Exact traveling wave solutions of nonlinear variable coefficients evolution equations with forced terms using the generalized $\left(\frac{G'}{G}\right)$−expansion method

ELSAYED ZAYED
Zagazig University
Department of Mathematics
Zagazig
Egypt
e.m.e.zayed@hotmail.com

MAHMOUD ABDELAZIZ
Zagazig University
Department of Mathematics
Zagazig
Egypt
mamabdelaziz@hotmail.com

Abstract: The exact traveling wave solutions of the nonlinear variable coefficients Burgers-Fisher equation and the generalized Gardner equation with forced terms can be found in this article using the generalized $(\frac{G'}{G})$-expansion method. As a result, hyperbolic, trigonometric and rational function solutions with parameters are obtained. When these parameters are taken special values, the solitary wave solutions are derived from the hyperbolic function solutions. It is shown that the proposed method is direct, effective and can be applied to many other nonlinear evolution equations in the mathematical physics.

Key–Words: Nonlinear evolution equations; Generalized $(\frac{G'}{G})$-expansion method; Variable coefficients Burgers-Fisher equation with the forced term; Variable coefficients generalized Gardner equation with the forced term, Exact solutions.

1 Introduction

The investigation of the exact solutions of nonlinear evolution equations (NLEEs) plays an important role in the study of nonlinear physical phenomena. It becomes one of the most exciting and extremely active areas of research investigation. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been presented, such as the inverse scattering method [1], the Hirota's bilinear method [2], the Bäcklund transformation [3], the Painlevé expansion [4], the sine-cosine method [5], the homogeneous balance method [6], the homotopy perturbation method [7–9], the variational iteration method [10–13], the Adomian decomposition method [14], the tanh function method [15,16], the algebraic method [17,18], the Jacobi elliptic function expansion method [19,20], the F-expansion method [21,22], the auxiliary equation method [23,24], the Exp-function method [25,26], the $\left(\frac{G'}{G}\right)$−expansion method [27–42] and so on. However, to our knowledge, most of aforementioned methods are related to the constant-coefficient models. Recently, the study of variable-coefficients NLEEs is attracted much attention because most of real nonlinear physical equations possess variable coefficients. Very recently, Wang et al. [31] first introduced the $(\frac{G'}{G})$-expansion method to look for traveling wave solutions of NLEEs. This method is based on the assumptions that the traveling wave solutions can be expressed by a polynomial in $(\frac{G'}{G})$, where G satisfies the following second order linear ordinary differential equation:

$$G'' + \lambda G' + \mu G = 0, \tag{1}$$

and λ, μ are arbitrary constants. The objective of this article is to apply the generalized $(\frac{G'}{G})$ -expansion method to improve the work made in [31]. Zhang et al [35] first proposed the generalized $(\frac{G'}{G})$-expansion method to construct exact solutions of the mKdV equation with variable coefficients. As applications of the proposed method, we will consider the following two models:

I. The nonlinear Burgers-Fisher equation with variable coefficients and forced term [32] has the form:

$$u_t - d(t)u_{xx} + a(t)uu_x + b(t)(u^2 - u) = R(t), \tag{2}$$

where $d(t), a(t), b(t)$ and $R(t)$ are arbitrary differentiable functions of t. The function $R(t)$ is called the forced term of Eq. (2). It is obvious that Eq. (2) is the Burgers-Fisher equation for $R(t) = 0, d(t), a(t)$ and $b(t)$ are constants. It is the variable coefficient Fisher equation when $d(t) = R(t) = 0$, which is used to describe nonlinear phenomena such as thermonuclear fusion and hydronium physics etc. It is also the variable coefficient Burgers equation when $b(t) = R(t) = 0$, which often emerged in Mathematics and Physics.

II. The nonlinear generalized Gardner equation with variable coefficients and forced term has the form:

$$\begin{aligned} u_t &+ a(t)uu_x + b(t)u^2u_x + h(t)u_{xxx} + d(t)u_x \\ &+ f(t)u = F(t), \end{aligned} \tag{3}$$

where $a(t), b(t), h(t), d(t), f(t)$ and $F(t)$ are arbitrary differentiable functions of t. The function $F(t)$ is called the forced term of Eq. (3). This equation is the variable coefficients generalized Gardner equation when $F(t) = 0$, see reference [43]. It is widely used in various branches of physics, such as plasma physics, fluid physics and quantum field theory. It also describes a variety of wave phenomena in plasma and solid state. Finally, it includes considerably interesting equations, such as KdV equation, mKdV equation and Gardner equation.

2 Description of the generalized $(\frac{G'}{G})$-expansion method

For a given NLEE with independent variables $X = (x, y, z, t)$ and dependent variable u, we consider the PDE

$$Q(u, u_t, u_x, u_y, u_z, u_{xt}, u_{yt}, u_{zt}, u_{tt}, u_{xx}, ...) = 0 . \tag{4}$$

The solution of Eq. (4) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$\begin{aligned} u(X) &= \alpha_0(X) + \sum_{i=1}^{m} \alpha_i(X) \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \\ & \alpha_m(X) \neq 0, \end{aligned} \tag{5}$$

where $G = G(\xi)$ satisfies Eq. (1) while $\xi = \xi(X)$ and $\alpha_i(X)$ are all analytic functions of X to be determined later and $' \equiv \frac{d}{d\xi}$. To determine $u(X)$ explicitly, we consider the following four steps:

Step1. Determine the positive integer m by balancing the highest order nonlinear term(s) and the highest order partial derivatives of $u(X)$ in Eq. (4).

Step2. Substitute (5) along with Eq. (1) into Eq. (4) and collect all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of Eq. (4) is converted into a polynomial in $(\frac{G'}{G})$. Then set each coefficient of this polynomial to zero to derive a set of over-determined differential equations for $\alpha_0(X)$, $\alpha_i(X)$ and $\xi(X)$.

Step3. Solve the system of over-determined differential equations obtained in Step2 for $\alpha_i(X)$, $(i = 0, 1, ..., m)$ and $\xi(X)$ by the use of *Maple* or *Mathematica*.

Step4. Use the results obtained in above steps to derive a series of fundamental solutions of Eq. (4) depending on $(\frac{G'}{G})$ since the solutions of Eq. (1) have been well known for us, then we can obtain the exact solutions of Eq. (4).

3 Applications

In this section, we determine the exact traveling wave solutions of the nonlinear Burgers-Fisher equation and the nonlinear generalized Gardner equation with variable-coefficients and the forced terms which are attracted much attention.

3.1 Example 1. The variable-coefficient Burgers-Fisher equation with the forced term

In order to obtain the exact solutions of Eq. (2), we assume that the solution of this equation can be written in the form

$$u(x,t) = v(x,t) + \int R(t)dt, \tag{6}$$

where $v(x,t)$ is a differentiable function of x, t while $R(t)$ is an integrable function of t. Substituting (6) into Eq. (2), we have

$$\begin{aligned} \upsilon_t &- d(t)\upsilon_{xx} + a(t)\upsilon\upsilon_x + a(t)\beta(t)\upsilon_x \\ &+ b(t)\left\{\upsilon^2 + [2\beta(t) - 1]\,\upsilon\right\} \\ &+ b(t)\beta(t)\,[\beta(t) - 1] = 0, \end{aligned} \tag{7}$$

where

$$\beta(t) = \int R(t)dt. \tag{8}$$

By balancing υ_{xx} with $\upsilon\upsilon_x$ in Eq. (7), we get $m = 1$. In order to search for explicit solutions, we suppose that Eq. (7) has the following formal solution:

$$\begin{aligned} \upsilon(x,t) &= \alpha_0(t) + \alpha_1(t)\left(\frac{G'}{G}\right), \\ \alpha_1(t) &\neq 0\,, \end{aligned} \tag{9}$$

where $G = G(\xi)$ satisfies Eq. (1) and $\xi = p(t)x + q(t)$. The functions $p(t)$ and $q(t)$ are differentiable functions of t to be determined. From (1) and (9) we have:

$$\begin{aligned} \upsilon_t &= -\alpha_1(t)\left[xp(t)^{\cdot} + q(t)^{\cdot}\right]\left(\frac{G'}{G}\right)^2 \\ &- \left\{\lambda\alpha_1(t)\left[xp(t)^{\cdot} + q(t)^{\cdot}\right] - \alpha_1(t)^{\cdot}\right\}\left(\frac{G'}{G}\right) \\ &- \mu\alpha_1(t)\left[xp(t)^{\cdot} + q(t)^{\cdot}\right] + \alpha_0(t)^{\cdot}, \end{aligned} \tag{10}$$

$$\upsilon_x = -\alpha_1(t)p(t)\left[\left(\frac{G'}{G}\right)^2 + \lambda\left(\frac{G'}{G}\right) + \mu\right], \tag{11}$$

$$\begin{aligned} \upsilon_{xx} &= \alpha_1(t)p(t)^2\left[2\left(\frac{G'}{G}\right)^3 + 3\lambda\left(\frac{G'}{G}\right)^2\right. \\ &+ \left.\left[\lambda^2 + 2\mu\right]\left(\frac{G'}{G}\right) + \mu\lambda\right], \end{aligned} \tag{12}$$

$$\begin{aligned} \upsilon_{xxx} &= -\alpha_1(t)p(t)^3\left[6\left(\frac{G'}{G}\right)^4 + 12\lambda\left(\frac{G'}{G}\right)^3\right. \\ &+ \left[7\lambda^2 + 8\mu\right]\left(\frac{G'}{G}\right)^2 + \lambda(\lambda^2 + 8\mu)\left(\frac{G'}{G}\right) \\ &+ \left.\mu(\lambda^2 + 2\mu)\right], \end{aligned} \tag{13}$$

where $^{\cdot} = \frac{d}{dt}$. Substituting (9)–(12) into Eq. (7) and collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand side of Eq. (7) is converted into a polynomial in $x^i\left(\frac{G'}{G}\right)^j$, $(i = 0, 1,\ j = 0, 1, 2, 3)$. Setting each coefficient of this polynomial to zero, we get the following set of over-determined differential equations:

$$\begin{aligned} x^0\left(\tfrac{G'}{G}\right)^3 &: -2d(t)\alpha_1(t)p(t)^2 \\ &\quad -a(t)\alpha_1(t)^2p(t) = 0, \end{aligned}$$

$$\begin{aligned} x^0\left(\tfrac{G'}{G}\right)^2 &: -\alpha_1(t)q(t)^{\cdot} - 3\lambda d(t)\alpha_1(t)p(t)^2 \\ &\quad -\lambda a(t)\alpha_1(t)^2p(t) + b(t)\alpha_1(t)^2 \\ &\quad -a(t)\alpha_1(t)p(t)[\alpha_0(t) + \beta(t)] = 0, \end{aligned}$$

$$\begin{aligned} x^0\left(\tfrac{G'}{G}\right)^1 &: -\lambda\alpha_1(t)q(t)^{\cdot} + \alpha_1(t)^{\cdot} - b(t)\alpha_1(t) \\ &\quad -(\lambda^2 + 2\mu)d(t)\alpha_1(t)p(t)^2 \\ &\quad -\lambda a(t)\alpha_1(t)p(t)[\alpha_0(t) + \beta(t)] \\ &\quad -\mu a(t)\alpha_1(t)^2p(t) \\ &\quad +2b(t)\alpha_1(t)[\alpha_0(t) + \beta(t)] = 0, \end{aligned}$$

$$x^0\left(\tfrac{G'}{G}\right)^0 : -\mu\alpha_1(t)q(t)^{\cdot} + \alpha_0(t)^{\cdot}$$

$$-\mu\lambda d(t)\alpha_1(t)p(t)^2+2b(t)\alpha_0(t)\beta(t)$$
$$-\mu a(t)\alpha_1(t)p(t)[\alpha_0(t)+\beta(t)]$$
$$-b(t)[\alpha_0(t)+\beta(t)]+b(t)\alpha_0(t)^2$$
$$+b(t)[\alpha_0(t)^2+\beta(t)^2]=0,$$

$$x^1\left(\frac{G'}{G}\right)^2 \ : \ -\alpha_1(t)p(t)^{\cdot}=0,$$

$$x^1\left(\frac{G'}{G}\right)^1 \ : \ -\lambda\alpha_1(t)p(t)^{\cdot}=0,$$

$$x^1\left(\frac{G'}{G}\right)^0 \ : \ -\mu\alpha_1(t)p(t)^{\cdot}=0. \qquad (14)$$

Solving the system (14) by the *Maple* or *Mathematica*, we have

case1.

$$\begin{aligned}\alpha_0(t) &= \frac{-c_1\lambda d(t)}{a(t)}-\frac{1}{4}\int b(t)\left\{\left[\frac{a(t)S(t)}{b(t)d(t)}\right]^2\right.\\ &+ \left.\frac{4c_1^2(\lambda^2-4\mu)d(t)^2}{a(t)^2}-1\right\}dt,\end{aligned}$$

$$\alpha_1(t) = \frac{-2c_1d(t)}{a(t)}, \ \ p(t)=c_1,$$

$$\begin{aligned}\beta(t) &= \frac{1}{2}-\frac{a(t)S(t)}{b(t)d(t)}+\frac{1}{4}\int b(t)\left\{\left[\frac{a(t)S(t)}{b(t)d(t)}\right]^2\right.\\ &+ \left.\frac{4c_1^2(\lambda^2-4\mu)d(t)^2}{a(t)^2}-1\right\}dt,\end{aligned}$$

$$\begin{aligned}q(t) &= -2c_1\int\frac{b(t)d(t)}{a(t)}dt\\ &- \frac{c_1}{2}\int\left[a(t)-\frac{S(t)a(t)^2}{b(t)d(t)}\right]dt+c_2,\end{aligned}$$

$$a(t) = a(t), \ \ b(t)=b(t), \ \ d(t)=d(t), \qquad (15)$$

where $S(t) = \frac{d}{dt}\left[\frac{d(t)}{a(t)}\right]$ and c_1, c_2 are arbitrary constants such that $c_1 \neq 0$.

case2.

$$\begin{aligned}\alpha_0(t) &= \frac{1}{2}(1+c_2\lambda)\\ &- \frac{1}{4}[c_2^2(\lambda^2-4\mu)-1]\int b(t)dt,\end{aligned}$$

$$\alpha_1(t) = c_2, \ \ p(t)=c_1, \ \ a(t)=\frac{-2c_1d(t)}{c_2},$$

$$\beta(t) = \frac{1}{4}[c_2^2(\lambda^2-4\mu)-1]\int b(t)dt,$$

$$q(t) = c_2\int b(t)dt+\frac{c_1^2}{c_2}\int d(t)dt+c_3,$$

$$b(t) = b(t), \ \ d(t)=d(t), \qquad (16)$$

where c_1, c_2, c_3 are arbitrary constants such that c_1, $c_2 \neq 0$. Let us now write down the exact solutions corresponding to case1. The exact solutions corresponding to case 2 are similar which are omitted here. Substituting (15) into (9), we have

$$\begin{aligned}v(x,t) &= \frac{-c_1d(t)}{a(t)}\left\{\lambda+2\left(\frac{G'(\xi)}{G(\xi)}\right)\right\}\\ &- \frac{1}{4}\int b(t)\left\{\left[\frac{a(t)S(t)}{b(t)d(t)}\right]^2\right.\\ &+ \left.\frac{4c_1^2(\lambda^2-4\mu)d(t)^2}{a(t)^2}-1\right\}dt.\end{aligned} \qquad (17)$$

Substituting (17) into (6), we have the solution of Eq. (2) in the form

$$\begin{aligned}u(x,t) &= \frac{-c_1d(t)}{a(t)}\left\{\lambda+2\left(\frac{G'(\xi)}{G(\xi)}\right)\right\}\\ &- \frac{1}{4}\int b(t)\left\{\left[\frac{a(t)S(t)}{b(t)d(t)}\right]^2\right.\end{aligned}$$

$$+ \quad \frac{4c_1^2(\lambda^2-4\mu)d(t)^2}{a(t)^2} - 1\Big\} dt$$
$$+ \quad \beta(t), \tag{18}$$

where

$$\xi = c_1\left\{x - 2\int \frac{b(t)d(t)}{a(t)}dt - \frac{1}{2}\int\left[a(t) - \frac{S(t)a(t)^2}{b(t)d(t)}\right]dt\right\} + c_2. \tag{19}$$

From the general solution of Eq. (1) we can find the ratio $\left(\frac{G'}{G}\right)$. Consequently, we have the following exact solutions of Eq. (2):

(I) If $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution in the form

$$u(x,t) = \frac{-c_1 d(t)\sqrt{\lambda^2-4\mu}}{a(t)} \times \left(\frac{A\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)+B\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{A\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)+B\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right) - \frac{1}{4}\int b(t)\left\{\left[\frac{a(t)S(t)}{b(t)d(t)}\right]^2 + \frac{4c_1^2(\lambda^2-4\mu)d(t)^2}{a(t)^2} - 1\right\}dt + \beta(t). \tag{20}$$

(II) If $\lambda^2 - 4\mu < 0$, we have the trigonometric function solution in the form

$$u(x,t) = \frac{-c_1 d(t)\sqrt{4\mu-\lambda^2}}{a(t)} \times \left(\frac{-A\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)+B\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{A\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)+B\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}\right) - \frac{1}{4}\int b(t)\left\{\left[\frac{a(t)S(t)}{b(t)d(t)}\right]^2 - \frac{4c_1^2(4\mu-\lambda^2)d(t)^2}{a(t)^2} - 1\right\}dt + \beta(t). \tag{21}$$

(III) If $\lambda^2 - 4\mu = 0$, we get the rational function solution in the form

$$u(x,t) = \frac{-2c_1Bd(t)}{a(t)(A+B\xi)} + \beta(t) - \frac{1}{4}\int b(t)\left\{\left[\frac{a(t)S(t)}{b(t)d(t)}\right]^2 - 1\right\}dt\,, \tag{22}$$

where A and B are arbitrary constants. Finally, we note that, if $\mu = 0, \lambda > 0, B = 0$ and $A \neq 0$ then we deduce from (20) that

$$u(x,t) = \frac{-c_1\lambda d(t)}{a(t)}\coth\left(\frac{\lambda\xi}{2}\right) + \beta(t) - \frac{1}{4}\int b(t)\left\{\left[\frac{a(t)S(t)}{b(t)d(t)}\right]^2 + \left[\frac{2c_1\lambda d(t)}{a(t)}\right]^2 - 1\right\}dt, \tag{23}$$

while if $\mu = 0, \lambda > 0, B \neq 0$ and $B^2 > A^2$ then we deduce from (20) that

$$u(x,t) = \frac{-c_1\lambda d(t)}{a(t)}\tanh\left(\xi_0 + \frac{\lambda\xi}{2}\right) + \beta(t) - \frac{1}{4}\int b(t)\left\{\left[\frac{a(t)S(t)}{b(t)d(t)}\right]^2 + \left[\frac{2c_1\lambda d(t)}{a(t)}\right]^2 - 1\right\}dt, \tag{24}$$

where $\xi_0 = \tanh^{-1}\left(\frac{A}{B}\right)$. Note that (23) and (24) represent the solitary wave solutions of the Burgers-Fisher equation (2).

3.2 Example 2. The generalized variable-coefficient Gardner equation with the forced term

In order to obtain the exact solutions of Eq. (3), we assume that the solution of this equation can be written in the form

$$u(x,t) = \omega(x,t) + \int F(t)dt, \tag{25}$$

where $\omega(x,t)$ is a differentiable function of x,t while $F(t)$ is an integrable function of t. Substituting (25) into Eq. (3), we have

$$\begin{aligned}\omega_t &+ [a(t)+2b(t)k(t)]\omega\omega_x + b(t)\omega^2\omega_x \\ &+ [d(t)+a(t)k(t)+b(t)k(t)^2]\omega_x \\ &+ h(t)\omega_{xxx} + f(t)[\omega+k(t)] = 0\,, \quad (26)\end{aligned}$$

where

$$k(t) = \int F(t)dt. \quad (27)$$

By balancing ω_{xxx} with $\omega^2\omega_x$ in Eq. (26), we get $m = 1$. In order to search for explicit solutions, we deduce that Eq. (26) has the same formal solution (9). Substituting (9)–(13) into Eq. (26) and collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand side of Eq. (26) is converted into a polynomial in $x^i\left(\frac{G'}{G}\right)^j$, $(i = 0,1, j = 0,...,4)$. Setting each coefficient of this polynomial to zero, we get the following set of over-determined differential equations:

$$\begin{aligned}x^0\left(\tfrac{G'}{G}\right)^4 \;:\;& -b(t)\alpha_1(t)^3p(t) \\ & -6h(t)\alpha_1(t)p(t)^3 = 0,\end{aligned}$$

$$\begin{aligned}x^0\left(\tfrac{G'}{G}\right)^3 \;:\;& -a(t)\alpha_1(t)^2p(t) - \lambda b(t)\alpha_1(t)^3p(t) \\ & -2b(t)\alpha_1(t)^2p(t)[\alpha_0(t)+k(t)] \\ & -12\lambda h(t)\alpha_1(t)p(t)^3 = 0,\end{aligned}$$

$$\begin{aligned}x^0\left(\tfrac{G'}{G}\right)^2 \;:\;& -\alpha_1(t)q(t)^{\cdot} - d(t)\alpha_1(t)p(t) \\ & -\lambda a(t)\alpha_1(t)^2p(t) \\ & -a(t)\alpha_1(t)p(t)[\alpha_0(t)+k(t)] \\ & -\mu b(t)\alpha_1(t)^3p(t) \\ & -2\lambda b(t)\alpha_1(t)^2p(t)[\alpha_0(t)+k(t)] \\ & -2b(t)\alpha_0(t)\alpha_1(t)p(t)\;k(t) \\ & -b(t)\alpha_1(t)p(t)[\alpha_0(t)^2+k(t)^2] \\ & -(7\lambda^2+8\mu)h(t)\alpha_1(t)p(t)^3 = 0,\end{aligned}$$

$$\begin{aligned}x^0\left(\tfrac{G'}{G}\right)^1 \;:\;& -\lambda\alpha_1(t)q(t)^{\cdot} - \lambda d(t)\alpha_1(t)p(t) \\ & -\lambda a(t)\alpha_1(t)p(t)[\alpha_0(t)+k(t)] \\ & -2\mu b(t)\alpha_1(t)^2p(t)[\alpha_0(t)+k(t)] \\ & -\lambda b(t)\alpha_1(t)p(t)[\alpha_0(t)^2+k(t)^2] \\ & -\lambda(\lambda^2+8\mu)h(t)\alpha_1(t)p(t)^3 \\ & -2\lambda b(t)\alpha_0(t)\alpha_1(t)p(t)k(t) \\ & +f(t)\alpha_1(t) - \mu a(t)\alpha_1(t)^2p(t) \\ & +\alpha_1(t)^{\cdot} = 0,\end{aligned}$$

$$\begin{aligned}x^0\left(\tfrac{G'}{G}\right)^0 \;:\;& -\mu\alpha_1(t)q(t)^{\cdot} - \mu\lambda d(t)\alpha_1(t)p(t) \\ & -\mu a(t)\alpha_1(t)p(t)[\alpha_0(t)+k(t)] \\ & -\mu(\lambda^2+2\mu)h(t)\alpha_1(t)p(t)^3 \\ & -\mu b(t)\alpha_1(t)p(t)[\alpha_0(t)^2+k(t)^2] \\ & +f(t)[\alpha_0(t)+k(t)] \\ & -2\mu b(t)\alpha_0(t)\alpha_1(t)p(t)k(t) \\ & +\alpha_0(t)^{\cdot} = 0,\end{aligned}$$

$$x^1\left(\tfrac{G'}{G}\right)^2 \;:\; -\alpha_1(t)p(t)^{\cdot} = 0,$$

$$x^1\left(\frac{G'}{G}\right)^1 \quad : \quad -\lambda\alpha_1(t)p(t)^{\cdot} = 0,$$

$$x^1\left(\frac{G'}{G}\right)^0 \quad : \quad -\mu\alpha_1(t)p(t)^{\cdot} = 0. \tag{28}$$

Solving the system (28) by the *Maple* or *Mathematica*, we have

$$\begin{aligned}
\alpha_0(t) &= \gamma(t)\left[c_2c_3 - \int \frac{f(t)k(t)}{\gamma(t)}dt\right], \\
\alpha_1(t) &= c_3\gamma(t), \;\; p(t) = c_1, \;\; h(t) = \frac{-c_3^2 b(t)\gamma(t)}{6c_1^2}, \\
a(t) &= b(t)\left[c_3(\lambda - 2c_2)\gamma(t) - 2k(t)\right. \\
&+ \left. 2\gamma(t)\int \frac{f(t)k(t)}{\gamma(t)}dt\right], \\
q(t) &= c_1\left\{\int b(t)\gamma(t)^2\left[\int \frac{f(t)k(t)}{\gamma(t)}dt\right]^2 dt\right. \\
&- 2\int b(t)\gamma(t)\left[c_3\left(c_2 - \frac{\lambda}{2}\right)\gamma(t) + k(t)\right] \times \\
&\left[\int \frac{f(t)k(t)}{\gamma(t)}dt\right] dt \\
&+ \frac{c_3^2}{6}\left(\lambda^2 + 2\mu + 6c_2^2 - 6c_2\lambda\right)\int b(t)\gamma(t)^2 dt \\
&+ c_3(2c_2 - \lambda)\int b(t)\gamma(t)k(t)dt \\
&+ \left.\int b(t)k(t)^2 dt - \int d(t)dt\right\}, \\
d(t) &= d(t), \;\; f(t) = f(t), \;\; k(t) = k(t), \\
b(t) &= b(t),
\end{aligned} \tag{29}$$

where $\gamma(t) = e^{-\int f(t)dt}$ and $c_1, c_2, c_3$are arbitrary constants such that $c_1, c_3 \neq 0$. Substituting (29) into (9), we have

$$\omega(x,t) = \gamma(t)\left\{c_3\left(\frac{G'(\xi)}{G(\xi)}\right) + c_2c_3 - \int \frac{f(t)k(t)}{\gamma(t)}dt\right\}. \tag{30}$$

Substituting (30) into (25), we have the solution of Eq. (3) in the form

$$\begin{aligned}
u(x,t) &= \gamma(t)\left\{c_3\left(\frac{G'(\xi)}{G(\xi)}\right) + c_2c_3 - \int \frac{f(t)k(t)}{\gamma(t)}dt\right\} \\
&+ k(t),
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
\xi &= c_1\left\{x + \int b(t)\gamma(t)^2\left[\int \frac{f(t)k(t)}{\gamma(t)}dt\right]^2 dt\right. \\
&- 2\int b(t)\gamma(t)\left[c_3\left(c_2 - \frac{\lambda}{2}\right)\gamma(t) + k(t)\right] \times \\
&\left[\int \frac{f(t)k(t)}{\gamma(t)}dt\right] dt \\
&+ \frac{c_3^2}{6}\left(\lambda^2 + 2\mu + 6c_2^2 - 6c_2\lambda\right)\int b(t)\gamma(t)^2 dt \\
&+ c_3(2c_2 - \lambda)\int b(t)\gamma(t)k(t)dt \\
&+ \left.\int b(t)k(t)^2 dt - \int d(t)dt\right\}.
\end{aligned} \tag{32}$$

Consequently, we have the following three types of exact solutions of Eq. (3):

(I) If $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution in the form

$$\begin{aligned}
u(x,t) &= \gamma(t)\left\{c_3\left[\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \times\right.\right. \\
&\left.\left(\frac{A\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + B\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{A\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + B\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right)\right] \\
&+ \left. c_2c_3 - \int \frac{f(t)k(t)}{\gamma(t)}dt\right\} + k(t).
\end{aligned} \tag{33}$$

(II) If $\lambda^2 - 4\mu < 0$, we have the trigonometric function solution in the form

$$u(x,t) = \gamma(t)\left\{c_3\left[\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\times\left(\frac{-A\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)+B\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{A\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)+B\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}\right)\right] + c_2c_3-\int\frac{f(t)k(t)}{\gamma(t)}dt\right\}+k(t). \quad (34)$$

(III) If $\lambda^2-4\mu=0$, we get the rational function solution in the form

$$u(x,t) = \gamma(t)\left[c_3\left(\frac{B}{A+B\xi}-\frac{\lambda}{2}\right)+c_2c_3 - \int\frac{f(t)k(t)}{\gamma(t)}dt\right]+k(t). \quad (35)$$

where A and B are arbitrary constants. Finally, we note that, if $\mu=0,\lambda>0,B=0$ and $A\neq 0$ then we deduce from (33) that

$$u(x,t) = \gamma(t)\left\{\frac{-c_3\lambda}{2}\left[1-\coth\left(\frac{\lambda\xi}{2}\right)\right]+c_2c_3 - \int\frac{f(t)k(t)}{\gamma(t)}dt\right\}+k(t), \quad (36)$$

while if $\mu=0,\lambda>0,B\neq 0$ and $B^2>A^2$ then we deduce from (33) that

$$u(x,t) = \gamma(t)\left\{\frac{-c_3\lambda}{2}\left[1-\tanh\left(\xi_0+\frac{\lambda\xi}{2}\right)\right] + c_2c_3-\int\frac{f(t)k(t)}{\gamma(t)}dt\right\}+k(t), \quad (37)$$

where $\xi_0=\tanh^{-1}\left(\frac{A}{B}\right)$. Note that (36) and (37) represent the solitary wave solutions of the generalized Gardner equation (3).

Remark. All solutions of this article have been checked with the *Maple* by putting them back into the original equations (2) and (3).

4 Conclusions

In this article, the generalized $\left(\frac{G'}{G}\right)$-expansion method is applied to obtain more general exact solutions for NLEEs. By using this method we have successfully obtained exact solutions with parameters of the variable coefficients Burgers-Fisher equation with the forced term and the variable coefficients generalized Gardner equation with the forced term. When these parameters are taken special values, the solitary wave solutions are derived from the hyperbolic solutions.

References:

[1] M.J. Ablowitz, P.A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scattering, Cambridge Univ. Press, New York, 1991.

[2] R. Hirota, Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons, Phys. Rev. Lett. 27 (1971) 1192-1194.

[3] M.R. Miurs, Backlund Transformation, Springer, Berlin, 1978.

[4] J. Weiss, M. Tabor, G. Carnevale, The Painlevé property for partial differential equations, J. Math. Phys. 24 (1983) 522-526.

[5] C.T. Yan, A simple transformation for nonlinear waves, Phys. Lett. A 224 (1996) 77-84.

[6] M.L. Wang, Exact solution for a compound KdV–Burgers equations, Phys. Lett. A 213 (1996) 279-287.

[7] M. El-Shahed, Application of He's homotopy perturbation method to Volterra's integro-differential equation, Int. J. Nonlinear Sci. Numer. Simul. 6 (2005) 163-168.

[8] J.H. He, Homotopy perturbation method for bifurcation of nonlinear problems, Int. J. Nonlinear Sci. Numer. Simul. 6 (2005) 207-208.

[9] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos Solitons and Fractals 26 (2005) 695-700.

[10] J.H. He, Variational iteration method—a kind of nonlinear analytical technique: some examples, Int. J. Nonlinear Mech. 34 (1999) 699-708.

[11] J.H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput. 114 (2000) 115-123.

[12] J.H. He, Variational principles for some nonlinear partial differential equations with variable coefficients, Chaos Solitons and Fractals 19 (2004) 847-851.

[13] J.H. He, Variational approach to (2 + 1)-dimensional dispersive long water equations, Phys. Lett. A 335 (2005) 182-184.

[14] T.A. Abassy, M.A. El-Tawil, H.K. Saleh, The solution of KdV and mKdV equations using adomian Pade approximation, Int. J. Nonlinear Sci. Numer.Simul. 5 (2004) 327-340.

[15] E.M.E. Zayed, H.A. Zedan, K.A. Gepreel, Group analysis and modified extended Tanh-function to find the invariant solutions and soliton solutions for nonlinear Euler equations, Int. J. Nonlinear Sci. Numer.Simul. 5 (2004) 221-234.

[16] H.A. Abdusalam, On an improved complex Tanh-function method, Int. J. Nonlinear Sci. Numer. Simul. 6 (2005) 99-106.

[17] J.Q. Hu, An algebraic method exactly solving two high dimensional nonlinear evolution equations, Chaos Solitons and Fractals 23 (2005) 391-398.

[18] Y. Chen, Q. Wang, B. Li, A series of soliton-like and double-like periodic solutions of a (2 + 1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equation, Commun. Theor. Phys. 42 (2004) 655-660.

[19] S.K. Liu, Z.T. Fu, S.D. Liu, Q. Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A 289 (2001) 69-74.

[20] Z.T. Fu, S.K. Liu, S.D. Liu, Q. Zhao, New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations, Phys. Lett. A 290 (2001) 72-76.

[21] J.B. Liu, K.Q. Yang, The extended F-expansion method and exact solutions of nonlinear PDEs, Chaos Solitons and Fractals 22 (2004) 111-121.

[22] S. Zhang, New exact solutions of the KdV–Burgers–Kuramoto equation, Phys. Lett. A 358 (2006) 414-420.

[23] S. Zhang, T.C. Xia, A generalized new auxiliary equation method and its applications to nonlinear partial differential equations, Phys. Lett. A 363 (2007) 356-360.

[24] S. Zhang, T.C. Xia, A generalized auxiliary equation method and its application to (2 + 1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equations, J. Phys. A: Math. Theore. 40 (2007) 227-248.

[25] J.H. He, X.H. Wu, Exp-function method for nonlinear wave equations, Chaos Solitons and Fractals 30 (2006) 700-708.

[26] J.H. He, M.A. Abdou, New periodic solutions for nonlinear evolution equations using Exp-function method, Chaos Solitons and Fractals 34 (2007) 1421-1429.

[27] E.M.E. Zayed, K.A. Gepreel, The $\left(\frac{G'}{G}\right)$-expansion method for finding travelling wave solutions of nonlinear PDEs in mathematical physics, J. Math. Phys. 50 (2009) 013502-013513.

[28] E.M.E. Zayed, K.A. Gepreel, Some applications of the $\left(\frac{G'}{G}\right)$-expansion method to non-linear partial differential equations, Appl. Math. Comput. 212 (2009) 1-13.

[29] E.M.E. Zayed, K.A. Gepreel, Three types of travelling wave solutions of nonlinear evolution equations using the $\left(\frac{G'}{G}\right)$-expansion method, Int. J. Nonlinear Sci. 7 (2009) 501-512.

[30] E.M.E. Zayed, S. Al-Joudi, Applications of an Improved $\left(\frac{G'}{G}\right)$-expansion Method to Nonlinear PDEs in Mathematical Physics, AIP Conf. Proc. Amer. Institute of Phys. 1168 (2009) 371-376.

[31] M.L. Wang, X.Z. Li, J.L. Zhang, The $\left(\frac{G'}{G}\right)$-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A 372 (2008) 417-423.

[32] H. Zhang, D. Lu, Exact Solutions of the Variable Coefficient Burgers-Fisher Equation with Forced Term, Int. J. Nonlinear Sci. 9 (2010) 252-256.

[33] H. Zedan, New classes of solutions for a system of partial differential equations by $\left(\frac{G'}{G}\right)$-expansion method, Nonlinear Sci. Lett. A, 1 (3), (2010) 219-238.

[34] J. Zhang, X. Wei, Y. Lu, A generalized $\left(\frac{G'}{G}\right)$-expansion method and its applications, Phys. Lett. A 372 (2008) 3653-3658.

[35] S. Zhang, J.L. Tong, W. Wang, A generalized $\left(\frac{G'}{G}\right)$-expansion method for the mKdV equation with variable coefficients, Phys. Lett. A 372 (2008) 2254-2257.

[36] S. Zhang, W. Wang and J.L. Tong, A generalized $\left(\frac{G'}{G}\right)$-expansion method and its application to the (2+1)-dimensional Broer–Kaup equations, Appl. Math. Comput. 209 (2009) 399-404.

[37] M. Song, Y. Ge, Application of the $\left(\frac{G'}{G}\right)$-expansion method to (3+1)-dimensional nonlinear evolution equations, Comput. Math. Appl. 60 (2010) 1220-1227.

[38] E.M.E. Zayed, M.A.M. Abdelaziz, Exact Solutions for Nonlinear PDEs with Variable Coefficients Using the Generalized $\left(\frac{G'}{G}\right)$-Expansion Method and the Exp-Function Method, Int. Rev. Phys. 4 (2010) 161-171.

[39] E.M.E. Zayed, M.A.M. Abdelaziz, Travelling wave solutions for the Burgers Equation and the KdV Equation with variable coefficients Using the Generalized $\left(\frac{G'}{G}\right)$-Expansion Method, Z. Naturforsch. 65a (2010) in press.

[40] E.M.E. Zayed, M.A.M. Abdelaziz, Exact Solutions for The Generalized Zakharov-Kuznetsov Equation with Variable Coefficients Using The Generalized $\left(\frac{G'}{G}\right)$-expansion method, AIP Conf. Proc. Amer. Institute of Phys. 1281 (2010) 2216-2219.

[41] E.M.E. Zayed, S. Al-Joudi, An Improved $\left(\frac{G'}{G}\right)$-expansion Method for Solving Nonlinear PDEs in Mathematical Physics, AIP Conf. Proc. Amer. Institute of Phys. 1281 (2010) 2220-2224.

[42] E.M.E. Zayed, Equivalence of The $\left(\frac{G'}{G}\right)$-expansion Method and The Tanh-Coth Function Method, AIP Conf. Proc. Amer. Institute of Phys. 1281 (2010) 2225-2228.

[43] L.H. Zhang, L.H. Dong, L.M. Yan, Construction of non-travelling wave solutions for the generalized variable-coefficient Gardner equation, Appl. Math. Comput. 203 (2008) 784-791.