, which is defined by

$$V(t) = p_0(t) + p_1(t) + p_2(t).$$

Since

$$\begin{aligned} & |p_0(t) - p_0^0| + |p_1(t) - p_1^0| + |p_2(t) - p_2^0| \\ \leq & ||P(t) - \widehat{P}_0|| \leq 2e^{-\omega_1 t}, \end{aligned}$$

while $p_0^0+p_1^0+p_2^0=\frac{I_1I_2+\lambda_0I_2+\lambda I_1}{Z}$, we have

$$|V(t)-\frac{I_1I_2+\lambda_0I_2+\lambda I_1}{Z}|\le||P(t)-\widehat{P}_0||\le 2e^{-\omega_1 t}.$$

Thus when $t>\frac{3\ln 2}{\omega_1}$, we can get that $V(t)=\frac{I_1I_2+\lambda_0I_2+\lambda I_1}{Z}+O(t)$, $|O(t)|\le 0.25$. Obviously, the probability of system failure is

$$\int_0^\infty p_F(x,t)dx+\int_0^\infty p_H(y,t)dy+\int_0^\infty p_C(z,t)dz.$$

It has an estimate $1-V(t)=1-\frac{I_1I_2+\lambda_0I_2+\lambda I_1}{Z}\pm 0.25$.

Note that the Z defined as (36) is a decrease function with respect to the repair rate $\alpha(x),\beta(y),\gamma(y)$. When the repair rates are strength, the availability of the system increases. and hence the reliability of the system is enhanced.

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