Analysis of a deteriorating cold standby system with priority

Lixia Ma
Tianjin University
Department of Mathematics
Tianjin, 300072
P. R. China
lixiama@tju.edu.cn

Genqi Xu
Tianjin University
Department of Mathematics
Tianjin, 300072
P. R. China
gqxu@tju.edu.cn

Nikos E. Mastorakis
Military Institutes of University Education
Hellenic Naval Academy
Terma Hatzikyriakou, Piraeus,
Greece
mastor@hna.gr

Abstract: A deteriorating cold standby repairable system consisting of two dissimilar components and one repairman is studied in this paper. Suppose that the life of each component satisfies the exponentially distribution and repair time of the component satisfies the general distribution, the component 1 has priority in use and repair. Firstly, a mathematical model is built via the differential and partial differential equations. And then using the C_0-semigroup theory of bounded linear operators, the existence and uniqueness of the solution, the non-negative steady-state solution and the exponential stability of the system are derived. Based on the stability result, some reliability indices of the system and an optimization problem are presented at the end of the paper.

Key–Words: C_0 Semigroup, Well-Posedness, Asymptotic Stability, Exponential Stability, Availability.

1 Introduction

With the development of the modern technology and extensive use of electronic products, the reliability problem of the reparable systems has become a hot topic. It is well known that the redundancy can enhance the reliability of the system. In the earlier study, the standby component doesn't fail in its standby period since they are machine system. However, the electronic product doesn't satisfy this property. In fact, for an electronic product, it has less failure rate if it is in use, otherwise, it has greater failure rate after certain time, which means that it deteriorates. Therefore, in this paper we will discuss the reliability of a system consisting of the electronic product with repair. For the reliability problem of system, it has been a hot topic in engineering and Mathematics. Using mathematical model and Markov renewal theory, one studied the indices of reliability of the reparable system, for example, see [1], [2], [3]. Notice that, in the earlier research, the component after repair is "as good as new", only recent a deteriorating case is considered in ([8]).

Although some nice results have been obtained, there exists certain difficulty to obtain some index of the system, for instance, how long time one can see the stable state of the system. The mainly difficulty comes from the mathematical model; this is because one cannot give an exact solution to the model. To overcome this difficulty, many authors have worked on the analysis of the reparable systems using functional analysis method (see [4]-[6], etc), more precisely, semigroup theory of bounded linear operators ([7]) to prove the well-posedness and the stability of the system. In this paper, we will give a new model about a cold standby repairable system, and then discuss the system by functional analysis method.

The rest is organized as follows. In section 2, a mathematical model for the system under consideration is established. And then in section 3, the existence and uniqueness of nonnegative time-dependent solution of the system are presented via C_0 semigroup theory. In section 4, by analyzing spectral distribution of the system operator, the main results on stability of the system are obtained. In section 5 some indices of stationary state of the system and the estimation of instantaneous availability of the system are given. Finally, an optimal repair rate problem is studied.

2 Modeling for a system with priority

Firstly, we describe the system under consideration. Suppose that a system consists of two dissimilar components and one repairman, the component 1 is the main working unit and the component 2 is cold standby unit. The component 1 has priority in use and repair. The system satisfies the following assumptions:

Assumption 1. Initially, the two components are both new, and component 1 is in a working state while component 2 is in a standby state.

Assumption 2. Assume that both components after repair are "as good as new".

Assumption 3. The component 1 has priority in working and repair. When both components are good, the component 1 has the higher use priority than the component 2, even if component 2 is working, it must be switched immediately into the standby state as soon as component 1 after repaired so that the component 1 becomes the working state; When both components fail (i.e. the system is down), component 1 has the higher repair priority than component 2, even if the repairman is repairing component 2, he must switch to repair the component 1. He will work on the repair of component 2 after completing the repair on component 1.

Assumption 4. The standby component will perhaps fail in the standby time for some reason. The failure and repair times of both components follow exponential distribution and general distribution respectively. λ_j and ε denote the failure rate of component $j(j = 1, 2)$ and the standby component; $\mu_j(x)$ denote the repair rate of component $j(j = 1, 2)$.

Assumption 5. All failures are independent of each other.

Under these assumptions, we can divide the system into the following states:

0: The components 1 and 2 are in good condition;

1: The component 1 is failure under repair and the component 2 is working;

2: The component 1 is working and component 2 is failure under repairing;

3: The components 1 and 2 are failure under repair.

$P_0(t)$ denotes the probability that component 1 is in working state and component 2 is in cold standby state at time t;

$p_1(t, x)dx$ represents the probability that the repairman is dealing with component 1 with the elapsed time lying in $[x, x + dx)$ and component 2 is in work at time t;

$p_2(t, x)dx$ represents the probability that the repairman is dealing with component 2 with the elapsed time lying in $[x, x + dx)$ and component 1 is in work at time t;

$p_3(t, x)dx$ represents the probability that the repairman is dealing with component 1 with the elapsed time lying in $[x, x + dx)$ and component 2 is waiting for repair at time t.

By the supplementary variables technique, the model of the system can be formulated as

$$\begin{cases} \frac{dP_0(t)}{dt} = -(\lambda_1 + \varepsilon)P_0(t) + \int_0^\infty \mu_1(x)p_1(x, t)dx \\ \qquad + \int_0^\infty \mu_2(x)p_2(x, t)dx, \\ \frac{\partial p_1(x,t)}{\partial t} = -\frac{\partial p_1(x,t)}{\partial x} - (\mu_1(x) + \lambda_2)p_1(x, t), \\ \frac{\partial p_2(x,t)}{\partial t} = -\frac{\partial p_2(x,t)}{\partial x} - (\mu_2(x) + \lambda_1)p_2(x, t), \\ \frac{\partial p_3(x,t)}{\partial t} = -\frac{\partial p_3(x,t)}{\partial x} - \mu_1(x)p_3(x, t). \end{cases} \tag{1}$$

The boundary conditions are

$$\begin{cases} p_1(0, t) = \lambda_1 P_0(t), \\ p_2(0, t) = \varepsilon P_0(t) + \int_0^\infty \mu_1(x)p_3(x, t)dx, \\ p_3(0, t) = \lambda_1 \int_0^\infty p_2(x, t)dx + \lambda_2 \int_0^\infty p_1(x, t)dx, \end{cases} \tag{2}$$

where $(x, t) \in \mathbb{R}_+ \times \mathbb{R}_+$.

3 The well-possedness of the system solution

In this section, we will discuss the well-posed-ness of the system. To this end, we formulate the system (1) into a suitable Banach space. Based on the practical background of our model, we can assume that $\mu_j(x)(j = 1, 2)$ satisfy

$$M = \sup_{x\in\mathbb{R}_+} \mu_j(x) < \infty; \ \int_0^\infty \mu_j(s)ds = \infty. \tag{3}$$

In the sequel, we denote by $\mathbb{R}_+ = [0, \infty)$. From the physical meaning of the problem, we take the state space $\mathbb{X} = \mathbb{R} \times [L^1(\mathbb{R}_+)]^3$, equipped with the norm $||P|| = |P_0| + \sum_{j=1}^{3} \|p_j||_{L^1}$, for each $P = (P_0, p_1, p_2, p_3) \in \mathbb{X}$. Obviously, $\mathbb{X}$ is a Banach space.

Define an operator $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ by

$$\mathcal{A}\begin{pmatrix} P_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \end{pmatrix} = \begin{pmatrix} -(\lambda_1 + \varepsilon)P_0 + \sum_{j=1}^{2} \int_0^\infty \mu_j(x)p_j(x)dx \\ (-\frac{d}{dx} - \mu_1(x) - \lambda_2)p_1(x), \\ (-\frac{d}{dx} - \mu_2(x) - \lambda_1)p_2(x), \\ (-\frac{d}{dx} - \mu_1(x))p_3(x) \end{pmatrix} \tag{4}$$

with domain

$$D(\mathcal{A}) = \left\{ P \in \mathbb{X} \;\middle|\; \begin{array}{l} p_j(x) \text{ is absolutely continuous,} \\ p_j(x), p_j'(x) \in L^1(\mathbb{R}_+), \\ p_1(0) = \lambda_1 P_0, \\ p_2(0) = \varepsilon P_0 + \int_0^\infty \mu_1(x)p_3(x)dx, \\ p_3(0) = \lambda_1 \int_0^\infty p_2(x)dx + \lambda_2 \int_0^\infty p_1(x)dx, \\ j = 1, 2, 3, \end{array} \right\} \tag{5}$$

Obviously, $\mathcal{A}$ is a linear operator in $\mathbb{X}$.

By the definition of $\mathcal{A}$, the system (1) can be rewritten as an abstract Cauchy problem in Banach space $\mathbb{X}$:

$$\begin{cases} \frac{dP(t)}{dt} = \mathcal{A}P(t), \quad t \geq 0, \\ P(0) = (1, 0, 0, 0). \end{cases} \tag{6}$$

where $P(t) = (P_0(t), p_1(x,t), p_2(x,t), p_3(x,t))$.

By Theorem II 6.7 and Definition II 6.8 of [9], we know that the well-possedness of the system (1) is equivalent to the operator $\mathcal{A}$ being the generator of some C_0 semigroup $T(t)$. Therefore, the main task of this section is to verify that the operator $\mathcal{A}$ generates a C_0 semigroup. In particular, the semigroup $T(t)$ generated by the operator $\mathcal{A}$ is also positive and contractive.

Theorem 1. *Let $\mathcal{A}$ be defined by (4) and (5). Then the system operator $\mathcal{A}$ is a closed and densely linear operator in $\mathbb{X}$.*

The proof is a direct verification (for example, see [**?**]), so we omit.

Theorem 2. *$\mathcal{A}$ is a dissipative operator in $\mathbb{X}$, and $\gamma \in \rho(\mathcal{A})$ for $\Re\gamma > 0$.*

Proof: We know that the dual space of $\mathbb{X}$ is $\mathbb{X}^* = \mathbb{R} \times (L^\infty(\mathbb{R}_+))^3$ by the direct verification, and the norm for $Q \in \mathbb{X}^*$ is given by $\|Q\| = \max\{|q_1|, \|q_j\|_\infty, j = 2,3,4\}$.

Step I. We show $\mathcal{A}$ is dissipative.

For any $P \in D(\mathcal{A})$, we choose

$$\begin{aligned} Q_P &= (\|P\|\mathrm{sign}(P_0), \|P\|\mathrm{sign}(p_1(x)), \\ &\quad \|P\|\mathrm{sign}(p_2(x)), \|P\|\mathrm{sign}(p_3(x)) \in \mathbb{X}^* \end{aligned}$$

Obviously, $Q_P \in \mathcal{F}(P) = \{Q \in \mathbb{X}^* | (P, Q_P) = ||Q_P||^2 = \|P\|^2\}$. Moreover,

$$\begin{aligned} &(\mathcal{A}P, Q_P) \\ &= \|P\|\{-(\lambda_1+\varepsilon)|P_0| + \mathrm{sign}(P_0)\sum_{j=1}^{2}\int_0^\infty \mu_j(x)p_j(x)dx \\ &- \sum_{j=1}^{3}\int_0^\infty p_j'(x)\mathrm{sign}(p_j(x))dx - \int_0^\infty(\mu_1(x)+\lambda_2)|p_1(x)|dx \\ &- \int_0^\infty(\mu_2(x)+\lambda_1)|p_2(x)|dx - \int_0^\infty \mu_1(x)|p_3(x)|dx\}, \end{aligned}$$

while

$$\begin{aligned} &\mathrm{sign}(P_0)\int_0^\infty \mu_j(x)p_j(x)dx - \int_0^\infty \mu_j(x)|p_j(x)|dx \le 0, \\ &\int_0^\infty p_j'(x)sign(p_j(x))dx = -|p_j(0)|, \quad (j=1,2) \end{aligned}$$

we have

$$\begin{aligned} &\tfrac{(\mathcal{A}P, Q_P)}{||P||} \\ &\le -(\lambda_1+\varepsilon)|P_0| + \sum_{j=1}^{3}|p_j(0)| \\ &- \int_0^\infty \lambda_2|p_1(x)|dx - \int_0^\infty \lambda_1|p_2(x)|dx \\ &- \int_0^\infty \mu_1(x)|p_3(x)|dx \\ &\le -(\lambda_1+\varepsilon)|P_0| + \lambda_1|P_0| + |\varepsilon P_0 \\ &+ \int_0^\infty \mu_1(x)p_3(x)dx| + |\lambda_1\int_0^\infty p_2(x)dx \\ &+ \lambda_2\int_0^\infty p_1(x)dx| - \int_0^\infty \lambda_2|p_1(x)|dx \\ &- \int_0^\infty \lambda_1|p_2(x)|dx - \int_0^\infty \mu_1(x)|p_3(x)|dx \\ &\le 0, \end{aligned}$$

i.e., $\mathcal{A}$ is dissipative.

Step II. We prove that $\{\gamma \in \mathbb{C} \mid \Re\gamma > 0\} \subset \rho(\mathcal{A})$.

For any $F = (f_0, f_1, f_2, f_3) \in \mathbb{X}$, we consider the resolvent equation $(\gamma I - \mathcal{A})P = F$, that is

$$\begin{cases} (\gamma+\lambda_1+\varepsilon)P_0 - \int_0^\infty \mu_1(x)p_1(x)dx \\ \qquad - \int_0^\infty \mu_2(x)p_2(x)dx = f_0, \\ \frac{dp_1(x)}{dx} + (\gamma+\mu_1(x)+\lambda_2)p_1(x) = f_1(x), \\ \frac{dp_2(x)}{dx} + (\gamma+\mu_2(x)+\lambda_1)p_2(x) = f_2(x), \\ \frac{dp_3(x)}{dx} + (\gamma+\mu_1(x))p_3(x) = f_3(x), \\ p_1(0) = \lambda_1 P_0, \\ p_2(0) = \varepsilon P_0 + \int_0^\infty \mu_1(x)p_3(x)dx, \\ p_3(0) = \lambda_1\int_0^\infty p_2(x)dx + \lambda_2\int_0^\infty p_1(x)dx. \end{cases} \tag{7}$$

Solving the differential equations in (7) yields

$$\begin{cases} p_1(x) &= p_1(0)e^{-\int_0^x(\gamma+\mu_1(s)+\lambda_2)ds} \\ &+ \int_0^x f_1(r)e^{-\int_r^x(\gamma+\mu_1(s)+\lambda_2)ds}dr, \\ p_2(x) &= p_2(0)e^{-\int_0^x(\gamma+\mu_2(s)+\lambda_1)ds} \\ &+ \int_0^x f_2(r)e^{-\int_r^x(\gamma+\mu_2(s)+\lambda_1)ds}dr, \\ p_3(x) &= p_3(0)e^{-\int_0^x(\gamma+\mu_1(s))ds} \\ &+ \int_0^x f_3(r)e^{-\int_r^x(\gamma+\mu_1(s))ds}dr. \end{cases} \tag{8}$$

By assumption to $\mu_j(x)$, we have

$$\begin{aligned} &\int_0^\infty \left|\int_0^x f_1(r)e^{-\int_r^x(\gamma+\mu_1(s)+\lambda_2)ds}dr\right|dx \\ \le\ &\int_0^\infty \int_0^x |f_1(r)|e^{-\int_r^x(\gamma+\mu_1(s)+\lambda_2)ds}drdx \\ =\ &\int_0^\infty |f_1(r)|dr\int_r^\infty e^{-\int_r^x(\gamma+\mu_1(s)+\lambda_2)ds}dx \\ \le\ &\frac{1}{\Re\gamma+\lambda_2}\|f_1\|_{L^1} \le \frac{1}{\Re\gamma}\|f_1\|_{L^1}, \end{aligned}$$

so, $p_1 \in L^1(\mathbb{R}_+)$ and

$$\begin{aligned} &\int_0^\infty \mu_1(x)p_1(x)dx \\ =\ &\int_0^\infty f_1(x)dx + p_1(0) - (\gamma+\lambda_2)\int_0^\infty p_1(x)dx \end{aligned}$$

is finite.

Similarly, we can prove that $p_2,\ p_3 \in L^1(\mathbb{R}_+)$ and $\int_0^\infty \mu_2(x)p_2(x)dx < +\infty$. Now we substitute (8) into (7) into boundary conditions (2) and get algebraic equations about $(P_0, p_1(0), p_2(0), p_3(0))$:

$$\begin{cases} (\gamma+\lambda_1+\varepsilon)P_0 - p_1(0)G_1(\gamma) - p_2(0)G_2(\gamma) \\ \qquad\qquad = f_0 + H_1(\gamma) + H_2(\gamma), \\ -\lambda_1 P_0 + p_1(0) = 0, \\ -\varepsilon P_0 + p_2(0) - p_3(0)G_3(\gamma) = H_3(\gamma) \\ -\lambda_2 G_4(\gamma)p_1(0) - \lambda_1 G_5(\gamma)p_2(0) + p_3(0) \\ \qquad\qquad = -\lambda_2 H_4(\gamma) - \lambda_1 H_5(\gamma), \end{cases} \tag{9}$$

where

$$\begin{aligned}
&G_1(\gamma)=\int_0^\infty \mu_1(x)e^{-\int_0^x(\gamma+\mu_1(s)+\lambda_2)ds}dx,\\
&H_1(\gamma)=\int_0^\infty \mu_1(x)\int_0^x f_1(r)e^{-\int_r^x(\gamma+\mu_1(s)+\lambda_2)ds}drdx,\\
&G_2(\gamma)=\int_0^\infty \mu_2(x)e^{-\int_0^x(\gamma+\mu_2(s)+\lambda_1)ds}dx,\\
&H_2(\gamma)=\int_0^\infty \mu_2(x)\int_0^x f_2(r)e^{-\int_r^x(\gamma+\mu_2(s)+\lambda_1)ds}drdx,\\
&G_3(\gamma)=\int_0^\infty \mu_1(x)e^{-\int_0^x(\gamma+\mu_1(s))ds}dx,\\
&H_3(\gamma)=\int_0^\infty \mu_1(x)\int_0^x f_3(r)e^{-\int_r^x(\gamma+\mu_1(x))ds}dr,\\
&G_4(\gamma)=\int_0^\infty e^{-\int_0^x(\gamma+\mu_1(s)+\lambda_2)ds}dx,\\
&H_4(\gamma)=\int_0^\infty \int_0^x f_1(r)e^{-\int_r^x(\gamma+\mu_1(s)+\lambda_2)ds}dr,\\
&G_5(\gamma)=\int_0^\infty e^{-\int_0^x(\gamma+\mu_2(s)+\lambda_1)ds}dx,\\
&H_5(\gamma)=\int_0^\infty \int_0^x f_2(r)e^{-\int_r^x(\gamma+\mu_2(s)+\lambda_1)ds}dr,
\end{aligned}\tag{10}$$

Denote the coefficient matrix of (9) by $\triangle(\gamma)$ and the determinant by $D(\gamma)$, then

$$\begin{aligned}&\triangle(\gamma)\\ =&\begin{pmatrix}\gamma+\lambda_1+\varepsilon & -G_1(\gamma) & -G_2(\gamma) & 0\\ -\lambda_1 & 1 & 0 & 0\\ -\varepsilon & 0 & 1 & -G_3(\gamma)\\ 0 & -\lambda_2 G_4(\gamma) & -\lambda_1 G_5(\gamma) & 1\end{pmatrix}\end{aligned}$$

When $\Re\gamma>0$, it holds that

$$\begin{aligned}
&G_1(\gamma)+\lambda_2 G_4(\gamma)\\
&=\int_0^\infty(\mu_1(x)+\lambda_2)e^{-\int_0^x(\gamma+\mu_1(s)+\lambda_2)ds}dx<1,\\
&G_2(\gamma)+\lambda_1 G_5(\gamma)\\
&=\int_0^\infty(\mu_2(x)+\lambda_1)e^{-\int_0^x(\gamma+\mu_2(s)+\lambda_1)ds}dx<1.
\end{aligned}\tag{11}$$

This shows that $\triangle(\gamma)$ is column strictly diagonal dominant, so $D(\gamma)\neq 0$ and linear equations (9) has a unique solution $(P_0,p_1(0),p_2(0),p_3(0))$. Thus, it can be derived from (8) that the resolvent equations (7) has a unique solution $(P_0,p_1(x),p_2(x),p_3(x))\in D(\mathcal{A})$, which means $\gamma I-\mathcal{A}$ is surjection. Because $\gamma I-\mathcal{A}$ is closed and $D(\mathcal{A})$ is dense in $\mathbb{X}$, then $(\gamma I-\mathcal{A})^{-1}$ exists and is bounded by Inverse Operator Theorem. So $\gamma\in\rho(\mathcal{A})$. □

As a direct result of Lumer-Phillips Theorem (see, [7]), we have the following result.

Theorem 3. *Let $\mathcal{A}$ and $\mathbb{X}$ be defined as before. Then $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathbb{X}$. Hence the equation (6) has a unique solution.*

Next, we will prove the existence of positive solutions to (1) since it is a practical problem. We complete the proof by showing $\mathcal{A}$ generates a positive semigroup on $\mathbb{X}$.

Theorem 4. *$\mathcal{A}$ generates a positive C_0-semigroup of contractions on $\mathbb{X}$.*

Proof: According to the positive semigroup theory (see [11]), $\mathcal{A}$ generates a positive C_0-semigroup of contractions if and only if $\mathcal{A}$ is a dispersive and $\mathcal{R}(I-\mathcal{A})=\mathbb{X}$. Since Theorem 2 has asserted that $\mathcal{R}(I-\mathcal{A})=\mathbb{X}$, we only need to prove $\mathcal{A}$ is a dispersive operator. Take $\Phi=(\mathrm{sign}_+(P_0),\mathrm{sign}_+(p_1(x)),\mathrm{sign}_+(p_2(x)),\mathrm{sign}_+(p_3(x)))$ instead of Q_P, where $\mathrm{sign}_+(P)=\begin{cases}1, & p>0;\\ 0, & p\le 0.\end{cases}$ The others is similar to the process of proving the dissipativity of the operator $\mathcal{A}$, so we omit. □

Theorem 5. *The semigroup $T(t)$ generated by $\mathcal{A}$ is random one.*

Proof: It is sufficient to prove that $T(t)$ is positive conservative according to definition of random semigroup. For any positive vector $P\in D(\mathcal{A})$, $P=(P_0,,p_1(x),p_2(x),p_3(x))$, we have $T(t)P>0$ and $T(t)P\in D(\mathcal{A})$, since $T(t)$ is positive. Set $P(t,\cdot)=(P_0(t),p_1(x,t),p_2(x,t),p_3(x,t))$, obviously, $T(t)P$ satisfies the equation $\frac{dT(t)P}{dt}=\mathcal{A}P,\ \forall t>0$.

Integrating the differential equations in (1) from 0 to ∞ with respect to x, we have

$$\frac{dP_0(t)}{dt}+\sum_{j=1}^{3}\frac{d}{dt}\int_0^\infty p_j(x,t)dx=0,$$

which shows that $P_0(t)+\sum_{j=1}^{3}\int_0^\infty p_j(x,t)dx$ is a constant. Since $T(t)P$ is continuous in t, we obtain $||T(t)P||=P_0(t)+\sum_{j=1}^{3}\int_0^\infty p_j(x,t)dx=||P||$. So $T(t)$ is a random semigroup, which coincides with the physical meaning. □

4 Stability of the system

4.1 Asymptotical stability

In this section, we will discuss the stability of the system by analyzing the spectra of the system operator $\mathcal{A}$.

Lemma 6. *Set $\mu_\gamma=\int_0^\infty\mu(x)e^{-\int_0^x(i\gamma+\mu(\xi))d\xi}dx$, then*

$$\begin{cases}\mu_\gamma=1, & if\ \gamma=0,\\ \mu_\gamma\neq 1, & if\ \gamma\neq 0,\end{cases}\quad and\quad |\mu_\gamma|<1,\ \ \gamma\neq 0.$$

Theorem 7. *$\gamma_0=0$ is a simple eigenvalue of $\mathcal{A}$. Moreover if for any $\xi\in\mathbb{R}_+$,*

$$\sup_{\xi\ge 0}\int_\xi^\infty e^{-\int_\xi^x\mu_1(s)ds}dx<\infty.\tag{12}$$

There is no other spectrum of $\mathcal{A}$ besides zero in the imaginary axis.

Proof: Let us consider the equation $\mathcal{A}P = 0$, i.e.,

$$\begin{cases} (\lambda_1+\varepsilon)P_0 - \int_0^\infty \mu_1(x)p_1(x)dx \\ \qquad - \int_0^\infty \mu_2(x)p_2(x)dx = 0, \\ \frac{dp_1(x)}{dx} + (\mu_1(x)+\lambda_2)p_1(x) = 0, \\ \frac{dp_2(x)}{dx} + (\mu_2(x)+\lambda_1)p_2(x) = 0, \\ \frac{dp_3(x)}{dx} + \mu_1(x)p_3(x) = 0, \\ p_1(0) = \lambda_1 P_0, \\ p_2(0) = \varepsilon P_0 + \int_0^\infty \mu_1(x)p_3(x)dx, \\ p_3(0) = \lambda_1\int_0^\infty p_2(x)dx + \lambda_2\int_0^\infty p_1(x)dx. \end{cases} \tag{13}$$

Since $1-\lambda_1 G_5(0) > 0$, directly solving the equations we get

$$\begin{cases} p_1(0) = \lambda_1 P_0, \\ p_2(0) = \frac{\varepsilon+\lambda_1\lambda_2 G_4(0)}{1-\lambda_1 G_5(0)} P_0, \\ p_3(0) = \frac{\varepsilon\lambda_1 G_5(0)+\lambda_1\lambda_2 G_4(0)}{1-\lambda_1 G_5(0)} P_0, \end{cases} \tag{14}$$

and

$$\begin{cases} p_1(x) = \lambda_1 P_0 e^{-\int_0^x(\mu_1(s)+\lambda_2)ds}, \\ p_2(x) = P_0\frac{\varepsilon+\lambda_1\lambda_2 G_4(0)}{1-\lambda_1 G_5(0)} e^{-\int_0^x(\mu_2(s)+\lambda_1)ds}, \\ p_3(x) = P_0\frac{\varepsilon\lambda_1 G_5(0)+\lambda_1\lambda_2 G_4(0)}{1-\lambda_1 G_5(0)} e^{-\int_0^x \mu_1(s)ds}. \end{cases} \tag{15}$$

So 0 is an eigenvalue of $\mathcal{A}$ with geometric multiplicity one and $P = (P_0, p_1(x), p_2(x), p_3(x))$ given by (15) is the eigenvector of $\mathcal{A}$ corresponding to 0.

Next, we will prove that the algebraic multiplicity of 0 is one.

Obviously, the dual space of $\mathbb{X}$ is $\mathbb{X}^* = \mathbb{R} \times [L^\infty(\mathbb{R}_+)]^3$. Let $\mathcal{A}^*$ be the dual operator of $\mathcal{A}$. By the definition of the dual operator, we can obtain

$$\mathcal{A}^*Q = \begin{pmatrix} -(\lambda_1+\varepsilon)q_0 + \lambda_1 q_0 + \varepsilon q_2(0) \\ \mu_1(x)q_0 + q_1'(x) - (\mu_1(x)+\lambda_2)q_1(x) + \lambda_2 q_3(0) \\ \mu_2(x)q_0 + q_2'(x) - (\mu_2(x)+\lambda_1)q_2(x) + \lambda_1 q_3(0) \\ q_3'(x) - \mu_1(x)q_3(x) + \mu_1(x)q_2(0) \end{pmatrix}$$

where $Q = (q_0, q_1(x), q_2(x), q_3(x))$ and

$$D(\mathcal{A}^*) = \left\{ Q \in \mathbb{X}^* \;\middle|\; \begin{array}{c} q_j(x), q_j'(x) \in L^\infty(\mathbb{R}_+) \\ j = 1,2,3 \end{array} \right\}.$$

Clearly, $Q = (1,1,\cdots,1) \in D(\mathcal{A}^*)$ and $\mathcal{A}^*Q = 0$, so 0 is an eigenvalue of $\mathcal{A}^*$ and $Q = (1,1,1,1,1,1)$ is a corresponding eigenvector. Let P be an eigenfunction of $\mathcal{A}$ given by (15), then$(P,Q) = 1$. Therefore, 0 is a simple eigenvalue of $\mathcal{A}^*$.

Now, we show that $\{ib \mid b \neq 0\} \subset \rho(\mathcal{A})$.

For $\forall b \in \mathbb{R}, b \neq 0, \forall F \in \mathbb{X}$, let us consider the equation $(ibI - \mathcal{A})P = F$, (i.e., replace γ with ib in (7)) and solving the differential equation in (7) yields

$$\begin{cases} p_1(x) = p_1(0)e^{-\int_0^x(ib+\mu_1(s)+\lambda_2)ds} \\ + \int_0^x f_1(r)e^{-\int_r^x(ib+\mu_1(s)+\lambda_2)ds}dr, \\ p_2(x) = p_2(0)e^{-\int_0^x(ib+\mu_2(s)+\lambda_1)ds} \\ + \int_0^x f_2(r)e^{-\int_r^x(ib+\mu_2(s)+\lambda_1)ds}dr, \\ p_3(x) = p_3(0)e^{-\int_0^x(ib+\mu_1(s))ds} \\ + \int_0^x f_3(r)e^{-\int_r^x(ib+\mu_1(s)ds}dr. \end{cases} \tag{16}$$

It is easy to see that $p_1, p_2 \in L^1(\mathbb{R}_+)$. For p_3,

$$\int_0^\infty |p_3(x)|dx \le |p_3(0)|\int_0^\infty e^{-\int_0^x \mu_1(s)ds}dx + \int_0^\infty |f_3(r)|dr\int_r^\infty e^{-\int_r^x \mu_1(s)ds}dx$$

By the assumption (12), we have

$$\int_0^\infty |f_3(r)|dr\int_r^\infty e^{-\int_r^x \mu_1(s)ds}dx < \infty \tag{17}$$

for any $f_3 \in L^1(\mathbb{R}_+)$, so $p_3 \in L^1(\mathbb{R}_+)$.

According to the proof of Theorem 2, the algebraic equation about $(P_0, p_1(0), p_2(0), p_3(0))$ and the coefficient matrix is $\triangle(ib)$ having the same form with (26). When $b \neq 0$, from Lemma 6 we know that

$$\begin{aligned} & |G_1(ib) + \lambda_2 G_4(ib)| \\ = & \left|\int_0^\infty (\mu_1(x)+\lambda_2)e^{-\int_0^x(ib+\mu_1(s)+\lambda_2)ds}dx\right| < 1 \end{aligned}$$

and $|G_2(ib) + \lambda_1 G_5(ib)| < 1, |G_3(ib)| < 1$. Thus, $\triangle(ib)$ is also column strictly diagonal dominant, so $D(ib) \neq 0$ and by the same method in Theorem 2, $ib \in \rho(\mathcal{A})$. That is $i\mathbb{R}\backslash\{0\} \subset \rho(\mathcal{A})$. $\square$

Remark 8. *The condition (12) is necessary for $i\mathbb{R}\backslash\{0\} \subset \rho(\mathcal{A})$, otherwise the estimate (17) doesn't hold evidently. Let us consider a counterexample. From this example we also see that the existence of expectation of the repair time does not ensure the condition (12). If (12) doesn't hold, maybe we have $i\mathbb{R} \subset \sigma(\mathcal{A})$.*

Example 4.1. *Let us consider function $\mu(x) = \frac{\alpha}{2+x}$ with $\alpha > 1$. We have*

$$\int_r^\infty e^{-\int_r^x \mu(s)ds}dx = \int_r^\infty \frac{(2+r)^\alpha}{(2+x)^\alpha}dx = \frac{2+r}{\alpha-1}.$$

So $\sup_{r\ge 0}\int_r^\infty e^{-\int_r^x \mu(s)ds}dx = \infty$.

For $f(r) = \frac{1}{(2+r)^{\frac{4}{3}}}e^{-ibr} \in L^1(\mathbb{R}_+)$, we have

$$\begin{aligned} & \int_0^\infty \left|\int_0^x e^{-\int_r^x(ib+\mu(s))ds}f(r)dr\right|dx \\ & = \int_0^\infty \frac{dr}{(2+r)^{\frac{4}{3}}}\int_r^\infty e^{-\int_r^x \frac{\alpha}{2+s}ds}dx \\ = & \int_0^\infty \frac{1}{(2+r)^{\frac{4}{3}}}\frac{2+r}{\alpha-1}dr = \infty. \end{aligned}$$

Summary discussion above, we have the following assertion.

Theorem 9. *Let space $\mathbb{X}$ and operator $\mathcal{A}$ be defined as before, the condition (12) hold. Then the following statements are true*

1)$\mathcal{A}$ generates a positive C_0-semigroup of contractions $T(t)$;

2) There exists a unique dynamic solution $P(t)$ for any initial value $P(0)$. In particular, when the initial value is nonnegative, the dynamic solution $P(t)$ is positive.

3) The system (1) has a positive steady-state $\hat{P} = (P_0, p_1(x), p_2(x), p_3(x))$ which is given by (15) with $P_0 = \frac{1}{Z}$, where

$$\begin{aligned} Z &= 1 + \lambda_1 \int_0^\infty e^{-\int_0^x (\mu_1(s)+\lambda_2)ds} \\ &+ \tfrac{\varepsilon+\lambda_1\lambda_2 G_4(0)}{1-\lambda_1 G_5(0)} \int_0^\infty e^{-\int_0^x (\mu_2(s)+\lambda_1)ds} \\ &+ \tfrac{\varepsilon\lambda_1 G_5(0)+\lambda_1\lambda_2 G_4(0)}{1-\lambda_1 G_5(0)} \int_0^\infty e^{-\int_0^x \mu_1(s)ds} \end{aligned} \tag{18}$$

Furthermore, the dynamic solution $P(t)$ converges to the nonnegative steady-state $\hat{P}$ in the sense of norm, i.e.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} T(t)P(0) = (P(0), Q)\hat{P}.$$

Proof: Recalling [10], the eigenfunction corresponding to eigenvalue 0 of the system operator defined by (15) is the stable solution the system (1), we denote it by $\hat{P}$. Note that $||\hat{P}|| = 1$, then the result is derived immediately. □

4.2 The exponential convergence

In the previous subsection we see that the dynamic solution of the system converges to the steady state when condition (12) holds. In this subsection, we consider the rate of convergence under some conditions about $\mu_j(x)(j = 1, 2)$ stronger than (12).

Since

$$\sup_{r\geq 0} \int_r^\infty e^{-\int_r^x \mu_1(s)ds} dx = \sup_{r\geq 0} \int_0^\infty e^{-\int_0^x \mu_1(s+r)ds} dx,$$

we define non-negative real number $\widehat{\mu}_1$ by

$$\widehat{\mu}_1 = \sup\{\eta \geq 0 \mid \sup_{r\geq 0} \int_0^\infty e^{\eta x - \int_0^x \mu_1(s+r)ds} dx < \infty\}. \tag{19}$$

In what follows, we always assume that $\widehat{\mu}_1 > 0$, which implies that condition (12) holds. Obviously, when $\eta < \widehat{\mu}_1$, the integral for $\forall\, r \geq 0$,

$$\sup_{r\geq 0} \int_0^\infty e^{-\int_0^x (\mu_1(s+r)-\eta)ds} dx < \infty, \tag{20}$$

while for $\eta > \widehat{\mu}_1$, it must be

$$\int_r^\infty e^{-\int_r^x (\mu_1(s)-\eta)ds} dx = \infty.$$

Set

$$\widehat{\mu} = \min\{\widehat{\mu}_1, \lambda_1\}. \tag{21}$$

Theorem 10. *Let $\mathbb{X}$, $\mathcal{A}$ and $\widehat{\mu}$ be defined as before. Then we have*

(I) The half-plane $\{\gamma \in \mathbb{C} \mid \Re\gamma + \widehat{\mu} < 0\}$ are in the spectrum of $\mathcal{A}$;

(II) The set $\{\gamma \in \mathbb{C} \mid \Re\gamma + \widehat{\mu} > 0, D(\gamma) \neq 0\}$ is in the resolvent set of $\mathcal{A}$ and the set $\{\gamma \in \mathbb{C} \mid \Re\gamma + \widehat{\mu} > 0, D(\gamma) = 0\}$ consists of all eigenvalues of $\mathcal{A}$;

(III) $\forall\delta > 0$, there are at most finitely many eigenvalues of $\mathcal{A}$ in the region $\{\gamma \in \mathbb{C} \mid \Re\gamma + \widehat{\mu} \geq \delta\}$;

(IV) There exists a constant $\omega_1 > 0$ such that the region $\{\gamma \in \mathbb{C} \mid \Re\gamma > -\omega_1\}$ has only one eigenvalue $\gamma_0 = 0$, thus it is strictly dominant.

Proof For any γ and $F = (f_0, f_1(x), f_2(x), f_3(x)) \in \mathbb{X}$, we consider the resolvent equation $(\gamma I - \mathcal{A})P = F$, that is

$$\begin{cases} (\gamma + \lambda_1 + \varepsilon)P_0 - \int_0^\infty \mu_1(x)p_1(x)dx \\ \qquad - \int_0^\infty \mu_2(x)p_2(x)dx = f_0, \\ \frac{dp_1(x)}{dx} + (\gamma + \mu_1(x) + \lambda_2)p_1(x) = f_1(x), \\ \frac{dp_2(x)}{dx} + (\gamma + \mu_2(x) + \lambda_1)p_2(x) = f_2(x), \\ \frac{dp_3(x)}{dx} + (\gamma + \mu_1(x))p_3(x) = f_3(x), \\ p_1(0) = \lambda_1 P_0, \\ p_2(0) = \varepsilon P_0 + \int_0^\infty \mu_1(x)p_3(x)dx, \\ p_3(0) = \lambda_1 \int_0^\infty p_2(x)dx + \lambda_2 \int_0^\infty p_1(x)dx. \end{cases} \tag{22}$$

Solving the differential equations in (22) yields

$$\begin{cases} p_1(x) &= p_1(0)e^{-\int_0^x (\gamma+\mu_1(s)+\lambda_2)ds} \\ &+ \int_0^x f_1(r)e^{-\int_r^x (\gamma+\mu_1(s)+\lambda_2)ds} dr, \\ p_2(x) &= p_2(0)e^{-\int_0^x (\gamma+\mu_2(s)+\lambda_1)ds} \\ &+ \int_0^x f_2(r)e^{-\int_r^x (\gamma+\mu_2(s)+\lambda_1)ds} dr, \\ p_3(x) &= p_3(0)e^{-\int_0^x (\gamma+\mu_1(s))ds} \\ &+ \int_0^x f_3(r)e^{-\int_r^x (\gamma+\mu_1(s))ds} dr, \end{cases} \tag{23}$$

When $\Re\gamma + \widehat{\mu} > 0$, using (20) the first part in the expression $p_1(x), p_2(x), p_3(x)$ are bounded. For the second terms in the expression $p_1(x), p_2(x)$ are also bounded. For the second term of $p_3(x)$,

$$\begin{aligned} &\int_0^\infty \left| \int_0^x f_3(r)e^{-\int_r^x (\gamma+\mu_1(s))ds} dr \right| dx \\ \leq\ & \int_0^\infty |f_3(r)|dr \int_r^\infty e^{-\int_r^x (\Re\gamma+\mu_1(s))ds} dx \\ \leq\ & M_2(\Re\gamma)\|f_3\|_{L^1}, \end{aligned}$$

where $M_2(\Re\gamma) = \sup_{r\geq 0}\int_0^\infty e^{-\int_0^x(\Re\gamma+\mu_1(s+r))ds}dx < \infty$. Therefore, $p_j \in L^1(\mathbb{R}_+), j=1,2,3$.

Obviously, when $\Re\gamma + \widehat{\mu} < 0$, at least $p_3(x)$ in (23) is not in $L^1(\mathbb{R}_+)$. Thus (I) follows.

(II) Now we substitute (23) into boundary conditions in (22) and get the algebraic equations about $(P_0, p_1(0), p_2(0), p_3(0))$:

$$\begin{cases} (\gamma+\lambda_1+\varepsilon)P_0 - p_1(0)G_1(\gamma) - p_2(0)G_2(\gamma) \\ \qquad = f_0 + H_1(\gamma) + H_2(\gamma), \\ -\lambda_1 P_0 + p_1(0) = 0, \\ -\varepsilon P_0 + p_2(0) - p_3(0)G_3(\gamma) = H_3(\gamma) \\ -\lambda_2 G_4(\gamma)p_1(0) - \lambda_1 G_5(\gamma)p_2(0) + p_3(0) \\ \qquad = -\lambda_2 H_4(\gamma) - \lambda_1 H_5(\gamma), \end{cases} \tag{24}$$

The coefficient determinant of the equations (24) is

$$D(\gamma) = \begin{vmatrix} \gamma+\lambda_1+\varepsilon & -G_1(\gamma) & -G_2(\gamma) & 0 \\ -\lambda_1 & 1 & 0 & 0 \\ -\varepsilon & 0 & 1 & -G_3(\gamma) \\ 0 & -\lambda_2 G_4(\gamma) & -\lambda_1 G_5(\gamma) & 1 \end{vmatrix}$$

Then the algebraic equations have solution if and only if $D(\gamma) \neq 0$.

When $D(\gamma) \neq 0$, the equations (24) has unique solution $(P_0, p_1(0), p_2(0), p_3(0))$, furthermore, the vector determined by (23) $(P_0, p_1(x), p_2(x), p_3(x)) \in D(\mathcal{A})$, and $(\gamma I - \mathcal{A})P = F$, so $\gamma \in \rho(\mathcal{A})$.

When $D(\gamma) = 0$, the equations (24) has nonzero solution $(P_0, p_1(0), p_2(0), p_3(0))$, furthermore, the vector $(P_0, p_1(x), p_2(x), p_3(x)) \in D(\mathcal{A})$, and $\mathcal{A}P = \gamma P$, so γ is an eigenvalues of $\mathcal{A}$.

(III) Note that

$$\begin{aligned} &G_1(\gamma) + \lambda_2 G_4(\gamma) \\ &= \int_0^\infty (\mu_1(x)+\lambda_2)e^{-\int_0^x(\gamma+\mu_1(s)+\lambda_2)ds}dx, \\ &= 1 - \int_0^\infty \gamma e^{-\int_0^x(\gamma+\mu_1(s)+\lambda_2)ds}dx = 1 - \gamma G_4(\gamma) \\ &G_2(\gamma) + \lambda_1 G_5(\gamma) \\ &= 1 - \int_0^\infty \gamma e^{-\int_0^x(\gamma+\mu_2(s)+\lambda_1)ds}dx = 1 - \gamma G_5(\gamma), \\ &G_3(\gamma) = 1 - \int_0^\infty \gamma e^{-\int_0^x(\gamma+\mu_1(s))ds}dx. \end{aligned} \tag{25}$$

and

$$D(\gamma) = \begin{vmatrix} \gamma & \gamma G_4(\gamma) & \gamma G_5(\gamma) & \gamma\Pi(\gamma) \\ -\lambda_1 & 1 & 0 & 0 \\ -\varepsilon & 0 & 1 & -G_3(\gamma) \\ 0 & -\lambda_2 G_4(\gamma) & -\lambda_1 G_5(\gamma) & 1 \end{vmatrix},$$

where

$$\Pi(\gamma) = \int_0^\infty e^{-\int_0^x(\gamma+\mu_1(\xi))d\xi}dx. \tag{26}$$

So, for all $-\widehat{\mu} + \delta \leq \Re\gamma \leq 0$, by condition (3)

$$\begin{aligned} &|G_3(\gamma)| = |\int_0^\infty \mu_1(x)e^{-(\Re\gamma+\widehat{\mu})x - i\Im\gamma x - \int_0^x(\mu_1(\xi)-\widehat{\mu})d\xi}dx| \\ &\leq M\int_0^\infty e^{-(\Re\gamma+\widehat{\mu})x - \int_0^x(\mu_1(\xi)-\widehat{\mu})d\xi}dx \end{aligned} \tag{27}$$

By Riemman Lemma, we have $\lim_{\Im\gamma\to\infty} G_3(\gamma) = 0$.

Similarly,

$$\begin{aligned} &G_4(\gamma) = \int_0^\infty e^{-(\Re\gamma+\lambda_2+\widehat{\mu})x - i\Im\gamma x - \int_0^x(\mu_1(\xi)-\widehat{\mu})d\xi}dx \\ &G_5(\gamma) = \int_0^\infty e^{-(\Re\gamma+\lambda_1+\widehat{\mu})x - i\Im\gamma x - \int_0^x(\mu_2(\xi)-\widehat{\mu})d\xi}dx \\ &\Pi(\gamma) = \int_0^\infty e^{-(\Re\gamma+\widehat{\mu})x - i\Im\gamma x - \int_0^x(\mu_1(\xi)-\widehat{\mu})d\xi}dx \end{aligned} \tag{28}$$

The Riemman Lemma asserts that $\lim_{\Im\gamma\to\infty}\Pi(\gamma) = 0$, $\lim_{\Im\gamma\to\infty} G_4(\gamma) = 0$, $\lim_{\Im\gamma\to\infty} G_5(\gamma) = 0$. Hence,

$$\lim_{\Im\gamma\to\infty}\frac{D(\gamma)}{\gamma} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -\lambda_1 & 1 & 0 & 0 \\ -\varepsilon & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

The limit is uniformly in the region $-\widehat{\mu}+\delta \leq \Re\gamma < 0$. Moreover $D(\gamma)$ is analytic function. So $D(\gamma)$ has at most finite number of zeros in $-\widehat{\mu}+\delta \leq \Re\gamma < 0$. Thus $\mathcal{A}$ has at most finite number eigenvalue in the region $-\widehat{\mu}+\delta \leq \Re\gamma < 0$. Moreover, there is no other spectrum of $\mathcal{A}$ besides zero in the imaginary axis according to Theorem 11. Thus (III) is true.

(IV) Set $H_1(\gamma) = \frac{D(\gamma)}{\gamma}$. So $\gamma \neq 0$ is a zero of $D(\gamma)$ if and only if it is that of $H_1(\gamma)$. Since $\overline{H_1(\gamma)} = H_1(\overline{\gamma})$, its zeros are symmetrically with respect to the real axis. Note that $\widehat{\mu} > 0$ implies $H_1(ib) \neq 0, b \in \mathbb{R}$. Let the zeros of $H_1(\gamma)$ in the region $-\widehat{\mu}+\delta \leq \Re\gamma \leq 0$ be $\gamma_k, k = 1, 2, \cdots, m$. We can set

$$\omega_1 = \min_{1\leq k\leq m}|\Re\gamma_k|.$$

There is no zero of $H_1(\gamma)$ as $\Re s > -\omega_1$. Hence there is only one eigenvalue $\gamma_0 = 0$ of $\mathcal{A}$ in the region $\{\gamma \in \mathbb{C} \mid \Re\gamma > -\omega_1\}$. □

According to the finite expansion theorem of semigroups, we have the following result.

Theorem 11. *Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before, and let $T(t)$ be the semigroup generated by $\mathcal{A}$. Suppose that $\widehat{\mu} > 0$ and $0 < \omega_1 < |\Re\gamma_1|$. Then for any initial $P(0)$, we have*

$$\|P(t) - \langle P(0), Q\rangle\widehat{P}\| \leq 2e^{-\omega_1 t}\|P(0)\| \tag{29}$$

where $P(t) = T(t)P(0)$.

Proof: Since the Riesz spectral project corresponding to γ_0 is given by

$$E(\gamma_0,\mathcal{A})F=\frac{1}{2\pi i}\int_{|s|=\varepsilon}\mathcal{R}(s,\mathcal{A})Fds=\langle F,Q\rangle\widehat{P}_0$$

for $\forall F\in\mathbb{X}$. This leads to $\|E(\gamma_0,\mathcal{A})\|=1$. Since $T(t)$ is a dissipative semigroup in $(I-E(\gamma_0,\mathcal{A}))\mathbb{X}$, and the condition $\widehat{\mu}>0$ ensures that the resolvent $R(\gamma,\mathcal{A})F=(P_0,p_1(x),p_2(x),p_3(x)$ is bounded uniformly in the region $-\widehat{\mu}+\delta\le\Re\gamma\le 0$ when $|\gamma|$ is large enough. So we have

$$\begin{aligned}&\|P(t)-\langle P(0),Q\rangle\widehat{P}\|\\ =\;&\|T(t)(I-E(\gamma_0,\mathcal{A}))P(0)\|\\ \le\;& e^{-\omega_1 t}\|(I-E(\gamma_0,\mathcal{A}))P(0)\|\\ \le\;& 2e^{-\omega_1 t}\|P(0)\|.\end{aligned}$$

The result is derived. □

4.3 Special case

In this subsection we discuss the special case that $\mu_1(x)$, $\mu_2(x)$ are the constant functions. In this case, $G_3(s)=\frac{\mu_1}{(\gamma+\mu_1)}$,$G_4(\gamma)=\frac{\mu_1}{(\gamma+\mu_1+\lambda_2)}$ and $G_5(\gamma)=\frac{\mu_2}{(\gamma+\mu_2+\lambda_1)}$. Hence

$$D(\gamma)=\frac{\gamma H_2(\gamma)}{(\gamma+\mu_1)(\gamma+\mu_1+\lambda_2)(\gamma+\mu_2+\lambda_1)}$$

and

$$H_2(\gamma)=\gamma^3+a_2\gamma^2+a_1\gamma+a_0$$

where

$$\begin{aligned}a_0&=\mu_1\mu_2(\lambda_1+\lambda_2)+\lambda_1\lambda_2(\mu_1-\mu_2)-\lambda_1^2\lambda_2\\&\quad+\mu_1^2\mu_2+\varepsilon(\mu_1^2+\lambda_2\mu_1-\lambda_1\mu_1-\lambda_1\lambda_2),\\ a_1&=\mu_1^2+2\lambda_1\mu_1+\mu_2\lambda_1+\lambda_2\mu_1+\lambda_2\mu_2\\&\quad+2\mu_1\mu_2+2\varepsilon\mu_1+\varepsilon\lambda_2-\varepsilon\lambda_1,\\ a_2&=2\lambda_1+2\lambda_2+2\mu_1+\mu_2+\varepsilon,\end{aligned}$$

are real coefficients. Clearly, $H_2(\gamma)$ has three zeros.

5 Reliability analysis of the system

5.1 Some indices deriving from Steady-State

From Theorem 9, we obtain the steady state of the system. Based on this, some indices of reliability can be derived.

Theorem 12. *The normal steady-state availability of the system is*

$$A_v^N=\frac{1}{Z}\tag{30}$$

where Z is given by (18).

Proof: By the expression of steady state (18) and definition of the normal availability $A_v^N=P_0$, we know the result holds. □

According to the definition of abnormal availability

$$A_v^U=\int_0^\infty p_1(x)dx+\int_0^\infty p_2(x)dx,$$

we have

Theorem 13. *The abnormal steady-state availability of the system is*

$$\begin{aligned}A_v^U&=\int_0^\infty p_1(x)dx+\int_0^\infty p_2(x)dx\\&=\tfrac{\lambda_1}{Z}[\int_0^\infty e^{-\int_0^x(\mu_1(s)+\lambda_2)ds}dx\\&\quad+\int_0^\infty\tfrac{\varepsilon+\lambda_1\lambda_2G_4(0)}{1-\lambda_1G_5(0)}e^{-\int_0^x(\mu_2(s)+\lambda_1)ds}dx]\end{aligned}\tag{31}$$

where $G_4(0),G_5(0)$ is given by (10).

The steady-state availability of the system is

$$\begin{aligned}A_v&=A_v^N+A_v^U\\&=\tfrac{1}{Z}+\tfrac{\lambda_1}{Z}[\int_0^\infty e^{-\int_0^x(\mu_1(s)+\lambda_2)ds}dx\\&\quad+\int_0^\infty\tfrac{\varepsilon+\lambda_1\lambda_2G_4(0)}{1-\lambda_1G_5(0)}e^{-\int_0^x(\mu_2(s)+\lambda_1)ds}dx]\end{aligned}\tag{32}$$

Remark : From the expression of Z, we can know if the system has stronger repair rate $\mu_j(x)$, then $\int_0^\infty e^{-\int_0^x\mu_j(s)ds}dx$ has a smaller value, thus the normal steady-state availability of the system will be larger, the failure probability of the system $p_3(x)$ will be smaller, and system normal reliability be enhanced.

The average failure number of the system at time t is called the instantaneous failure frequency, denoted by $W_f(t)$, and its limit as $t\to\infty$ is called steady-state failure frequency of the system, denoted by W_f. From [2] and the model of the system, we know

$$W_f=\lim_{t\to\infty}p_3(t,0)=p_3(0)\tag{33}$$

We immediately have

Theorem 14. *The steady-state failure frequency of the system is*

$$W_f=\frac{\varepsilon\lambda_1G_5(0)+\lambda_1\lambda_2G_4(0)}{[1-\lambda_1G_5(0)]Z}.$$

5.2 The estimation of instantaneous availability

The instantaneous availability of the system is the probability of the system being in work, which is defined by $A_v(t)=P_0(t)+\int_0^\infty p_1(x,t)dx+\int_0^\infty p_2(x,t)dx$. From Theorem 11,

$$|A_v(t)-A_v|\;\le\;||P(t)-\widehat{P}||\le 2e^{-|\omega|t},$$

where A_v is given by (32), we have

$$\begin{aligned} A_v(t) &= A_v(t) - A_v + A_v \le |A_v(t) - A_v| + A_v \\ &\le ||P(t) - \widehat{P}|| + A_v \\ &\le 2e^{-|\omega|t} + A_v. \end{aligned}$$

Because when $t > \frac{3\ln 10 - \ln 5}{|\omega|}$, $||P(t) - \widehat{P}|| \le 0.01$. So, we have $A_v(t) \le 0.01 + A_v$.

Obviously, the failure probability of the system is $\int_0^\infty p_3(x,t)dx = 1 - A_v(t)$. It has an estimate$1 - A_v(t) \ge 0.99 - A_v$, when $t > \frac{3\ln 10 - \ln 5}{|\omega|}$.

6 Optimal repair rate for steady-state reliability

In this section, we shall consider a optimization problem for the repair rate $\mu(x) = (\mu_1(x), \mu_2(x))$. Suppose $\widehat{P}_0$ is the expectable probability of zero state in steady-state and $P_0(\mu)$ is the first component of solution of system corresponding to $\mu(x)$. Take the index functional

$$S(\mu) = |P_0(\mu)) - \widehat{P}_0|^2.$$

Our aim is to find $\mu(x) = (\mu_1(x), \mu_2(x))$ such that it minimizes $S(\mu)$.

According to the assumptions (3), we let the admissible set be U

$$U = \left\{ \begin{array}{c} (\mu_1(x), \mu_2(x)) \in [L^\infty(\mathbb{R}_+)]^3 \ \Big| \\ \frac{\ln(1+x)}{1+x} \le \mu_1(x), \mu_2(x) \le M \end{array} \right\}$$

Clearly U is a closed and convex set in $[L^\infty(\mathbb{R}_+)]^3$. Our object is to find $\mu^* \in U$ such that

$$S(\mu^*) = \inf_{\mu \in U} S(\mu). \tag{34}$$

μ^* is said to be optimal repair rate of the system.

To solve the problem (34), let

$$W = \left\{ \begin{array}{c} P_0 \in \mathbb{R} \ \big| \ \exists \mu \in U \text{ such that} \\ (P_0, p_1(x), p_2(x), p_3(x)) \in \mathbb{R} \times (L^1(\mathbb{R}_+))^3 \\ \text{is a nonnegative solution corresponding to } \mu(x) \\ P_0 + \int_0^\infty p_1(x)dx \\ + \int_0^\infty p_2(x)dx + \int_0^\infty p_3(x)dx = 1 \end{array} \right\}. \tag{35}$$

We firstly prove that there exists a $P_0^* \in W$ satisfying

$$S(P_0^*) = \inf_{P_0 \in W} S(P_0) = \inf_{P_0 \in W} |P_0 - \widehat{P}_0|^2. \tag{36}$$

Theorem 15. *W is a bounded and w^*-closed set in $\mathbb{R}$.*

Proof: From the previous section we know W is a bounded set. Now we prove that W is a closed set.

Let $P_0^{(n)} \in W$, and $P_0^{(n)} \to \widetilde{P}_0$. Then there exist a sequence $\mu^{(n)}(x) \in U$ such that $P^{(n)}(x)$ are the nonnegative solution to (1) with boundary conditions (2). Since $U \subset [L^\infty(\mathbb{R}_+)]^3$ is bounded and then w^*-sequence compact, there exist a subsequence of $\mu^{(n)}(x)$ without loss generality itself such that $\mu^{(n)}(x) \xrightarrow{w^*} \widetilde{\mu}(x) = (\widetilde{\mu_1}(x), \widetilde{\mu_2}(x))$. It is easy to see that $\widetilde{\mu_1}(x)$ and $\widetilde{\mu_2}(x)$ are nonnegative functions. For any $x \in \mathbb{R}_+$ fixed, the function $\chi_{[0,x]}(s) \in L^1(\mathbb{R}_+)$, the w^* convergence of $\mu^{(n)}$ deduces that

$$\begin{array}{c} \lim\limits_{n\to\infty} \int_0^\infty \chi_{[0,x]}(s)\mu_1^{(n)}(s)ds = \int_0^x \widetilde{\mu_1}(s)ds, \\ \lim\limits_{n\to\infty} \int_0^\infty \chi_{[0,x]}(s)\mu_2^{(n)}(s)ds = \int_0^x \widetilde{\mu_2}(s)ds. \end{array}$$

Since $(\mu_1^{(n)}, \mu_2^{(n)}) \in U$ implies that

$$\begin{aligned} \frac{1}{2}[\ln(1+x)]^2 &\le \int_0^x \mu_1^{(n)}(s)ds, \int_0^x \mu_2^{(n)}(s)ds \\ &\le Mx, \end{aligned}$$

so it holds that

$$\frac{1}{2}[\ln(1+x)]^2 \le \int_0^x \widetilde{\mu_1}(s)ds, \int_0^x \widetilde{\mu_2}(s)ds \le Mx.$$

Therefore, $\widetilde{\mu}(x) \in U$. On the other hands, from (15) we get that

$$\left\{ \begin{array}{l} p_1^{(n)}(x) = \lambda_1 P_0^{(n)} e^{-\int_0^x (\mu_1^{(n)}(s) + \lambda_2)ds}, \\ p_2^{(n)}(x) = P_0^{(n)} \frac{\varepsilon + \lambda_1\lambda_2 G_4(0)}{1 - \lambda_1 G_5(0)} e^{-\int_0^x (\mu_2^{(n)}(s) + \lambda_1)ds}, \\ p_3^{(n)}(x) = P_0^{(n)} \frac{\varepsilon\lambda_1 G_5^{(n)}(0) + \lambda_1\lambda_2 G_4^{(n)}(0)}{1 - \lambda_1 G_5^{(n)}(0)} e^{-\int_0^x \mu_1^{(n)}(s)ds}. \end{array} \right. \tag{37}$$

The convergence of $P_0^{(n)}$ implies that for each $x \in \mathbb{R}_+$, $p_1^{(n)}(x)$, $p_2^{(n)}(x)$ and $p_3^{(n)}(x)$ are convergent and the limit are respectively

$$\begin{array}{l} \widetilde{p}_1(x) = \lambda_1 e^{-\int_0^x (\widetilde{\mu_1}(s) + \lambda_2)ds} \widetilde{P}_0, \\ \widetilde{p}_2(x) = \frac{\varepsilon + \lambda_1\lambda_2 G_4(0)}{1 - \lambda_1 G_5(0)} e^{-\int_0^x (\widetilde{\mu_2}(s) + \lambda_1)ds} \widetilde{P}_0, \\ \widetilde{p}_3(x) = \frac{\varepsilon\lambda_1 G_5(0) + \lambda_1\lambda_2 G_4(0)}{1 - \lambda_1 G_5(0)} e^{-\int_0^x \widetilde{\mu_1}(s)ds} \widetilde{P}_0. \end{array}$$

Clearly, it holds that

$$\begin{aligned} \widetilde{p}_1(0) &= \lambda_1 \widetilde{P}_0 \\ \widetilde{p}_2(0) &= \varepsilon \widetilde{P}_0 + \int_0^\infty \widetilde{\mu}_1(x)\widetilde{p}_3(x)dx, \\ \widetilde{p}_3(0) &= \lambda_1 \int_0^\infty \widetilde{p}_2(x,t)dx + \lambda_2 \int_0^\infty \widetilde{p}_1(x)dx. \end{aligned}$$

From above expression we see that $\widetilde{p}_1(x)$, $\widetilde{p}_2(x)$ and $\widetilde{p}_3(x)$ are absolute continuous and $\widetilde{P_0}, \widetilde{p_1}, \widetilde{p_2}, \widetilde{p}_3(x)$, satisfy the differential equations in (1).

Finally, we prove that the integral $\int_0^\infty e^{-\int_0^x \widetilde{\mu_1}(s)ds}dx$ is finite. Note that the relation $\frac{1}{2}[\ln(1+x)]^2 \le \int_0^x \mu_1(s)ds \le Mx$, we have

$$\int_0^\infty e^{-\int_0^x \mu_1^{(n)}(s)ds}dx \le \int_0^\infty e^{-\frac{1}{2}[\ln(1+x)]^2}dx < \infty.$$

The Fatou lemma asserts that

$$\int_0^\infty e^{-\int_0^x \widetilde{\mu_1}(s)ds}dx \le \int_0^\infty e^{-\frac{1}{2}[\ln(1+x)]^2}dx.$$

Thus,

$$\begin{aligned} Z &= 1+\lambda_1\int_0^\infty e^{-\int_0^x(\widetilde{\mu_1}(s)+\lambda_2)ds} \\ &+\frac{\varepsilon+\lambda_1\lambda_2G_4(0)}{1-\lambda_1G_5(0)}\int_0^\infty e^{-\int_0^x(\widetilde{\mu_2}(s)+\lambda_1)ds} \\ &+\frac{\varepsilon\lambda_1G_5(0)+\lambda_1\lambda_2G_4(0)}{1-\lambda_1G_5(0)}\int_0^\infty e^{-\int_0^x\widetilde{\mu_1}(s)ds} \end{aligned} \tag{38}$$

has meaning. Therefore, $(\widetilde{P}_0, \widetilde{p}_1(x), \widetilde{p}_2(x), \widetilde{p}_3(x))$ is a nonnegative solution of (1) corresponding to $\widetilde{\mu}(x) = (\widetilde{\mu_1}(x), \widetilde{\mu_2}(x))$, and satisfies condition

$$\widetilde{P}_0(\widetilde{\mu}) + \sum_{j=1}^{3}\int_0^\infty \widetilde{p}_j(x)dx = 1.$$

So $\widetilde{P}_0 \in W$, and W is a closed set. □

Theorem 16. *W is a convex set, and $S(P_0)$ is a strictly convex functional on W.*

Proof: Let $P_0^{(1)}, P_0^{(2)} \in W$ and $P_0^{(1)} \ne P_0^{(2)}$. By the definition of W there exist $\mu^{(i)} = (\mu_1^{(i)}, \mu_2^{(i)}), i = 1, 2$ corresponding to $P_0^{(i)}$. For any $0 < \tau < 1$, we set $P^{(i)} = (P_0^{(i)}, p_1^{(i)}(x), p_2^{(i)}(x), p_3^{(i)}(x))$ and

$$P_\tau(x) = \tau P^{(1)}(x) + (1-\tau)P^{(2)}(x).$$

A direct verification shows that P_τ satisfy the differential equation (1) corresponding to $\mu_\tau = \tau\mu^{(1)} + (1-\tau)\mu^{(2)}$. So W is a convex set. Further, we have

$$\begin{aligned} & S(\tau P_0^{(1)} + (1-\tau)P_0^{(2)}) \\ = \ & |\tau P_0^{(1)} + (1-\tau)P_0^{(2)} - \widehat{P_0}|^2 \\ < \ & \tau|P_0^{(1)} - \widehat{P_0}|^2 + (1-\tau)|P_0^{(2)} - \widehat{P_0}|^2 \\ = \ & \tau S(P_0^{(1)}) + (1-\tau)S(P_0^{(2)}). \end{aligned}$$

Therefore, $S(P_0)$ is a strictly convex functional in W. □

Theorem 17. *There exists a unique $P_0^* \in W$ such that*

$$S(P_0^*) = \inf_{p_0\in W} S(P_0).$$

Proof: Since $W \subset \mathbb{R}_+$ is a bounded and closed set, the existence and uniqueness of $P_0^* \in W$ follows from the theory of the convex function on convex set. □

Theorem 18. *If $\widehat{P}_0 \notin W$, then there exists a unique $\mu^* \in U$ such that*

$$S(\mu^*) = \inf_{\mu\in U} S(\mu).$$

Proof: Note that $S(P_0) = |P_0 - \widehat{P}_0|^2$, $S(\mu) = |P_0(\mu) - \widehat{P}_0|^2$ and $P_0(\mu) = \frac{1}{Z(\mu)}$ where

$$\begin{aligned} Z &= 1+\lambda_1\int_0^\infty e^{-\int_0^x(\mu_1(s)+\lambda_2)ds} \\ &+\frac{\varepsilon+\lambda_1\lambda_2G_4(0)}{1-\lambda_1G_5(0)}\int_0^\infty e^{-\int_0^x(\mu_2(s)+\lambda_1)ds} \\ &+\frac{\varepsilon\lambda_1G_5(0)+\lambda_1\lambda_2G_4(0)}{1-\lambda_1G_5(0)}\int_0^\infty e^{-\int_0^x\mu_1(s)ds} \end{aligned} \tag{39}$$

where $G_4(0) = \int_0^\infty e^{-\int_0^x(\mu_1(s)+\lambda_2)ds}dx$, $G_5(0) = \int_0^\infty e^{-\int_0^x(\mu_2(s)+\lambda_1)ds}dx$. Obviously, $Z(\mu)$ depends continuously on $(\mu_1, \mu_2) \in U$. In particular, it holds that

$$\begin{aligned} & Z(\mu) \ge 1 + \frac{\lambda_1}{M+\lambda_2} \\ & + \frac{\varepsilon(M+\lambda_2)+\lambda_1\lambda_2}{M(M+\lambda_2)} + \frac{\varepsilon\lambda_1(M+\lambda_2)+\lambda_1\lambda_2(M+\lambda_1)}{M^2((M+\lambda_2)} \end{aligned} \tag{40}$$

where M is defined as in U. Therefore, $\forall\mu \in U$,

$$\begin{aligned} & P_0(\mu) = \frac{1}{Z(\mu)} \\ & \le \frac{M^2(M+\lambda_2)}{M^2((M+\lambda_2)+\lambda_1M^2+\varepsilon(M+\lambda_1)(M+\lambda_2)+\lambda_1\lambda_2M+\lambda_1\lambda_2(M+\lambda_1)}. \end{aligned}$$

Therefore, when $\widehat{P}_0 \notin W$, it must have

$$\begin{aligned} & \widehat{P}_0 > \\ & \frac{M^2(M+\lambda_2)}{M^2((M+\lambda_2)+\lambda_1M^2+\varepsilon(M+\lambda_1)(M+\lambda_2)+\lambda_1\lambda_2M+\lambda_1\lambda_2(M+\lambda_1)}. \end{aligned}$$

so $S(\mu)$ arrives minimum at $\mu^* = M(1, 1)$. □

Acknowledgements: The research is supported by the National Science Natural Foundation in China (NSFC-60874034).

References:

[1] K. C. Who, A Reparable Multistate Device With Arbitrarily Distributed Repair Times, *Microelectronics and Reliability*, 21(1981),2, pp.255–256.

[2] J. H. Cao and K. Cheng, *Introduction to Reliability Mathematics*, Science Press, Beijing, China, 1986.

[3] R. B. Liu, Y. H. Tang and C. Y. Luo, A new kind of N-unit series repairable system and its reliability analysis. *Applied Mathematical Modelling*, 20 (2007), 1,pp.164–170.

[4] W. L. Wang, and G. Q. Xu, Stability analysis of a complex standby system with constant waiting and different repairman criteria incorporating environmental failure, *Applied Mathematical Modelling*, 33 (2009), pp.724–743.

[5] L. N. Guo, H. B. Xu, C. Gao and G. T. Zhu, Stability analysis of a new kind series system. *IMA Journal of Applied Mathematics*, 75(2010), pp.439–460.

[6] Z. F. Shen et al. Exponential asymptotic property of a parallel repairable system with warm standby under common-cause failure. *Mathematical Analysis and applications*, 341(2008), 1, pp.457-466.

[7] A. Pazy, *Semigroup of linear operators and application to partial differential equations*, Springer-Verlag, Berlin, New York,1982.

[8] Z. Y. Lin and G. J. Wang, A geometric process repair model for a repairable cold standby system with priority in use and repair. *Reliability Engineering and System Safety*, 94(2009), pp.1782–1787.

[9] K. J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Math, Vol.194, Springer-Verlag, 2000.

[10] G. Gupur, X. Z. Li and G. T. Zhu, *Functional Analysis Method in Queuing Theory*, Hertfordshire, Research Information Ltd, United Kingdom, 2001.

[11] R. Nagel, *One-Parameter Semigroup of Positive Operators*, Lecture Notes in Mathematics, Springer, New York, 1986.