β_0-excellent graphs

A. P. Pushpalatha, G. Jothilakshmi
Thiagarajar College of Engineering
Madurai - 625 015
India
Email: gjlmat@tce.edu, appmat@tce.edu

S. Suganthi, V. Swaminathan
Ramanujan Research Centre
Saraswathi Narayanan College
Madurai - 625 022
India
sulnesri@yahoo.com

Abstract: Claude Berge [1] in 1980, introduced B graphs. These are graphs in which every vertex in the graph is contained in a maximum independent set of the graph. Fircke et al [3] in 2002 made a beginning of the study of graphs which are excellent with respect to various graph parameters. For example, a graph is domination excellent if every vertex is contained in a minimum dominating set. The B-graph of Berge was called β_0 excellent graph. β_0 excellent trees were characterized [3]. A graph is just β_0 excellent if every vertex belongs to exactly one maximum independent set of the graph.This paper is devoted to the study of β_0 excellent graphs and just β_0 excellent graphs.

Key–Words: β_0-excellent and just β_0 excellent, Harary graphs, Generalized Petersen graph

1 Introduction

Let μ be a parameter and let $G = (V, E)$ be simple graph. A vertex $v \in V(G)$ is said to be μ-good if v belongs to a μ-minimum (μ-maximum) set of G according as μ is a super hereditary (hereditary) parameter. v is said to be μ-bad if it is not μ-good. A graph G is said to be μ-excellent if every vertex of G is μ-good. G is μ-commendable if number of μ-good vertices in G is strictly greater than the number μ-bad vertices of G and there should be at least one μ-bad vertex in G. G is said to be μ-fair if number of μ-good vertices in G is equal to the number of μ-bad vertices in G and G is said to be μ-poor if number of μ-bad vertices in G is strictly greater than the number of μ-good vertices in G.

γ-excellent trees and total domination excellent trees have been studied in [3], [8]. β_0-excellent trees was also dealt with in some of the theorems in [3]. Continuing the study on γ-excellent graphs, N.Sridharan and Yamuna [4, 5, 6] , made an extensive work in this area. They have defined excellent, very excellent, just total excellent, rigid very excellent graphs with respect to the domination parameter and made a substantial contribution in this area.

This paper starts with the definition of β_0-excellent graphs. In the first section, general results on β_0- excellent graphs are proved.The second section is devoted to β_0-excellence in Cartesian Product of graphs. The third section deals with β_0-excellence of Harary graphs. The fourth section is devoted to the study of just β_0-excellent graphs.

Definition 1.1. ***Double star*** *is a graph obtained by taking two stars and joining the vertices of maximum degrees with an edge. If the stars are $K_{1,r}$ and $K_{1,s}$, then the double star is denoted by $D_{r,s}$.*

Definition 1.2. *A fan F_n is defined as the graph join $P_{n-1} + K_1$,where $n \geq 3$ and P_{n-1} is the path graph on $n - 1$ vertices.*

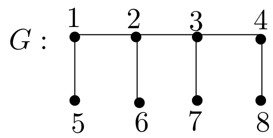
2 β_0-excellent graphs

Definition 2.1. *Let $G = (V, E)$ be a simple graph. Let $u \in V(G)$. u is said to be β_0-good if u is contained in a β_0-set of G.*

Definition 2.2. *u is said to be β_0-bad if there exists no β_0-set of G containing u.*

Definition 2.3. *A graph G is said to be β_0-excellent if every vertex of G is β_0-good.*

Example 2.4.



The β_0-sets of G are $\{1, 3, 6, 8\}$, $\{5, 6, 7, 8\}$, $\{2, 4, 5, 7\}$. Hence all the vertices are β_0-good. Hence G is β_0-excellent.

Theorem 2.5. *(1) K_n is β_0-excellent.*

(2)The central vertex of $K_{1,n}$ is β_0-bad and every other vertex is β_0-good.

(3) C_n is β_0-excellent.

(4) P_n is β_0-excellent if and only if n is even.

(5) In a Double star $D_{r,s}$, all the pendent vertices are β_0-good but the two supporting vertices are β_0-bad. Hence $D_{r,s}$ is not a β_0-excellent graph.

(6) $K_{m,n}$ is β_0-excellent if and only if $m = n$.

(7) In W_n, the central vertex is β_0-bad, while other vertices are β_0-good.

(8) $\overline{K_n}$ is a β_0-excellent graph.

(9) $F_n, n \geq 3$ is not β_0-excellent.

Remark 2.6. *Suppose G has a unique β_0-set. Then G is β_0-excellent if and only if $G = \overline{K_n}$.*

Remark 2.7. *If G has a full degree vertex and if $G \neq K_n$, then G is not β_0-excellent.*

Theorem 2.8. *For any graph G, $G \circ K_1$ is β_0-excellent.*

Definition 2.9. *A graph is said to be β_0-fair(β_0-poor) graph if the number of β_0-good vertices is greater than(less than) the number of β_0-bad vertices.*

Example 2.10. *Let G be the graph obtained from $K_{1,3}$ by subdividing all pendent edges exactly once. Then G is β_0-fair.*

Example 2.11. *In $G = K_4 - \{e\}$, exactly two vertices are β_0-good and remaining vertices are β_0-bad. If $n \geq 5$, then $G = K_n - \{e\}$ is β_0-poor, since the number of β_0-bad vertices is greater than number of β_0-good vertices.*

Theorem 2.12. *Every non β_0- excellent graph can be embedded in a β_0-excellent graph.*

If G is a non β_0-excellent graph, then $G \circ K_1$ is a β_0-excellent graph in which G is embedded.

Remark 2.13. *Suppose $G = K_{n+1}$. Then $\beta_0(G \circ K_1) - \beta_0(G) = n$, which means the difference between the independence number of the graph, in which the given graph is embedded and the given graph is large.*

Definition 2.14. *A graph G is said to be vertex transitive if given any two vertices $u, v (u \neq v)$ of G, there is an automorphism ϕ of G such that $\phi(u) = v$. If G is vertex transitive, then it is regular.*

Theorem 2.15. *Any vertex transitive graph is β_0-excellent.*

Proof. Let G be a vertex transitive graph. Let S be a β_0-set of G. Let $u \in V(G)$. Suppose $u \notin S$. Select any vertex $v \in S$. As G is vertex transitive, there exists an automorphism ϕ of G which maps v to u. Let $S' = \{\phi(w) : w \in S\}$. Since S is a β_0-set and ϕ is an automorphism, S' is a β_0-set. Since $v \in S$, $\phi(v) = u \in S'$. Therefore G is β_0-excellent.

Theorem 2.16. *Let G be a non β_0-excellent graph. Then there exists a graph H in which the following conditions are true.*

(i) H is β_0-excellent.

(ii) G is an induced subgraph of H.

(iii) $\beta_0(H) = \beta_0(G)$.

Proof. Let G be a non -β_0-excellent graph. Let $B = \{b_1, b_2, \ldots, b_k\}$ be the set of all β_0-bad vertices of G. Let $V_1, V_2, \ldots, V_k$ be a set of independent sets of maximum cardinalities containing $b_1, b_2, \ldots, b_k$ respectively.

Let $|V_i| = t_i$, $1 \leq i \leq k$. Then $t_i < \beta_0(G)$, for all i, $1 \leq i \leq k$. Let $W_i = \left\{w_{i_1}, w_{i_2}, \ldots, w_{i_{\beta_0 - t_i}}\right\}$, $1 \leq i \leq k$. Add each element of W_i, $1 \leq i \leq k$ as a vertex to the vertex set of G. Let the new sets of vertices $W_1, W_2, \ldots, , W_k$ be made a complete k-partite graph. Join each vertex of W_i with every vertex of $V - V_i$, $1 \leq i \leq k$. Let H be the resulting graph. Then $V_i \cup W_i$ is an independent set of H of cardinality β_0. Any β_0-set of G continues to be an independent set of H of cardinality β_0. There is no other independent set of H of cardinality greater than β_0. Therefore $\beta_0(H) = \beta_0(G)$. Each new vertex added to G and each b_i is contained in a maximum independent set of H. Therefore H is a β_0-excellent graph. Clearly, G is an induced subgraph of H and $\beta_0(H) = \beta_0(G)$.

Theorem 2.17. *Let G, H be β_0-excellent graphs with $V(G) \cap V(H) = \phi$. Then*

(i) $G \cup H$ is β_0-excellent.

(ii) $G + H$ is β_0-excellent if and only if $\beta_0(G) = \beta_0(H)$.

Proof. (i) Any β_0-set of $G \cup H$ is of the form $S_1 \cup S_2$, where S_1 is a β_0-set of G and S_2 is a β_0-set of H. Hence $G \cup H$ is β_0-excellent.

(ii) Let $\beta_0(G) < \beta_0(H)$. Then any β_0-set of $G + H$ is a β_0-set of H and conversely. If $\beta_0(G) = \beta_0(H)$, then any β_0-set of G and any β_0-set of H are β_0-sets of $G + H$ and conversely. Therefore $G + H$ is β_0-excellent if and only if $\beta_0(G) = \beta_0(H)$.

Definition 2.18. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs Then their **Cartesian Product** $G_1 \Box G_2$ is defined to be the graph whose vertex set is $V_1 \Box V_2$ and edge set is $\{((u_1, v_1), (u_2, v_2))$: either $u_1 = u_2$ and $v_1 v_2 \in E_2$ or $v_1 = v_2$ and $u_1 u_2 \in E_1\}$.

Theorem 2.19. *Let H be a graph.*

(i) Let $n \geq \chi(H)$. Then $\beta_0(K_n \Box H) = |V(H)|$ and $K_n \Box H$ is β_0-excellent.

(ii) Let $n < \chi(H)$. Then $\beta_0(K_n \Box H) = t$, where t is the maximum cardinality of an union of n-disjoint independent sets in H.

Proof. (i) Let $n \geq \chi(H)$. Let $\prod = \{V_1, V_2, \ldots, V_{\chi(H)}\}$ be a chromatic partition of H. Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. Then $S = \{(u_1, v) : v \in V_1\} \cup \{(u_2, v) : v \in V_2\} \cup \ldots \cup \{(u_{\chi(G)}, v) : v_{\chi(G)} \in V_{\chi(G)}\}$ is an independent set of $K_n \square H$. Therefore $\beta_0(K_n \square H) \geq |V(H)|$. But $\beta_0(K_n \square H) \leq \beta_0(K_n)|V(H)| = |V(H)|$. Therefore $\beta_0(K_n \square H) = |V(H)|$. Any set of χ-vertices of K_n will produce a β_0-set of $K_n \square H$. Hence $K_n \square H$ is β_0-excellent.

(ii) Let $n < \chi(H)$. Let $S_1, S_2, \ldots, S_n$ be disjoint independent sets in H such that $\sum_{i=1}^{n} |S_i|$ is maximum. Let $t = \sum_{i=1}^{n} |S_i|$. Then $T = \{(u_1, v) : v \in S_1\} \cup \{(u_2, v) : v \in S_2\} \cup \ldots \cup \{(u_n, v) : v \in S_n\}$ is an independent set of $K_n \square H$. Therefore $\beta_0(K_n \square H) \geq \sum_{i=1}^{n} |S_i| = |T| = t$. Let S be a maximum independent set of $K_n \square H$. Let $X_i = S \cap (\{u_i\} \times V(H)), 1 \leq i \leq n$. Let $Y_i = \{v \in V(H) : (u_i, v) \in X_i, 1 \leq i \leq n)$. Then Y_i's are independent and disjoint in H. $|S| = \sum_{i=1}^{n} |X_i| = \sum_{i=1}^{n} |Y_i| \leq \sum_{i=1}^{n} |S_i| = |T|$. Therefore $t = |T| \geq \beta_0(K_n \square H) = |T| = t$.

Illustration 2.20. *Let H be $K_{5,3,5,2}$. Then $K_3 \square H$ is not β_0-excellent. (Here $\chi(H) = 4 > 3$).*

Theorem 2.21. *$K_n \square H$ is β_0-excellent if and only if every vertex of H belongs to the union of disjoint independent sets of H of maximum cardinality.*

Proof. Suppose every vertex of H belongs to the union of disjoint independent sets of H of maximum cardinality. Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. $(u_i, v) \in V(K_n \square H), 1 \leq i \leq n$. Then $v \in V(H)$. Then there exist disjoint independent sets $S_1, S_2, \ldots, S_n$ of H such that $\sum_{i=1}^{n} |S_i| = t$ is maximum and $v \in S_j$, for some $j, 1 \leq j \leq n$.

Then $T = \{(u_1, v) : v \in S_1\} \cup \{(u_2, v) : v \in S_2\} \ldots \cup \{(u_i, v) : v \in S_j\} \cup \{(u_j, v) : v \in S_i\} \ldots \cup \{(u_n, v) : v \in S_n\}$ is a maximum independent set of $K_n \square H$ containing (u_i, v). Therefore $K_n \square H$ is β_0-excellent.

Conversely, Suppose every vertex of H belongs to the union of disjoint independent sets of H of maximum cardinality. Then there exists a vertex $v \in H$ such that v does not belong to any union of n disjoint independent sets of H of maximum cardinality. Since any maximum independent set of $K_n \square H$ is obtained from n disjoint independent sets of H, with the union having maximum cardinality, (u_i, v), $1 \leq i \leq n$ will not belong to any maximum independent set of $K_n \square H$. Therefore $K_n \square H$ is not β_0-excellent.

Theorem 2.22. *Let H be a graph. Then $\overline{K_n} \square H$ is β_0-excellent if and only if H is β_0-excellent.*

Proof. Suppose H is β_0-excellent. Then $\beta_0(\overline{K_n} \square H) = n.\beta_0(H)$. Any β_0-set of H gives rise to a β_0-set of $\overline{K_n} \square H$. Therefore $\overline{K_n} \square H$ is β_0-excellent. Suppose H is not β_0-excellent. Let $u \in V(H)$ be such that u is not contained in any β_0-set of H. Suppose S is a β_0-set of $\overline{K_n} \square H$ containing (v, u), for some $v \in V(\overline{K_n})$. Therefore $|S| = n.\beta_0(H)$. Also S is of the form $V(G) \times T$, where T is a β_0-set of H. Therefore $u \in T$, a contradiction.

Theorem 2.23. *Let $G \neq \overline{K_n}$ and let G be a β_0-excellent graph. Let $H = P_{2n}$. Then $G \square H$ is β_0-excellent if (i) or (ii)is satisfied. $G \square H$ is not β_0-excellent if (iii) is satisfied.*

(i) For any β_0-set S of G, there exists a β_0-set of G in $V - S$

(ii) Let the cardinality of the union of any two disjoint non-maximum independent set of $G \leq |S| + \beta_0(< V - S >)$, for any β_0-set S of G. For every β_0-set S of G, $V - S$ does not contain β_0-set of G and for any β_0-set S of G, the maximum number of independent elements in $V - S$ is the same.

(iii) If any two β_0-sets of G are not disjoint and there exists a β_0-set S of G such that the maximum number of independent elements in $V - S$ is greater than the maximum number of independent elements in the complement of any other β_0-set, then $G \square H$ is not β_0-excellent.

Proof. (i) Let G have two disjoint β_0-sets. Then $\beta_0(G \square P_{2n}) = 2n\beta_0(G)$. For: Let S_1, S_2 be two disjoint β_0-sets of G. Let $V(P_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$. $\{(x_i, v_i) : x_i \in S_1\} \cup \{(y_i, v_{i+1}) : y_i \in S_2\}$ is an independent set in $G \square P_{2n}$. Thus $\{(x_i, v_1) : x_i \in S_1\} \cup \{(y_i, v_2) : y_i \in S_2\} \cup \{(x_i, v_3) : x_i \in S_1\} \cup \{(y_i, v_4) : y_i \in S_2\} \cup \ldots \cup \{(x_i, v_{2n-1}) : x_i \in S_1\} \cup \{(y_i, v_{2n}) : y_i \in S_2\}$ is an independent set of $G \square P_{2n}$. Therefore $\beta_0(G \square P_{2n}) \geq 2n\beta_0(G)$.

But $\beta_0(G \square P_{2n}) \leq \beta_0(G)|V(P_{2n})| = 2n\beta_0(G)$. Hence $\beta_0(G \square P_{2n}) = 2n\beta_0(G)$. Let $(x, y) \in V(G \square P_{2n})$. Then there exists a β_0-set S_1 of G containing x. Also by hypothesis, $V - S_1$ contains a β_0-set of G,say S_2. $\bigcup_{t=1}^{n} (S_1 \times \{v_{2t-1}\}) \cup \bigcup_{t=1}^{n} (S_2 \times \{v_{2t}\})$ and $\bigcup_{t=1}^{n} (S_2 \times \{v_{2t-1}\}) \cup \bigcup_{t=1}^{n} (S_1 \times \{v_{2t}\})$ are β_0-sets of $G\Box P_{2n}$. Hence there exists a β_0- set of $G\Box P_{2n}$ containing (x, y). Therefore $G\Box P_{2n}$ is β_0-excellent.

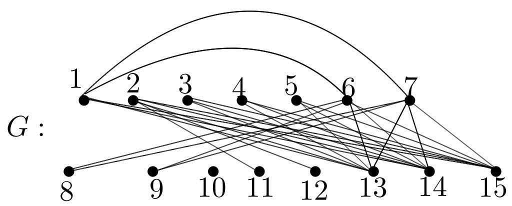
(ii) It can be easily proved that $\beta_0(G\Box P_{2n}) = (\beta_0(G) + k)n$, where k is the maximum number of independent elements in the complement of any β_0-set of G. In this case, $G\Box P_{2n}$ is β_0-excellent.[For: Let (x, y) be any element of $V(G\Box P_{2n})$. Then there exists a β_0-set S_1 of G containing x. Also $V - S_1$ contains an independent set of cardinality k. Let S_2 be a maximum independent set in $V - S_1$. $\bigcup_{t=1}^{n}(S_1 \times \{v_{2t-1}\}) \cup \bigcup_{t=1}^{n}(S_2 \times \{v_{2t}\})$ and $\bigcup_{t=1}^{n}(S_2 \times \{v_{2t-1}\}) \cup \bigcup_{t=1}^{n}(S_1 \times \{v_{2t}\})$ are the β_0-elements of $G\Box P_{2n}$. Hence there exists a β_0-set of $G\Box P_{2n}$ containing (x, y). Therefore $G\Box P_{2n}$ is β_0-excellent.

(iii) Suppose there exists a β_0-set S_1 of G such that the maximum number of independent elements say k in $V - S_1$ is greater than the maximum number of independent elements in the complement of any other β_0-set of G.

$\beta_0(G\Box P_{2n}) = (\beta_0(G) + k)n$. Let $u \in V(G) - S_1$. Then there exists a β_0-set S_2 of G containing u. The maximum number of independent elements in $V(G) - S_2$ is less than k. Therefore (u, v), where $v \in V(P_{2n})$ is not contained in any β_0-set of $G\Box P_{2n}$. Hence $G\Box P_{2n}$ is not β_0-excellent.

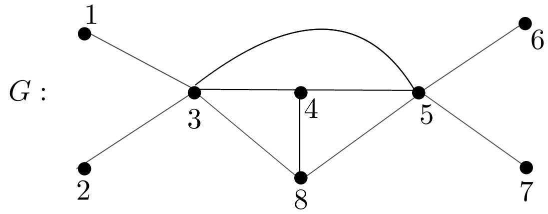
Remark 2.24. *There exist graphs in which the maximum number of independent elements in the complement of any β_0-set of G is greater than the maximum number of independent elements in the complement of any other β_0-set of G.*

Example 2.25.



The β_0-sets of G are $S_1 = \{1, 2, 3, 4, 5, 8, 9, 10\}$, $S_2 = \{3, 4, 5, 6, 7, 10, 11, 12\}$, $S_3 = \{8, 9, 10, 11, 12, 13, 14, 15\}$. Then $V - S_1 = \{6, 7, 11, 12, 13, 14, 15\}$, $V - S_2 = \{1, 2, 8, 9, 13, 14, 15\}$, $V - S_3 = \{1, 2, 3, 4, 5, 6, 7\}$. The set $\{11, 12, 13, 14, 15\}$ is a β_0-set in $V - S_1$; $\{8, 9, 13, 14, 15\}$ is a β_0-set in $V - S_2$ and $\{2, 3, 4, 5, 6, 7\}$ is a β_0-set in $V - S_3$. Hence S_3 satisfies the property described in the remark 2.24

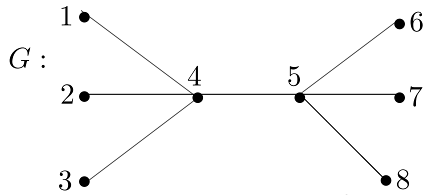
Example 2.26.



The β_0-sets of G are $S_1 = \{1, 2, 4, 6, 7\}, S_2 = \{1, 2, 8, 6, 7\}$. Clearly G is not β_0 - excellent. The maximum number of independent sets in $V - S_1$ and in $V - S_2$ is one. The sets $S_3 = \{1, 2, 4\}$ and $S_4 = \{8, 6, 7\}$ are not β_0-sets. The maximum number of independent sets in $V - (S_3 \cup S_4)$ is one. That is, there exist two disjoint independent sets of cardinality 3 each and the maximum number of independent elements in complement of their union is one.

$|S_3| + |S_4| + \beta_0(V - (S_3 \cup S_4)) = 7 > |S_1| + \beta_0(V - S_1) = |S_2| + \beta_0(V - S_2) = 6$.

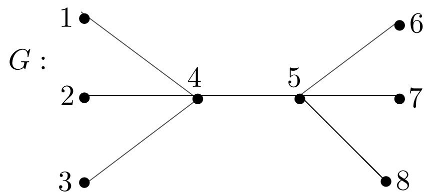
Example 2.27.



The β_0-set of G is $S = \{1, 2, 3, 6, 7, 8\}$. The 4-element disjoint independent sets are $\{1, 2, 3, 5\}$, $\{4, 6, 7, 8\}$. $\beta_0(G\Box P_{2n}) = 8n$. It can be shown that there is a β_0-set of G and a set of maximum number of elements in the complement, such that independent set generated contains $7n$ elements.

Though G is not $\beta_0$excellent, $G\Box P_{2n}$ is $\beta_0$excellent here. For:

Example 2.28.



Let $V(P_{2n}) = \{u_1, u_2, \ldots, u_{2n}\}$. Let $A = \{1, 2, 3, 5\}$ and $B = \{4, 6, 7, 8\}$. Two maximum independent sets of $G\Box P_{2n}$ are

$$\bigcup_{i\equiv 1(mod 2), 1\le i\le 2n} \{(u_i, 1), (u_i, 2), (u_i, 3), (u_i, 5)\}$$

$$\cup \bigcup_{j\equiv 0(mod\ \ 2), 2\le j\le 2n} \{(u_j, 4), (u_j, 6), (u_j, 7), (u_j, 8)\}$$

and

$$\bigcup_{i\equiv 1(mod 2), 1\le i\le 2n} \{(u_i, 4), (u_i, 6), (u_i, 7), (u_i, 8)\}$$

$\bigcup \quad \bigcup_{j\equiv 0(mod\ \ 2), 2\leq j\leq 2n} \{(u_j, 1), (u_j, 2), (u_j, 3), (u_j, 5)\}$.
Hence $G\Box P_{2n}$ is β_0-excellent, but G is not β_0-excellent.

Remark 2.29. *Suppose G is a graph in which $V(G) = A \cup B$, where A, B are independent and disjoint subsets are $V(G)$. Then $G\Box P_{2n}$ is β_0-excellent. (or) equivalently if G is bipartite graph, then $G\Box P_{2n}$ is β_0-excellent. Hence $T\Box P_{2n}$ is β_0-excellent, for any tree T and $C_{2n}\Box P_{2n}$ is β_0-excellent.*

Theorem 2.30. *Suppose G is of even order in which $V(G) = A \cup B$, $A \cap B = \phi$, A, B are independent and $|A| = |B|$. Then $G\Box P_{2n+1}$ is β_0-excellent.*

Proof. The following are β_0-sets of $G\Box P_{2n+1}$.

$$\bigcup_{i\equiv 1(mod\ \ 2),\ 1\leq i\leq 2n+1} P \bigcup \bigcup_{j\equiv 0(mod\ \ 2),\ 1\leq j\leq 2n} Q$$

and

$$\bigcup_{i\equiv 1(mod\ \ 2),\ 1\leq i\leq 2n+1} R \bigcup \bigcup_{j\equiv 0(mod\ \ 2),\ 1\leq j\leq 2n} S,$$

where $P = \{(v, u_i) : v \in A)\}$,
$Q = \{(v, u_j) : v \in B)\}, R = \{(v, u_i) : v \in B)\}$ and $S = \{(v, u_j) : v \in A)\}$.

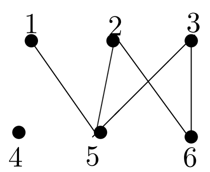
Hence $G\Box P_{2n+1}$ is β_0-excellent.

Corollary 2.31. *If G is bipartite graph with equi-cardinal bipartition, then $G\Box P_{2n+1}$ is β_0-excellent.*

Corollary 2.32. *(1) $C_n\Box P_{2n+1}$ is β_0-excellent. (2) If T is a tree with equi-cardinal bipartition, then $T\Box P_{2n+1}$ is β_0-excellent.*

Example 2.33.

G :

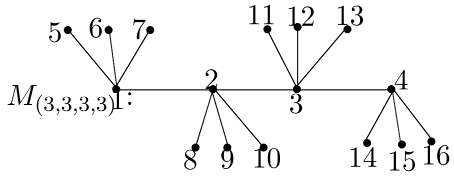


The graph G is of even order in which $V(G) = A \cup B$, $A \cap B = \phi$, A, B are independent and $|A| = |B|$, where $A = \{1, 2, 3\}, B = \{4, 5, 6\}$. Here G is not β_0-excellent (since $\beta_0(G) = 4$ and $\{1, 2, 3, 4\}$ is the unique β_0-set of G).

Example 2.34. *$D_{r,r}$ is a graph of even order in which $V(G) = A \cup B$, $A \cap B = \phi$, A, B are independent and $|A| = |B|$, where A, B are respectively the set of pendents adjacent to each of the two centers. $D_{r,r}$ is not β_0-excellent, since the two centers are β_0-bad vertices.*

Remark 2.35. *Consider the path P_t with each vertex of P_t as centers, add r-pendent vertices. Let the resulting graph be denoted by $M_{r,r,...,t-times}$. Then $M_{r,r,...,r}\Box P_{2n+1}$ is β_0-excellent, but $M_{r,r,...,r}$ is not β_0-excellent.*

Illustration 2.36.



$A = \{1, 2, 3, 5, 10, 11, 12, 13\}$, $B = \{4, 6, 7, 8, 9, 14, 15, 16\}$ are two disjoint independent sets and $A \cup B = V(G)$. $M_{3,3,3,3}$ is not β_0-excellent. But $M_{3,3,3,3}\Box P_{2n+1}$ is β_0-excellent.

Theorem 2.37. *$G\Box P_{2n}$ is β_0-excellent if and only if there exists an independent partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of G such that $\max_{1\leq i,j\leq k, i\neq j} \{|V_i \cup V_j|\}$ is attained for pairs (i, j) with $\bigcup_{|V_i\cup V_j|} \{i, j\} = \{1, 2, \ldots, k\}$.*

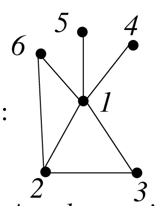
Proof. Any maximum independent subset of $G\Box P_{2n}$ is of the form $X_1 \cup X_2 \cup \cdots \cup X_{2n}$ where $X_i = A \times \{u_i\}$ if i is odd and $X_i = B \times \{u_i\}$ if i is even, A, B being disjoint independent sets of G such that $A \cup B$ has maximum cardinality. Suppose G has an independent partition satisfying the hypothesis. Then clearly, $G\Box P_{2n}$ is β_0- excellent.

Conversely, suppose $G\Box P_{2n}$ is β_0-excellent. Then every vertex $\{u, v\}, u \in V(G)$ and $v \in V(P_{2n})$ belongs to a β_0-set of $G\Box P_{2n}$. The structure of β_0-sets of $G\Box P_{2n}$ imply that there exist disjoint independent sets $V_1, V_2, \ldots, V_k$ in G whose union is $V(G)$, satisfying the condition in the theorem.

Corollary 2.38. *Let $\chi(G) \geq 3$. Then $G\Box P_{2n}$ is β_0-excellent if there exists a chromatic partition $\pi = \{V_1, V_2, \ldots, V_{\chi(G)}\}$ of G such that $\max_{1\leq i,j\leq \chi, i\neq j} \{|V_i \cup V_j|\}$ is attained for pairs (i, j) with $\bigcup_{|V_i\cup V_j|} \{i, j\} = \{1, 2, \ldots, \chi(G)\}$.*

Remark 2.39. *The converse of the above corollary is not true. Consider the graph G.*

G :



A chromatic partition of G is given by $\{\{2, 4, 5\}, \{3, 6\}, \{1\}\}$.

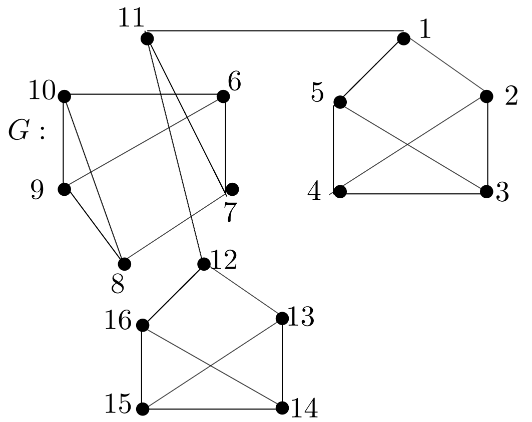
Corollary 2.40. *If G is a complete r-partite($r \geq 3$)graph with equi-cardinal partite sets, then $G\Box P_{2n}$ is β_0-excellent.*

Remark 2.41. *Let $G = \overline{K_2} + \overline{K_3} + \overline{K_2}$. Then G is not β_0-excellent, but $G\Box P_{2n}$ is β_0-excellent.*

Corollary 2.42. *Q_n is β_0-excellent,since $Q_n = Q_{n-1}\Box P_2$. Moreover $Q_n\Box P_{2n}$ is also β_0-excellent.*

Remark 2.43. *Let $G = \overline{K_4} + \overline{K_3} + \overline{K_2}$. Then G and $G\Box P_{2n}$ are not β_0-excellent.*

Theorem 2.44. *There exists a regular graph which is not β_0-excellent.*



For this graph G, the β_0-set is $\{2, 5, 6, 8, 11, 13, 16\}$ consisting of 7 vertices. The remaining vertices are contained in the independent sets $\{1, 3, 6, 8, 12, 14\}$, $\{1, 3, 7, 9, 12, 14\}$, $\{1, 3, 7, 9, 12, 15\}$ of cardinality 6 each. Thus this graph is 3-regular but not β_0-excellent.

Theorem 2.45. *Let G be a bipartite graph with bipartition V_1, V_2. Then $G\Box C_m$ is β_0-excellent.*

Proof. Case(i): Let $m = 2n$. Let $V(C_{2n}) = \{u_1, u_2, \ldots, u_{2n}\}$.

The maximum independent sets of $G\Box C_{2n}$ are
$$\bigcup_{j\equiv 0(mod2),\ 1\leq j\leq 2n} P \bigcup \bigcup_{j\equiv 0(mod2),\ 2\leq j\leq 2n} Q$$
and
$$\bigcup_{j\equiv 0(mod2),\ 1\leq j\leq 2n} R \bigcup \bigcup_{j\equiv 0(mod2),\ 2\leq j\leq 2n} S,$$
where $P = \{(v_i, u_j) : v_i \in V_1)\}, Q = \{(v_i, u_j) : v_i \in V_2)\}, R = \{(v_i, u_j) : v_i \in V_2)\}$ and $S = \{(v_i, u_j) : v_i \in V_1)\}$. Hence $G\Box C_m$ is β_0-excellent.

Case(ii): Let $m = 2n + 1$. Let $V(C_{2n+1}) = \{u_1, u_2, \ldots, u_{2n+1}\}$.

$\beta_0(G\Box C_{2n+1}) = n|V(G)|$.

The following are β_0-sets of $G\Box C_{2n+1}$.
$$\bigcup_{j\equiv 1(mod\ \ 2),\ 1\leq j\leq 2n} P_1 \bigcup \bigcup_{j\equiv 0(mod\ \ 2),\ 2\leq j\leq 2n} P_2;$$
$$\bigcup_{j\equiv 1(mod\ \ 2),\ 1\leq j\leq 2n} Q_1 \bigcup \bigcup_{j\equiv 0(mod\ \ 2),\ 2\leq j\leq 2n} Q_2;$$
$$\bigcup_{j\equiv 1(mod\ \ 2),\ 2\leq j\leq 2n+1} R_1 \bigcup \bigcup_{j\equiv 0(mod\ \ 2),\ 2\leq j\leq 2n} R_2$$
and
$$\bigcup_{j\equiv 1(mod\ \ 2),\ 2\leq j\leq 2n+1} S_1 \bigcup \bigcup_{j\equiv 0(mod\ \ 2),\ 2\leq j\leq 2n} S_2,$$
where$P_1 = \{(v_i, u_j) : v_i \in V_1)\}$, $P_2 = \{(v_i, u_j) : v_i \in V_2)\}$, $Q_1 = \{(v_i, u_j) : v_i \in V_2)\}$, $Q_2 = \{(v_i, u_j) : v_i \in V_1)\}$, $R_1 = \{(v_i, u_j) : v_i \in V_1)\}$, $R_2 = \{(v_i, u_j) : v_i \in V_2)\}$, $S_1 = \{(v_i, u_j) : v_i \in V_2)\}$ and $S_2 = \{(v_i, u_j) : v_i \in V_1)\}$. Hence $G\Box C_m$ is β_0-excellent.

Theorem 2.46. *Let G be a β_0-excellent graph. For any β_0-set S of G, let $V - S$ contain a β_0-set of G. Then $G\Box C_{2n}$ is β_0-excellent.*

Proof. The proof follows from the fact that for any β_0-set S of G and a β_0-set S_1 of G in $V - S$,
$$\bigcup_{i\equiv 1(mod\ \ 2),\ 1\leq i\leq 2n} P \bigcup \bigcup_{j\equiv 0(mod\ \ 2),\ 1\leq j\leq 2n} Q,$$
where $P = \{(v, u_i) : v \in S)\}$, $Q = \{(v, u_j) : v \in S_1)\}$ and
$$\bigcup_{i\equiv 1(mod\ \ 2),\ 1\leq i\leq 2n} R \bigcup \bigcup_{j\equiv 0(mod\ \ 2),\ 1\leq j\leq 2n} S,$$
where $R = \{(v, u_i) : v \in S_1)\}$, $S = \{(v, u_j) : v_i \in S)\}$ are β_0-sets of $G\Box C_{2n}$.

Theorem 2.47. *(i) $C_{2n}\Box C_{2k+1}$ is β_0-excellent. (ii)$C_{2n}\Box C_{2m}$ is β_0-excellent. (iii) $C_{2k+1}\Box C_{2n+1}, n \leq k$ is β_0-excellent.*

Corollary 2.48. *The following graphs are β_0-excellent.*

(i) $P_{2n}\Box P_{2k+1}$. Result follows from the fact that $G\Box P_{2k+1}$ is β_0-excellent if G is of even order and $V(G) = A \cup B, A \cap B = \phi, |A| = |B|$ and A, B are independent.

(ii) $P_{2n}\Box C_{2k+1}$. ($G\Box C_m$ is β_0-excellent if G is bipartite with partition V_1, V_2.)

(iii) $P_{2n+1}\Box C_{2k+1}$.

(since $G\Box C_m$ is β_0-excellent if G is bipartite with partition V_1, V_2.)

(iv) $P_{2n}\Box C_{2k}$.

(since $G\Box C_m$ is β_0-excellent if G is bipartite with partition V_1, V_2.)

(v) $P_{2n+1}\Box C_{2k}$.

(since $G\Box C_m$ is β_0-excellent if G is bipartite with partition V_1, V_2.)

Definition 2.49. ***Mycielski Graphs*** Let $G = (V, E)$ be a simple graph. The Mycielskian of G is the graph $\mu(G)$ with vertex set equal to the disjoint union $V \cup V' \cup \{u\}$ where $V' = \{x' : x \in V\}$ and the edge set $E \cup \{xy', x'y : xy \in E\} \cup \{y'u : y' \in V'\}$. The vertex x' is called the twin of the vertex and the vertex u is called the root of $\mu(G)$.

Theorem 2.50. *Let $G \neq K_2$ be a graph. Then $\mu(G)$ is not β_0-excellent.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$. Let $V(\mu(G)) = \{u_1, u_2, \ldots, u_n, u'_1, \ldots, u'_n, v\}$. Then $E(\mu(G)) = \bigcup_{i=1}^{n} \{u'_i u_j : u_j \in N_G(u_i), 1 \leq j \leq n\} \cup \{u'_i v : 1 \leq i \leq n\}$. It has been proved that $\beta_0(\mu(G)) = \max\{2\beta_0(G), |V(G)|\}$.

Suppose $\beta_0(G) < \frac{|V(G)|}{2}$. Then $\{u'_1, u'_2, \ldots, u'_n\}$ is the only β_0-set of $\mu(G)$.

Suppose $\beta_0(G) > \frac{|V(G)|}{2}$.

Let $\left\{u_{i_1}, u_{i_2}, \ldots, u_{i_{\beta_0}}\right\}$ be a β_0-set of G.Then$\left\{u_{i_1}, u_{i_2}, \ldots, u_{i_{\beta_0}}, u'_{i_1}, u'_{i_2}, \ldots, u'_{i_{\beta_0}}\right\}$ is a β_0-set of $\mu(G)$. Clearly v is not in any β_0-set of $\mu(G)$.

Suppose $\beta_0(G) = \frac{|V(G)|}{2}$. Suppose $\beta_0(G) = 1$ and $|V(G)| = 2$. Then $G = K_2$ in which case $\mu(G) = C_5$ which is β_0-excellent. Suppose $\beta_0(G) > 1$. Then for any β_0-set $\left\{u_{i_1}, u_{i_2}, \ldots, u_{i_{\beta_0}}\right\}$ of G, $\left\{u_{i_1}, u_{i_2}, \ldots, u_{i_{\beta_0}}, u'_{i_1}, u'_{i_2}, \ldots, u'_{i_{\beta_0}}\right\}$ is a β_0-set of $\mu(G)$. Also $\{u'_1, \ldots, u'_n\}$ is a β_0-set of $\mu(G)$.

v does not belong to any of these β_0-sets. Therefore $\mu(G)$ is not β_0-excellent, when $G \neq K_2$.

Definition 2.51. *Let G be a graph. G is said to be β_1-excellent if every edge of G belongs to a β_1-set of G.*

Remark 2.52. *G is β_1-excellent if and only if $L(G)$ is β_0-excellent.*

3 β_0-excellence of Harary graphs

Definition 3.1. ***Harary graphs** $H_{n,m}$ with n vertices and $m < n$ is defined as follows:*

Case(i): *n is even and $m = 2r$. Then $H_{n,2r}$ has n vertices $0, 1, 2, \cdots, n-1$ and i, j are joined if $i - r \leq j \leq i + r$, where the addition is taken under modulo n.*

Case(ii):*m is odd and n is even. Let $m = 2r + 1$. Then $H_{n,2r+1}$ is constructed by first drawing $H_{n,2r}$ and then adding edges joining vertex i to the vertex $i + \frac{n}{2}$, for $0 \leq i \leq \frac{n}{2}$.*

Case(iii): *m and n are odd. Let $m = 2r + 1$. Then $H_{n,2r+1}$ is constructed by drawing $H_{n,2r}$ and then adding edges joining vertex 0 to the vertices $\frac{n-1}{2}$ and $\frac{n-1}{2}$ and vertex i to $i + \frac{n+1}{2}$, for $1 \leq i \leq \frac{n-1}{2}$.*

Theorem 3.2. *Let $n > 2r$.*

$$\beta_0(H_{2r,n}) = \begin{cases} \frac{n-r}{r+1} & \text{if r+1 divides n-r} \\ \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1 & \text{if r+1 does not divide n-r} \end{cases}$$

Proof. Let $V(H_{2r,n}) = \{0, 1, 2, \ldots, n-1\}$.

Case(i):

Let $r + 1$ divides $n - r$. Consider $S = \{i, r + i + 1, 2r + i + 2, \ldots, tr + t + i\}$, where $t = \frac{n-r}{r+1} - 1$.

$tr + t + i = (n - r) - r - 1 + i = n - 2r - 1 + i$. Suppose $n - 2r - 1 + i = i - s$ (or) $i - s + n$, according as $i - s \geq 0$ (or) otherwise. Then $n - 2r - 1 = -s$ (or)$n - s$. That is $2r + 1 = s + n$(or)s. Since $s \leq r, 2r + 1 \neq s$. Therefore $2r + 1 = s + n$. But $s + n > 2r + 1$. Since $s \geq 1, n > 2r$, a contradiction. Therefore S is an independent set in $H_{2r,n}$. Therefore $\beta_0(H_{(2r,n)}) \geq t + 1$. Suppose S_1 is an independent set of $H_{2r,n}$ of cardinality $t + l, l \geq 2$.

Let $S_1 = \{a_1, a_2, \ldots a_{t+l}\}$. Let $a_1 < a_2 < \ldots, < a_{t+l}$.

$t + l = \frac{n-r}{r+1} - 1 + l \geq \frac{n-r}{r+1} + 1$ (since $l \geq 2$).

Let $a_1 = i$. Then $a_2 > i + r, a_3 > i + 2r, \ldots, a_{t+l} > i + (t + l - 1)r$. That is $a_{t+l} > i + \left(\frac{n-r}{r+1}\right) r$.

Let $1 \leq s \leq r$. a_{t+l} is adjacent to a_1 if and only if $i - s$ or $i - s + n > i + \left(\frac{n-r}{r+1}\right) r$. That is if and only if $s < -\left(\frac{n-r}{r+1}\right) r$, a contradiction since right hand side is negative and s is positive. (or) $i - s + n > i + \left(\frac{n-r}{r+1}\right) r$. This implies $n - s > \left(\frac{n-r}{r+1}\right) r$.

$n - r = q(r + 1) \Rightarrow q(r + 1) + r - s > qr$.

$qr + q + (r - s) > qr$. Since $s \leq r, r - s \geq 0$, one has $qr + q + (r - s) > qr$ (since $q \geq 1$), which is true. a_{t+l} is adjacent to a_1. Therefore S_1 is not independent. So $\beta_0(H_{2r,n}) \leq t + 1$. Therefore $\beta_0(H_{2r,n}) = t + 1$.

Case(ii):

Let $r + 1$ do not divide $n - r$. Consider $S = \{i, r + 1 + i, 2r + 2 + i, \ldots, tr + t + i\}$, where $t = \left\lfloor \frac{n-r}{r+1} \right\rfloor$. Let $n - r = q(r + 1) + \alpha, \alpha > 0, \alpha < r + 1$. Therefore $t = q$.

$tr + t + i = qr + q + i = q(r + 1) + i = n - r - \alpha + i$.

Let $1 \leq s \leq r$.

If $tr + t + i = i - s(or)i - s + n$, according as $i - s \geq 0$(or) otherwise , then $n - \alpha - r + i = i - s(or)i - s + n$. $n - \alpha - r + i = i - s$ (or)$i - s + n$. $n - \alpha - r = -s$ (or) $-\alpha - r = -s$. That implies $r + \alpha - n = s$ (or) $s = r + \alpha$ i.e, $s < 0$ or $s > r$(since $\alpha + r < n$), a contradiction. [$r + \alpha < n$, because

$n = r+q(r+1)+\alpha$. If $q = 0$, then $n = r+\alpha$, where $\alpha < r+1$. That is $n \leq 2r$, a contradiction. So $q \geq 1$. Therefore $n > r+\alpha$.] Thus S is an independent set in $H_{2r,n}$. Therefore $\beta_0(H_{2r,n}) \geq t+1$. Suppose S_1 is an independent set of $H_{2r,n}$ of cardinality $t+l, l \geq 2$. Let $S_1 = \{a_1, a_2, \ldots, a_{t+l}\}$. Let $a_1 < a_2 < \ldots < a_{t+l}$. $t+l = q+l = \left\lfloor \frac{n-r}{r+1} \right\rfloor + l > \frac{n-r}{r+1} + 1$.

Let $a_1 = i$. Then $a_2 > i+r, a_3 > i+2r, \ldots, a_{t+l} > i+(t+l-1)r > i + \left(\frac{n-r}{r+1}\right) r$. a_{t+l} is adjacent to a_1 if and only if $i-s$(or)$i-s+n$ is greater than $i+\left(\frac{n-r}{r+1}\right) r$. That is if and only if $-s$ (or)$-s+n > \left(\frac{n-r}{r+1}\right) r$.

But $-s > \left(\frac{n-r}{r+1}\right) r$ is not possible, since the right hand side is positive and left hand side is negative. Therefore $-s+n > \left(\frac{n-r}{r+1}\right) r$. That is $n-s > \left(q+\frac{\alpha}{r+1}\right) r$. That leads to $q(r+1)+\alpha+r-s > qr+\frac{r\alpha}{r+1}$ which means $q(r+1)+r-s > qr+\frac{r\alpha}{r+1} - \alpha = qr - \frac{\alpha}{r+1}$. That is $q(r+1)+r-s \geq qr$ [since $\frac{\alpha}{r+1} < 1$] , which is true, since $q(r+1)+r-s = qr+q+r-s \geq qr$, as $r-s \geq 0$. Therefore a_{t+l} is adjacent to a_1. Therefore S_1 is not an independent set. Therefore $\beta_0(H_{2r,n}) \leq t+1$. Therefore $\beta_0(H_{2r,n}) = t+1$.

Theorem 3.3. *Consider $H_{2r+1,n}$, where n is even.*

Then (i) If $2(r+1)$ does not divide n, then

$$\beta_0(H_{2r+1,n}) = \begin{cases} \frac{n-r}{r+1}, & \text{if r+1 divides n-r} \\ \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1, & \text{otherwise} \end{cases}.$$

(ii)If $2(r+1)$ divides n,then

$$\beta_0(H_{2r+1,n}) = \left\lfloor \frac{n-r}{r+1} \right\rfloor.$$

Proof. We observe the following

(i) Suppose $2(r+1)$ divides n. Then $r+1$ does not divide $n-r$.

Let $r+1$ divide $n-r$. Let $n = 2q(r+1)$ and $n-r = q_1(r+1)$. Therefore $2q(r+1) = r+q_1(r+1)$. That is $(2q-q_1)(r+1) = r$, a contradiction. Hence (i).

(ii) Suppose $2(r+1)$ does not divide n. Then $r+1$ divides $n-r$ if and only if $n = 2q(r+1)+r$, for some positive integer q.

Let $n = 2q(r+1)+\alpha$, where $0 < \alpha < 2(r+1)$. Then $n-r = 2q(r+1)+\alpha-r$.

Suppose $r+1$ divides $n-r$. Then $\alpha-r$ is divisible by $r+1$. Let $\alpha-r = k(r+1)$. If $k < 0, \alpha = r+k(r+1)$ implies that $\alpha < 0$, a contradiction. Hence $k \geq 0$. Thus $\alpha = k(r+1)+r$. Since $\alpha < 2(r+1), k = 1$, one has $\alpha = 2r+1$. Therefore $n = 2q(r+1)+2r+1$. That means that n is odd, a contradiction. Therefore $k = 0$. That is, $\alpha = r$. Therefore $n = 2q(r+1)+r$.

Conversely, if $n = 2q(r+1)+r$, then clearly $n-r$ is divisible by $r+1$.

Case(i):

Subcase (i): Suppose $2(r+1)$ does not divide n and $r+1$ divides $n-r$. Then by observation (ii), $n = 2q(r+1)+r$, for some positive integer q. For any integer $i, i+\frac{n}{2}$ is of the form $lr+l+i$ if and only if $l(r+1) = \frac{n}{2}$. That is if and only if 2(r+1) divides n , a contradiction.

Let $S = \{i, r+1+i, 2r+2+i, \ldots tr+t+i\}$, where $t = \frac{n-r}{r+1} - 1$. That is $t = 2q-1$.

$tr+t+i = (n-r)-r-1+i = n-2r-1+i$. Suppose $n-2r-1+i = i-s$ (or) $i-s+n$ according as $i-s \geq 0$ (or) otherwise. Then $n-2r-1 = -s$(or) $n-s$. That is $2r+1 = n+s$ (or) s. Since $s \leq r, 2r+1 \neq s$. Therefore $2r+1 = n+s$. But $s+n > 2r+1$, since $s \geq 1$ and $n > 2r$, a contradiction. Therefore S is an independent set in $H_{2r+1,n}$. Therefore $\beta_0(H_{2r+1,n}) \geq t+1 = \frac{n-r}{r+1}$. Since $H_{2r,n}$ is a spanning subgraph of $H_{2r+1,n}$, we get that $\beta_0(H_{2r+1,n}) \leq \beta_0(H_{2r,n}) = \frac{n-r}{r+1}$.

Therefore $\beta_0(H_{2r+1,n}) = \frac{n-r}{r+1}$.

Subcase (ii): $2(r+1)$ does not divide n and $r+1$ does not divide $n-r$.

By observation (ii), $n = 2q(r+1)+\alpha$, where $0 < \alpha < 2(r+1)$ and $\alpha \neq r$. Proceeding as in case (ii) of theorem 3.2, we get that $S = \{i, r+1+i, 2r+2+i, \ldots tr+t+i\}$, where $t = \left\lfloor \frac{n-r}{r+1} \right\rfloor$ is an independent set of $H_{2r+1,n}$. Therefore $\beta_0(H_{2r+1,n}) \geq \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1$. But $\beta_0(H_{2r+1,n}) \leq \beta_0(H_{(2r,n)}) = \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1$. Therefore $\beta_0(H_{2r+1,n}) = \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1$.

Case (ii): $2(r+1)$ divides n.

By observation (i), $r+1$ does not divide $n-r$. Let $\frac{n}{2(r+1)} = l$. Then i is adjacent to $i+\frac{n}{2}$ gives that i is adjacent to $lr+l+i$.

Let S be the set of all elements $i, r+1+i, 2r+2+i, \ldots, (l-1)(r+1)+i, l(r+1)+1+i, (l+1)(r+1)+1+i$,

$\ldots, t(r+1)+l+i$, where $t = \left\lfloor \frac{n-r}{r+1} \right\rfloor - 1$.

Let $n-r = q(r+1)+\alpha$, where $0 < \alpha < r+1$. Therefore $t = q-1$.

$$\begin{aligned} tr+t+1+i &= t(r+1)+1+i \\ &= (q-1)(r+1)+1+i \\ &= q(r+1)+i-r = n-r-\alpha + \end{aligned}$$
$i-r = n-2r-\alpha+i$.

Let $1 \leq s \leq r$. If $t(r+1)+1+i = i-s$ (or) $i-s+n$ (according as $i-s \geq 0$ (or) otherwise), then $n-2r-\alpha+i = i-s$(or) $i-s+n$. That is $n-2r-\alpha = -s$ (or) $n-s$. That is $n-2r-\alpha = -s$ (or) $n-2r-\alpha = n-s$. If $n-2r-\alpha = n-s$, then $s = 2r+\alpha$, a contradiction , since $s \leq r$. If $n-2r-\alpha = -s$, then $s = 2r+\alpha-n = 2r+\alpha-q(r+1)-\alpha-r$. That is $s = r-q(r+1) < 0$, a contradiction. Therefore S is an independent set in $H_{2r+1,n}$. Therefore $\beta_0(H_{2r+1,n}) \geq \left\lfloor \frac{n-r}{r+1} \right\rfloor$.

Let $S_1 = \{i, r+1+i, 2r+2+i, \ldots, (l-1)(r+1)+i, l(r+1)+1+i, (l+1)(r+1)+1+i, \ldots, t(r+1)+l+i\}$, where $t = \left\lfloor \frac{n-r}{r+1} \right\rfloor$. Let $n-r = q(r+1)+\alpha, 0 < \alpha < r+1$. Let $1 \leq s \leq r$. Then $t = q$. If $t(r+1)+1+i = i-s$ (or) $i-s+n$ (according as $i-s \geq 0$ (or) otherwise.). Then $q(r+1)+1+i = i-s$ (or)$i-s+n$. That is $q(r+1)+1 = -s$(or) $n-s$. If $q(r+1)+1 = -s$, then a contradiction , since L.H.S is positive. If $q(r+1)+1 = n-s$, then $n-r-\alpha+1 = n-s$. That is $s = r+\alpha-1$. But $s \leq r$. Therefore $\alpha \leq 1$. But $\alpha > 0$. Therefore $\alpha = 1$.

Therefore, $n-r = q(r+1)+1 = t(r+1)+1$. In this case , $t(r+1)+1+i$ is adjacent with i. Therefore S_1 is not independent.

Therefore $\beta_0(H_{2r+1,n}) \leq \left\lfloor \frac{n-r}{r+1} \right\rfloor$.

Therefore $\beta_0(H_{2r+1,n}) = \left\lfloor \frac{n-r}{r+1} \right\rfloor$.

Observation 3.4.

(i) $2(r+1)$ can not divide both $n+1, n-1$.

This is because in such a case $2(r+1)$ divides 2, a contradiction, since $2(r+1) \geq 4$.

(ii) If $2(r+1)$ divides $n-1$, then $r+1$ does not divide $n-r$.

This is because if $r+1$ divides $n-r$, then $n-r = a(r+1)$. $n = a(r+1)+r$. Let $n-1 = 2(r+1)l$. Then $2(r+1)l+1 = a(r+1)+r$. $2(r+1)l = a(r+1)+r-1$, a contradiction.

(iii) Suppose $2(r+1)$ divides $n-1$. Let $r+1$ do not divide $n-r$. Then $t(r+1)+1 < n-r$, where $t = \left\lfloor \frac{n-r}{r+1} \right\rfloor$.

Since $r+1$ does not divide $n-r$, $t(r+1)+1 \leq n-r$.Suppose $t(r+1)+1 = n-r$. Let $n-1 = 2q(r+1)$. Therefore $t(r+1)+r = n-1 = 2q(r+1)$. Thus $r+1$ divides r, a contradiction.

(iv) Suppose $2(r+1)$ divides $n+1$. Then $r+1$ divides $n-r$.

Let $n+1 = 2q(r+1)$. Therefore $n-r = 2q(r+1)-r-1 = (r+1)(2q-1)$. Therefore $r+1$ divides $n-r$.

Theorem 3.5. *Consider $H_{2r+1,n}$, where n is odd.*

(i) $2(r+1)$ does not divide $n-1$ as well as $n+1$. Then

$$\beta_0(H_{2r+1,n}) = \begin{cases} \frac{n-r}{r+1}, & \text{if r+1 divides n-r} \\ \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1, & \text{otherwise} \end{cases}$$

(ii) $2(r+1)$ divides $n-1$ but not $n+1$.

$\beta_0(H_{2r+1,n}) = \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1$.

(iii) $2(r+1)$ divides $n+1$ but not $n-1$.

Then $\beta_0(H_{2r+1,n}) = \frac{n-r}{r+1}$.

Proof. Case (i):

$2(r+1)$ does not divide $n-1$ as well as $n+1$.

Let $0 \leq i \leq n-1$.

Let $S = \{i, r+1+i, 2(r+1)+i, \ldots, t(r+1)+i\}$.

$l(r+1)+i = i+\frac{n+1}{2}, (i \geq 0)$. This implies $r+1$ divides $\frac{n+1}{2}$, a contradiction.

$l(r+1)+0 = 0+\frac{n-1}{2}$. This implies $r+1$ divides $\frac{n-1}{2}$, a contradiction. $l(r+1)+i$ is adjacent to $m(r+1)+i$, if $l(r+1)+i+\left(\frac{n+1}{2}\right) = m(r+1)+i$. This implies $(m-l)(r+1) = \frac{n+1}{2}$. This implies $r+1$ divides $\frac{n+1}{2}$, a contradiction.

Subcase(i): Let $r+1$ divide $n-r$. Let $t = \frac{n-r}{r+1}-1$.

$t(r+1)+i = n-r-r-1+i = n-2r-1+i$.

Suppose $t(r+1)+i = i-s$(or) $i-s+n, (1 \leq s \leq n)$ according as $i-s \geq 0$ (or) otherwise. Then $n-2r-1+i = i-s$ (or) $i-s+n$. That is $n-2r-1 = -s$ (or)$n-s$. Since $n > 2r+1, n-(2r+1)$ is positive and $-s$ is negative. Therefore $n-2r-1 = -s$ is not possible. $n-2r-1 = n-1$ gives $s = 2r+1, 1 \leq s \leq n$, a contradiction. Therefore $|S| = t+1 = \frac{n-r}{r+1}$. Therefore $\beta_0(H_{2r+1,n}) \geq \frac{n-r}{r+1}$. $H_{2r,n}$ is a spanning subgraph of $H_{2r+1,n}$.

$\beta_0(H_{2r+1,n}) \leq \beta_0(H_{2r,n}) \leq \frac{n-r}{r+1}$. Therefore $\beta_0(H_{2r+1,n}) = \frac{n-r}{r+1}$.

Subcase(ii):

Let $r+1$ do not divide $n-r$. Let $t = \left\lfloor \frac{n-r}{r+1} \right\rfloor$.

Proceeding as in case (ii) of theorem 3.3,

we get that $\beta_0(H_{2r+1,n}) \geq \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1$.

$H_{2r,n}$ is a spanning subgraph of $H_{2r+1,n}$. Therefore $\beta_0(H_{2r+1,n}) \leq \beta_0(H_{2r,n}) = \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1$. Therefore $\beta_0(H_{2r+1,n}) = \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1$.

Case (ii):

$2(r+1)$ divides $n-1$ but not $n+1$. By observation(ii), $r+1$ does not divide $n-r$ and $t(r+1)+1 < n-r$, where $t = \left\lfloor \frac{n-r}{r+1} \right\rfloor$. Let $0 \leq i \leq n-1$.

Let $i > 0$ and let $S = \{i, r+1+i, 2(r+1)+i, \dots, t(r+1)+i\}$, where $t = \left\lfloor \frac{n-r}{r+1} \right\rfloor$.

$l(r+1)+i = i + \frac{n+1}{2}$. This implies $r+1$ divides $\frac{n+1}{2}$, a contradiction.

$l(r+1)+i$ is adjacent to $m(r+1)+i$, if $l(r+1)+i+\left(\frac{n+1}{2}\right) =$
$m(r+1)+i$. This implies $(m-l)(r+1) = \frac{n+1}{2}$. This implies $r+1$ divides $\frac{n+1}{2}$, a contradiction. Proceeding as in Case(ii), we get that S is an independent set of cardinality $\left\lfloor \frac{n-r}{r+1} \right\rfloor + 1$.

Thus $\beta_0(H_{2r+1,n}) \geq \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1$.

$H_{2r,n}$ is a spanning subgraph of $H_{2r+1,n}$.

Therefore $\beta_0(H_{2r+1,n}) \leq \beta_0(H_{2r,n}) = \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1$. Therefore $\beta_0(H_{2r+1,n}) = \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1$.

Case(iii):

$2(r+1)$ divides $n+1$ but not $n-1$. Then $r+1$ divides $n-r$. 0 is adjacent to $\frac{n-1}{2}$ and $\frac{n+1}{2}$. Let $l_1 = \frac{n+1}{2(r+1)}$. 0 is adjacent to $l_1(r+1)$ and $l_1(r+1)-1$. Let S_0 be the set of all elements $0, r+1, \dots,$
$(l_1-1)(r+1), l_1(r+1)+1, \dots, t(r+1)+1$.
If $a(r+1) = b(r+1) + \frac{n+1}{2}$, where $a, b \leq (l_1-1)$. $(a-b)(r+1) = \frac{n+1}{2}$. This implies $a-b = \frac{n+1}{2(r+1)} = l_1$, a contradiction.

If $a(r+1) + \frac{n+1}{2} = b(r+1)+1$,where $a \leq l_1 - 1, b \geq l_1$, then $(b-a)(r+1) = \frac{n-1}{2}$. That is $r+1$ divides $\frac{n-1}{2}$, a contradiction.

If $a(r+1)+1 = b(r+1)+1+\frac{n+1}{2}$, where $a, b \geq l_1$ and $a > b$, then $a-b = \frac{n+1}{2(r+1)} = l_1$. Therefore $a = b + l_1 \geq l_1 + l_1 = 2l_1 = \frac{n+1}{(r+1)}$. Therefore $a(r+1) \geq n+1$.That is $t(r+1) \geq a(r+1) \geq n+1$. Therefore $t \geq \frac{n+1}{(r+1)}$, a contradiction, since $t = \frac{n-r}{r+1} - 1$. Therefore S_0 is an independent set of cardinality $t+1 = \frac{n-r}{r+1}$.

Let $i \neq 0$. i is adjacent to $\frac{n+1}{2} + i$.

Let $l_1 = \frac{n+1}{2(r+1)}$. Therefore i is adjacent to $l_1(r+1)+i$.

Let $S_i = \{i, i+r+1, i+2(r+1), \dots, (l_1-1)(r+1)+i, l_1(r+1)+1+i, \dots,$
$t(r+1)+1+i\}$.

If $a(r+1)+i = b(r+1)+i+\frac{n+1}{2}$ where $a, b \leq l_1 - 1$, then $a-b)(r+1) = \frac{n+1}{2}$. $a-b = \frac{n+1}{2(r+1)} = l_1$, a contradiction.

If $a(r+1)+i+\frac{n+1}{2} = b(r+1)+1+i$, where $a \leq l_1-1, b \geq l_1$, then $(b-a)(r+1) = \frac{n-1}{2}$, a contradiction.

If $a(r+1)+1+i = b(r+1)+1+\frac{n+1}{2}+i$ where $a, b \geq l_1$ and $a > b$,then $(a-b)(r+1) = \frac{n+1}{2}$. This implies $a-b = \frac{n+1}{2(r+1)} = l_1$.

Therefore $a = l_1 + b \geq l_1 + l_1 = 2l_1 = \frac{n+1}{r+1}$ and $a(r+1) \geq n+1$. That is $t(r+1) \geq a(r+1) \geq n+1$, which implies $t \geq \frac{n+1}{r+1}$, a contradiction, since $t = \frac{n-r}{r+1} - 1$. So S_i is independent set of cardinality $t+1 = \frac{n-r}{r+1}$. Therefore $\beta_0(H_{2r+1,n}) \geq \frac{n-r}{r+1}$.

$H_{2r,n}$ is a spanning subgraph of $H_{2r+1,n}$.

Therefore $\beta_0(H_{2r+1,n}) \leq \beta_0(H_{2r,n}) = \frac{n-r}{r+1}$.

Therefore $\beta_0(H_{2r+1,n}) = \frac{n-r}{r+1}$.

Theorem 3.6. *Consider $H_{2r+1,n}$, where n is odd. Let $2(r+1)$ divide $(n-1)$ but not $n+1$. If $t(r+1)+2 \geq n-r$, where $t = \frac{n-r}{r+1}$, then $H_{2r+1,n}$ is not β_0-excellent.*

Proof. $2(r+1)$ divides $n-1$ but not $n+1$. Let $l_1 = \frac{n-1}{2(r+1)}$. 0 is adjacent to $l_1(r+1)$. Also 0 is adjacent to $l_1(r+1)+1$,since $l_1(r+1)+1 = \frac{n-1}{2}+1 = \frac{n+1}{2}$. Let S_1 be the set of all elements $0, r+1, \dots, (l_1-1)(r+1), l_1(r+1) = 2, (l_1+1)(r+1)+2, \dots, t(r+1)+2$.

If $t(r+1)+2 < n-r$, then S_1 is an independent set of cardinality $\left\lfloor \frac{n-r}{r+1} \right\rfloor + 1$. Suppose $t(r+1)+2 = n-r$. Then S_1 is not independent. Let S_2 be the set of all elements $0, r+1, \dots, (l_1-1)(r+1), l_1(r+1)+2, (l_1+1)(r+1)+2, \dots, (t-1)(r+1)+2$ be an independent set. Therefore $|S_2| = t = \left\lfloor \frac{n-r}{r+1} \right\rfloor < \beta_0$.

Let S_1' be the set of all elements $0, -(r+1), -2)r+1), \dots, -(l_1-1)(r+1), -l_1(r+1)-2, \dots,$
$-t(r+1)-2$. If $t(r+1)+2 < n-r$, then $-t(r+1)-2 > r$. Therefore $-t(r+1)-2$ is not adjacent to 0. Therefore S_1' is an independent set of cardinality $\left\lfloor \frac{n-r}{r+1} \right\rfloor + 1 = \beta_0$.

Suppose $t(r+1)+2 = n-r$. Then S_1' is not independent.
Also S_2' be the set of all elements
$0, -(r+1), -2(r+1), \dots, -(l_1-1)(r+1), -l_1(r+1)-2, \dots,$
$-(t-1)(r+1)-2$ is an independent set of cardinality $\left\lfloor \frac{n-r}{r+1} \right\rfloor < \beta_0$. Therefore 0 does not belong to any β_0-set.

Illustration 3.7. *Consider $H_{5,7}$. $r = 2, n = 7, 2(r+1) = 6$ does not divide $n+1 = 8$, but 6 divides $n-1 = 7-1 = 6$.*

$t = \left\lfloor \frac{n-r}{r+1} \right\rfloor = 1, \beta_0 = \left\lfloor \frac{n-r}{r+1} \right\rfloor + 1 = \left\lfloor \frac{5}{3} \right\rfloor = 2.$

$S_0 = \{0\}$ *is a maximal independent set containing* 0 *and there is no β_0-set containing* 0.

$S_1 = \{1,4\}, S_2 = \{2,5\}, S_3 = \{3,6\}$ *are the β_0-sets of $H_{5,7}$.*

Remark 3.8. *$H_{2r,n}$ is β_0-excellent. $H_{2r+1,n}$ is not β_0-excellent if and only if n is odd and $2(r+1)$ divides $n-1$.*

4 JUST β_0- EXCELLENT GRAPHS

N. Sridharan and M. Yamuna [10] initiated the study of just excellence in graphs with respect to the domination parameter. A graph G is just γ excellent if every vertex is contained in a unique minimum dominating set. In this section, just β_0- excellent graphs are defined and studied.

4.1 Introduction

Partition of $V(G)$ into independent sets is the same as proper coloring of the graph. A chromatic partition is a partition of the vertex set into minimum number of independent sets. Such a partition may not contain a maximum independent set. For example, a double star contains a unique chromatic partition of cardinality two in which both the independent sets are not maximum. The question that naturally arises is that "'Does there exist a graph in which the vertex set can be partitioned into maximum independent sets ?"'. This leads to the concept of just β_0- excellent graphs. It is shown in this chapter that a graph of order n is just β_0- excellent if and only if $\beta_0(G)$ divides n, G has exactly $\frac{n}{\beta_0}$ distict β_0 sets and the maximum cardinality of a partition of $V(G)$ into independent sets is $\frac{n}{\beta_0}$. This section is devoted to the definition and properties of just β_0-excellent graphs, just β_0 excellence in product graphs, just β_0 excellence in Generalized Petersen graphs and just β_0 excellence in Harary graphs.

4.2 Definitions and Properties of just β_0- excellent graphs

Definition 4.1. *A graph G is said to be just β_0-excellent graph if for each $u \in V$, there exists a unique β_0-set of G containing u.*

Examples of just β_0-excellent graphs

(1) C_{2n} (2) K_n (3) $K_{n,n}$ (4) $P_m \Box P_n$, if $mn \equiv 0(mod 2)$.

Examples of not just β_0-excellent graphs

(1) C_{2n+1} (2) $K_{1,n}$ (3)P_n (4) The subdivision graph of $K_{1,n}$ (5) Petersen graph (6) W_n, $n \geq 5$ (7) $D_{r,s}$ (8) $G \circ K_1$, for any connected graph G.

(9) $F_n = P_{n-1} + K_1$.

Properties of just β_0-excellent graphs

1. Every just β_0-excellent graph is a β_0-excellent graph.

2. If G is justβ_0-excellent and $G \neq K_n$,then there is no vertex u such that $< V - N[u] >$ contains at least two maximum independent sets.

Proof. Since G is just β_0-excellent, given $u \in V(G)$, there exists a unique β_0-set S of G containing u. Suppose $V - N[u]$ contains at least two maximum independent sets. $G \neq K_n$.

Therefore $\beta_0(G) \geq 2$ and $\beta_0(< V - N[u] >) \geq 1$. $S - \{u\}$ is an independent set of $< V - N[u] >$ and hence $\beta_0(< V - N[u] >) \geq \beta_0(G) - 1$.

If $\beta_0(< V - N[u] >) = \beta_0(G)$, then any β_0-set of $< V - N[u] >$ together with u is an independent set of G of cardinality $\beta_0(G)+1$, a contradiction. Let T_1, T_2 be two maximum independent sets of $V - N[u]$. Then $T_1 \cup \{u\}$ and $T_2 \cup \{u\}$ are maximum independent sets of G, a contradiction.

3. Let G be just β_0-excellent. Then there exists a unique partition of $V(G)$ into β_0-sets of G.

Proof. Let $u \in V(G)$. Let S_1 be the unique β_0-set of G containing u.

If $V - S_1 = \phi$, then there is nothing to prove. Otherwise consider a vertex $v \in V - S_1$. v is contained in a unique β_0-set say S_2 of G. $S_1 \cap S_2 = \phi$, since G is just β_0-excellent. If $V - (S_1 \cup S_2) = \phi$, the process stops. Otherwise there exists $w \in V - (S_1 \cup S_2)$. There exists a unique β_0-set say S_3 of G containing w. Clearly $S_i \cap S_j = \phi$, $i \neq j$, $1 \leq i, j \leq 3$. Proceeding like this, we get a partition of $V(G)$ into β_0-sets of G.

4. $\beta_0(G)$ is a factor of n.

Proof. From the previous property, $n = m\beta_0(G)$, where m is the cardinality of the partition of $V(G)$ into β_0-sets.

5. Let G be a just β_0-excellent graph. Let $|V(G)| = n$. Then $n = \chi(G)\beta_0(G)$.

From property 4, $n = d\beta_0(G)$. Also $\frac{n}{\beta_0(G)} \leq \chi(G)$ and hence $d \leq \chi(G)$. Clearly $\chi(G) \leq d$. Hence $\chi(G) = d$.

6. In a just β_0-excellent graph G, $|V(G)| = \beta_0(G).\chi(G)$. The converse is not true.

Consider P_6. $\beta_0(P_6) = 3$, $\chi(P_6) = 2$. $|V(P_6)| = 6 = \beta_0(P_6).\chi(P_6)$. But P_6 is not a just β_0-excellent graph.

7. $\delta(G) \geq \frac{n}{\beta_0(G)} - 1$.

Proof. Let $\Pi = \{S_1, S_2, \ldots, S_m\}$ be a β_0-set partition of $V(G)$. Let $u \in S_i$. Then u is adjacent to at least one vertex in each S_j, $j \neq i$. Therefore $deg(u) \geq m - 1$. Therefore $\delta(G) \geq m - 1 =$

$\frac{n}{\beta_0(G)} - 1$.

8. $\frac{n}{\beta_0(G)} = 1$ if and only if $G = \overline{K_n}$.

9. If G has two or more disjoint β_0-sets, then G has no isolates.

Proof. Suppose G has two or more disjoint β_0-sets. Let $S_1, S_2, \ldots, S_t$ be the disjoint β_0-sets. Then $t \geq 2$. Suppose G has an isolate, say u. Let $u \in S_1$. Then $S_2 \cup \{u\}$ is an independent set of cardinality $\beta_0(G)+1$, a contradiction. (or) [Equivalently, any isolate vertex is contained in every β_0-set and hence if there are isolates, there can not be two or more disjoint β_0-sets.] Thus, if G is just β_0-excellent and G has an isolate, then $G = \overline{K_n}$ and conversely.

10. Let G be a just β_0-excellent graph. If $G \neq K_2$ and $G \neq \overline{K_n}$, then $\delta(G) \geq 2$.

Proof. Since $G \neq \overline{K_n}$ and since G is a just β_0-excellent graph, $\delta(G) \geq 1$.

Suppose u is a pendent vertex of G. Let $N(u) = \{v\}$. Since G is just β_0-excellent, there exists a β_0-set of D containing v. Therefore $v \in D$ and $u \notin D$. Suppose $\beta_0(G) = 1$. Then G is a complete graph. Since $G \neq K_2$ and $\delta(G) \geq 1$, $G = K_n$, $n \geq 3$. Therefore $\delta(G) \geq 2$. Therefore u is not a pendent vertex, a contradiction. Suppose $\beta_0(G) \geq 2$. Then $|D| \geq 2$. Therefore there exist $w \in D$, $w \neq v$. Let $D_1 = (D - \{v\}) \cup \{u\}$. Then D_1 ia a β_0-set of G and w is contained in two β_0-sets of G namely D and D_1, a contradiction. Therefore $\delta(G) \geq 2$.

Remark 4.2. *Any even cycle G is a just β_0-excellent graph with $\delta(G) = 2$. Any tree is not a just β_0-excellent graph.*

11. A graph G has exactly two disjoint β_0-sets whose union is $V(G)$ say V_1, V_2 if and only if for every non empty proper subset A of V_1 or V_2, $|N(A)| > |A|$.

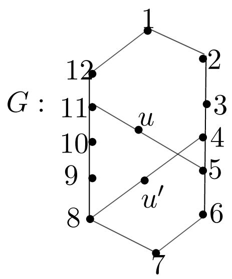
Proof. Suppose G has exactly two disjoint β_0-sets whose union is $V(G)$ say V_1, V_2. Let $A \subset V_1$. Suppose $|N(A)| \leq |A|$. Let $C = V_2 - N(A)$. If $C = \phi$, then $N(A) = V_2$. Thus $|A| \geq |N(A)| = |V_2| = \beta_0(G)$. But $A \subset V_1$, a contradiction. Thus $C \neq \phi$. $A \cup C$ is an independent set of G and $|A \cup C| = |A| + |C| = |A| + \beta_0(G) - |N(A)| \geq \beta_0(G)$, a contradiction, since G has exactly two disjoint β_0-sets whose union is $V(G)$. Therefore, $|N(A)| > |A|$. Conversely, let there be two disjoint β_0-sets whose union is $V(G)$ say V_1, V_2 and for any proper subset A of V_1 or V_2, $|N(A)| > |A|$.

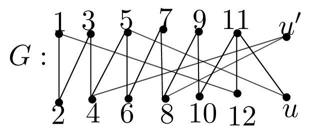
Let W be a β_0-set of G. $W \neq V_1$, and $W \neq V_2$. Let $W \cap V_1 = W_1$, $W \cap V_2 = W_2$. Then $W_1 \neq \phi, W_2 \neq \phi$. $|N(W_1)| > |W_1|$. $N(W_1) \cap W_2 = \phi$. (For: if $x \in N(W_1) \cap W_2$, then $x \in N(W_1)$ and $x \in W_2$. That is x is adjacent to every vertex in W_1 and $x \in W_2$. But $W_1 \cup W_2 = W$ is an independent set, a contradiction.) $|W_1| + |W_2| = \beta_0(G)$. Therefore $|N(W_1)| + |W_2| > \beta_0(G)$. That is $|V_2| > \beta_0(G)$, a contradiction. Hence the theorem.

Corollary 4.3. *A graph G has exactly two disjoint β_0-sets whose union is $V(G)$ if G is of even order and contains a spanning cycle $u_1, u_2, \ldots, u_{2n}$ such that whenever u_i, u_j are adjacent, then i, j are of opposite parity.*

Proof. Suppose G is of even order and contains a spanning cycle $u_1, u_2, \ldots, u_{2n}$ such that whenever u_i, u_j are adjacent, then i, j are of opposite parity. Then $\{u_1, u_3, \ldots, u_{2n-1}\}$, $\{u_2, u_4, \ldots, u_{2n}\}$ are the only β_0-sets of G whose union is $V(G)$. The converse is not true.

(i) Consider G.





There are exactly two disjoint β_0-sets $\{1, 3, 5, 7, 9, 11, u'\}, \{2, 4, 6, 8, 10, 12, u\}$ whose union is $V(G)$. G has no spanning cycle. For: consider $S = \{4, 8, 5, 11\}$. $\omega(G - S) = 5$ and the five components are $\{u\}, \{u'\}, \{9, 10\}, \{6, 7\}, \{1, 2, 3, 12\}$.

(If G has spanning cycle, then for any $S \subseteq V(G), \omega(G - S) \leq |S|$.)

(ii) Consider $C_{2n}, (n \geq 6)$. Let $V(C_{2n}) = \{u_1, u_2, \ldots, u_{2n}\}$. Add two more vertices u, u'. Join u with u_{2n-1}, u_{2n-3} and u' with u_{2n-4}, u_{2n-6}. Let G be the resulting graph. Let $S = \{u_{2n-1}, u_{2n-3}, u_{2n-4}, u_{2n-6}\}$. The components of $G - S$ are $\{u\}, \{u'\}, \{u_{2n-2}\}, \{u_{2n-5}\}, \{u_{2n}, 1, \ldots, u_{2n-7}\}$.

Therefore $\omega(G - S) > |S|$. Therefore G can not contain a spanning cycle. But G has exactly two β_0-sets namely $\{1, 3, 5, \ldots, (2n-1), u'\}, \{2, 4, \ldots, 2n, u\}$.

12. Let G have two disjoint β_0-sets V_1, V_2 whose union is $V(G)$. Then

(a) G has no isolates.

(b) $N(V_1) = V_2$ and $N(V_2) = V_1$.

(c)If $G \neq K_2$, then $\delta(G) \geq 2$.

Proof. (a) Suppose G has an isolate say u. Let $u \in V_1$. Then $V_2 \cup \{u\}$ is an independent set of cardinality $\beta_0(G)+1$, a contradiction. Therefore G has no isolates.

(b) Suppose $N(V_1) \subset V_2$. Let $v \in V_2 - N(V_1)$. Then v is an isolate of G, a contradiction. Therefore $N(V_1) = V_2$. Similarly, $N(V_2) = V_1$.

(c) Let $u \in V_1$. (Similar proof holds if $u \in V_2$). If $|V_1| = 1$, then $G = K_2$, a contradiction. Therefore $|V_1| > 1$. Let $A = \{u\}$. Since $|N(A)| > |A|$, $|N(A)| \geq 2$. Therefore $deg(u) \geq 2$. Therefore $\delta(G) \geq 2$.

13. Let G have exactly two disjoint β_0-sets $V_1(G)$ and $V_2(G)$ whose union is $V(G)$. Then G is connected.

Proof. Suppose G is disconnected. Let G_1 be a component of G and
$G_2 =< V(G) - V(G_1) >$. Let $V_1 \cap V(G_1) = A$, $V_2 \cap V(G_1) = D$, $V_1 \cap V(G_2) = C$ and $V_2 \cap V(G_2) = B$.
Since $\phi \neq A \subset V_1$. Then $|N(A)| > |A|$, (using property 11). $N(A) \subset D$. Therefore $V_2 - N(A) \supset B$ (since $B \cup D = V_2$). Therefore $|V_2 - N(A)| \geq |B|$. $|V_2| = |N(A)| + |V_2 - N(A)|$ and hence $\beta_0(G) > |A|+|B|$. Similarly, $|V_1| = \beta_0 > |C|+|D|$. Therefore $|A|+|B|+|C|+|D| < 2\beta_0(G)$. But $|V_1| = |A|+|C|$. $|V_2| = |B|+|D|$. $|V_1|+|V_2| = |A|+|B|+|C|+|D|$.

Then $2\beta_0(G) = |A| + |B| + |C| + |D|$, a contradiction. Therefore G is connected.

14. Every just β_0-excellent graph $G \neq \overline{K_n}$ is connected.

Proof. Suppose G is not connected. Since $G \neq \overline{K_n}$, one of the connected components of G, say G_1, has at least two vertices.

Claim: G_1 is a just β_0-excellent graph.

Let $u \in V(G_1)$. Then there exists a unique β_0-set say S of G containing u. Let $S_1 = S \cap V(G_1)$. Then S_1 is an independent set of G_1 containing u. Suppose S_1 is not a β_0-set of G_1. Then $|S_1| < \beta_0(G_1)$. Let $S_2 = S \cap V(G - G_1)$. Then $S = S_1 \cup S_2$ and S_1 and S_2 are disjoint. Therefore $\beta_0(G) = |S| = |S_1| + |S_2| < \beta_0(G_1) + \beta_0(< G - G_1 >)$. But $\beta_0(G) = \beta_0(G_1)+\beta_0(< G-G_1 >)$, a contradiction. Therefore S_1 is a β_0-set of G. G_1 is β_0-excellent graph.
Let $u \in V(G_1)$. Suppose A and B are β_0-sets of G_1 containing u. Let C be any β_0-set of $< G - G_1 >$. Then $A \cup C, B \cup C$ are β_0-sets of G containing u, a contradiction, since G is just β_0-excellent. Therefore G_1 is just β_0-excellent. Since G_1 is connected and of order ≥ 2, there are at least two β_0-sets in G_1. Let A_1, B_1 be two β_0-sets of G_1. Let C be a β_0-set of $< G-G_1 >$. Then $C \cup A_1, C \cup B_1$ are two β_0-sets of G containing C which is non empty, a contradiction. Therefore G is connected.

15. Let G be a just β_0-excellent graph.

Let $u \in V(G)$. Let S be the unique β_0-set of G containing u. Then $< pn[u,S] >$ is complete and $|pn[u,S]| \leq \chi(G)$.

Proof. Let $x, y \in pn(u,S)$. Then u is adjacent to x, y. Also x, y are not adjacent to any vertex of $S - \{u\}$. If x, y are not adjacent, then $(S - \{u\}) \cup \{x, y\}$ is an independent set of G of cardinality $\beta_0(G) + 1$, a contradiction. Therefore $N[u]$ is complete. Since G is just β_0-excellent, there exist at least $|N[u]|$ β_0-sets in G. Therefore $pn[u,S] \leq$ number of β_0-sets of $G = \chi(G)$.

16. There are graphs for which $pn[u,S] = \chi(G)$.

Consider $K_{4,4,4}$. Let $V(K_{4,4,4})$ be the set of all elements u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4 where $\{u_1, u_2, u_3, u_4\}$, $\{v_1, v_2, v_3, v_4\}$, $\{w_1, w_2, w_3, w_4\}$ are β_0-sets. Remove the edges $v_1u_2, v_1u_3, v_1u_4, w_1u_2, w_1u_3, w_1u_4$.

Let G be the resulting graph. G is just β_0-excellent having the three β_0-sets, $\{u_1, u_2, u_3, u_4\}, \{v_1, v_2, v_3, v_4\}$, and $\{w_1, w_2, w_3, w_4\}$. Let $S = \{u_1, u_2, u_3, u_4\}$. $pn[u,S] = \{u_1, v_1, w_1\}$. Then $|pn[u,S]| = 3 = \chi(G)$.

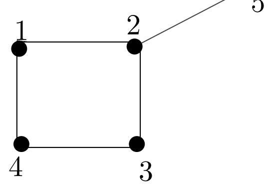
17. Let G be a bipartite just β_0-excellent graph and $G \neq K_2$. Let $u \in V(G)$. Let S be the unique β_0-set of G containing u. Then $pn[u,S] = \{u\}$.

Proof. Since G is bipartite, $\chi(G) = 2$. Since G is just β_0-excellent, number of β_0-sets of G = $\chi(G) = 2$. If for any $u \in V(G)$, $pn[u,S] \supset \{u\}$, then there exists $u \in pn[u,S], v \neq u$. Also, if V_1, V_2 is the bipartition and if $u \in V_1$, then $(V_1 - \{u\}) \cup \{v\}$ is a β_0-set, contradicting the fact that there are exactly two β_0-sets. Therefore $pn[u,S] = \{u\}$.

Remark 4.4. $|pn[u,S]| = 1 < 2 = \chi(G)$.

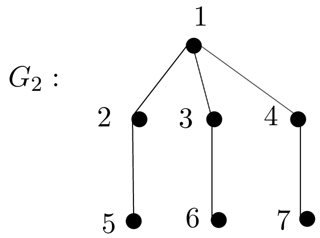
Example: 1

G_1 :



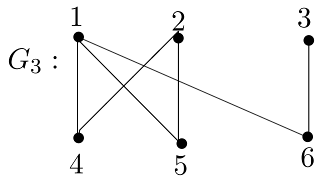
G_1 is γ-excellent but not β_0-excellent.

Example :2



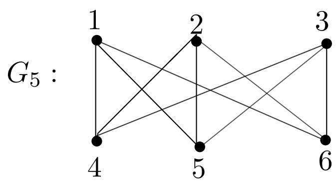
G_2 is neither γ nor β_0-excellent.

Example:3



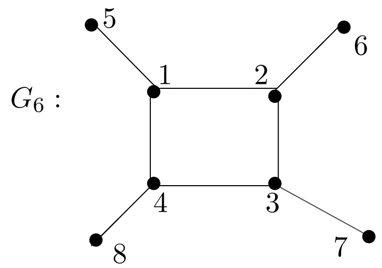
G_3 is β_0-excellent but not γ-excellent.

Example:4



G_5 is just β_0-excellent but not γ-just excellent.

Example:5



G_6 is neither just γ-excellent nor β_0-excellent.

Example:6 C_9 is just γ-excellent but not just β_0-excellent. C_9 is β_0-excellent.

Example:7 K_n is both just γ-excellent and just β_0-excellent.

Remark 4.5. *Q_n is just β_0-excellent ($\beta_0(Q_n) = 2^{n-1}$, each vertex is n-regular and $\chi(Q_n) = 2$).*

Theorem 4.6. *A graph G is just β_0-excellent if and only if*

(i) $\beta_0(G)$ divides n.

(ii) G has exactly $\frac{n}{\beta_0(G)}$ distinct β_0-sets.

(iii) The maximum cardinality of a partition of $V(G)$ into independent sets is $\frac{n}{\beta_0(G)}$.

Proof. Let G be a just β_0-excellent. Let $S_1, S_2, \ldots, S_m$ be the collection of distinct β_0-sets of G. Since G is just β_0-excellent, these sets are pairwise disjoint and their union is $V(G)$. Therefore (i),(ii) and (iii) follows.

Conversely, let G be a graph satisfying the conditions (i), (ii) and (iii). Let $n = m\beta_0(G)$. By condition (iii), there exist independent sets $V_1, V_2, \ldots, V_m$ such that they are pairwise disjoint and $V_1 \cup V_2 \cup \ldots \cup V_m = V$.
Therefore $n = \sum_{i=1}^{m} |V_i| \leq m\beta_0(G)$. Since $n = m\beta_0(G)$, each V_i is a maximum independent sets of G. Therefore $V = V_1 \cup V_2 \cup \ldots \cup V_m$ and V_i's are pairwise disjoint β_0-sets. Therefore G is β_0-excellent. Since G has exactly $\frac{n}{\beta_0(G)}$ $(= m)$ distinct β_0-sets, $V_1, V_2, \ldots, V_m$ are the only β_0-sets of G. Therefore G is just β_0-excellent.

Observation 4.7. *Let G be a just β_0-excellent graph. Then $\Delta(G) \leq (\chi(G) - 1)\beta_0(G)$.*
Proof. Let $u \in V(G)$. Let $deg(u) > (\chi(G) - 1)\beta_0(G)$. u is not adjacent to at least $\beta_0(G) - 1$ vertices. $deg_G(u) + deg_{\overline{G}}(u) = n - 1$.

Therefore $n - 1 > (\chi(G) - 1)\beta_0(G) + \beta_0(G) - 1 = \chi(G)\beta_0(G) - 1 = n - 1$, a contradiction.

Therefore $deg_G(u) \leq (\chi(G) - 1)\beta_0(G)$. Therefore $\Delta(G) \leq (\chi(G) - 1)\beta_0(G)$.

Remark 4.8. *The upper bound is reached in $G = K_{n_1,n_2,\ldots,n_r}$, where $n_1 = n_2, \ldots = n_r = n$.($\chi(G) = r, \beta_0(G) = n, deg(u) = (r-1)n) = (\chi(G) - 1)\beta_0(G)$.*

Theorem 4.9. *Let G, H be just β_0-excellent graphs and $G \neq \overline{K_n}$, $H \neq \overline{K_n}$.*

Then (i) $G \cup H$ is not just β_0-excellent.

(ii) $G + H$ is just β_0-excellent if and only if $\beta_0(G) = \beta_0(H)$.
Proof. (i) Since $G \neq \overline{K_n}, H \neq \overline{K_n}$, G has at least two β_0-sets and H has at least two β_0-sets. Let $u \in V(G)$. Then there exists a unique β_0-set S in G containing u. Let T_1, T_2 be two β_0- sets of H. Then $S \cup T_1, S \cup T_2$ are two β_0-sets of $G \cup H$ containing u. Therefore $G \cup H$ is not just β_0-excellent.
(ii) Suppose $G + H$ is just β_0-excellent. Then $G + H$ is β_0-excellent. Therefore $\beta_0(G) = \beta_0(H)$.
Conversely, let $\beta_0(G) = \beta_0(H)$. Any β_0-set of $G+H$ is either a β_0-set of G or a β_0-set of H. Since G, H are just β_0-excellent, we get that $G + H$ is just β_0-excellent.

Theorem 4.10. *Let G be a just β_0-excellent and let $G \neq \overline{K_n}$ and G be not a bipartite graph. Then $\beta_0(G) \leq \frac{n}{3}$, where $n = |V(G)|$.*

Proof. Since G is not bipartite and G is just β_0-excellent, there are at least three β_0-sets. Therefore $\frac{n}{\beta_0(G)} \geq 3 \Rightarrow \beta_0(G) \leq \frac{n}{3}$.

Remark 4.11. *$K_{r,r,r}$ is a just β_0-excellent graph in which $\beta_0(G) = \frac{n}{3}$.*

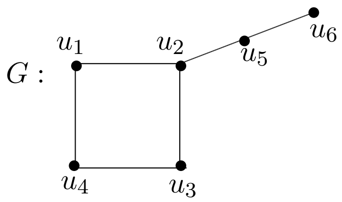
Theorem 4.12. *Every graph is an induced subgraph of a just β_0-excellent graph.*

Proof. Let G be a graph. Let $S_{11}, S_{12}, \ldots, S_{1k_1}$ be disjoint β_0-sets of G.

($k_1 \geq 1$). Let $G_1 = G - (S_{11} \cup S_{12} \cup \ldots \cup S_{1k_1})$. Let $S_{21}, S_{22}, \ldots, S_{2k_2}$ be disjoint β_0-sets of G_1. Proceeding in this manner, we get a partition π of $V(G)$ into independent sets such that the first set of k_1 independent sets are β_0-sets of G. Add new vertices such that each partite set in π have cardinality $\beta_0(G)+1$. Make the new vertices adjacent to all the vertices in the partite sets of π other than that in which they lie. It is easy to see that the resulting graph is just β_0-excellent with independence number $\beta_0(G)+1$.

Addition of vertices to G such that each partite set in π has cardinality $\beta_0(G)$ may not give a just β_0-excellent graph.

Example 4.13.



β_0-sets of G are $\{u_1, u_3, u_5\}$, $\{u_2, u_4, u_6\}$,$\{u_1, u_3, u_6\}$.

Here $\pi = \{\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}\}$ and $\cup_{S\in\pi} S = V(G)$. If we add no vertex, we get G itself which is β_0-excellent but not just β_0-excellent.

Definition 4.14. *Let G be any graph. Suppose G is not just β_0-excellent. Let H be a just β_0-excellent graph of minimum order containing G as an induced subgraph. Then $|V(H)| - |V(G)|$ is called just β_0-excellent embedding index of G and is denoted by $em_{\beta_0}(G)$.*

Remark 4.15. $em_{\beta_0}(G) \leq t(\beta_0 + 1) - n$.

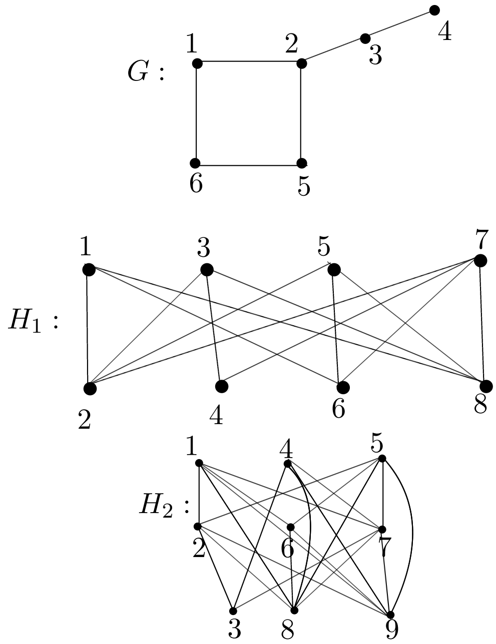
Definition 4.16. *Let G be a graph. Suppose G is not just β_0-excellent graph. Let H be a just β_0-excellent graph of minimum independence number containing G as an induced subgraph. Then $|\beta_0(H)| - |\beta_0(G)|$ is called just β_0-excellent embedding independent index of G and is denoted by $emi_{\beta_0}(G)$.*

Remark 4.17. *(1) Since G is an induced subgraph of H, $\beta_0(G) \leq \beta_0(H)$.*

(2) $0 \leq emi_{\beta_0}(G) \leq 1$.

(3) There are graphs in which $emi_{\beta_0}(G) = 0$.

Example 4.18.



$\beta_0(H_1) = 4$, $\beta_0(H_2) = 3$. G is an induced subgraph of both H_1 and H_2 . H_2 is a graph with minimum independence number containing G as an induced subgraph. Thus $emi_{\beta_0}(G) = 0$.

Remark 4.19. *If G is not β_0- excellent and G has a unique β_0-set, then $emi_{\beta_0}(G) = 0$.*

Remark 4.20. *If G is just β_0-excellent, then $H = G$ and hence $emi_{\beta_0}(G) = em_{\beta_0}(G) = 0$.*

Remark 4.21. *Let G be a non just β_0-excellent graph. G is said to belong to emi-C_1 class if $emi_{\beta_0}(G) = 0$ and emi-C_2 class if $emi_{\beta_0}(G) = 1$.*

Example 4.22. *(1) $K_{1,n}$ belongs to emi-C_1 class.*

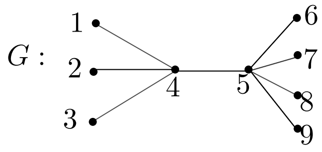
(2) C_{2n+1} belongs to emi-C_2 class .

Open Problem:

Characterize emi-C_1 class and emi-C_2 class.

Remark 4.23. *Consider $D_{r,s}$. It has a unique β_0-set. Any chromatic partition consists of two sets. If we consider a chromatic partition and add new vertices and edges as in the theorem, then we may not get a just β_0-excellent graph.*

Example 4.24.

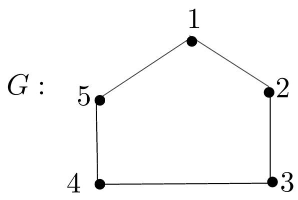


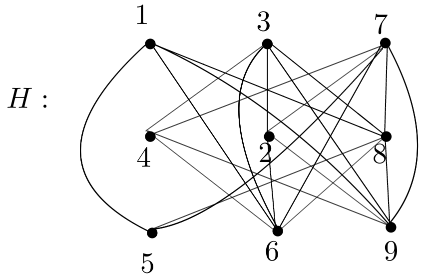
$\beta_0(H) = 7$ and $\{1,2,3,6,7,8,9\}$ is the unique β_0-set of H. Then H is not even β_0-excellent. Hence the partition of $V(G)$ into independent sets is to be done in the manner described in the theorem.

Remark 4.25. *Let π be the partition of $V(G)$ as in the theorem. Then the number of new vertices added is $|\pi|(\beta_0(G)+1)-n$.*

Proof. Let $\pi = \{V_1, V_2, \ldots, V_k, \ldots, V_t\}$, where $V_1, V_2, \ldots, V_k$ are β_0-sets ($k \geq 1$) of G and the remaining sets are independent having cardinality $< \beta_0(G)$. The number of vertices added to $G = k + \sum_{i=k+1}^{t}(\beta_0(G)+1-|V_i|) = k+(t-k)(\beta_0+1)-(n-k\beta_0) = t\beta_0+t-n = t(\beta_0+1)-n$.

Illustration 4.26.





The β_0-sets of C_5 are $\{1,3\}$, $\{1,4\}$,$\{2,4\}$,$\{2,5\}$. Hence C_5 is not β_0-excellent. But for H, the β_0-sets are $\{1,3,7\}$, $\{2,4,8\}$ and $\{5,6,9\}$. H is just β_0-excellent graph; $\beta_0(C_5) = 2$, $\beta_0(H) = 3$ and the number of new vertices added is 4.

4.3 Just β_0 excellence in Product graphs

Theorem 4.27. *Let H be a graph. If $n = \chi(H)$, then $K_n \Box H$ is just β_0-excellent and if $n > \chi(H)$, then $K_n \Box H$ is not just β_0-excellent.*

Proof follows from the theorem 2.19.

Observation 4.28. *Let H be a graph. $\overline{K_n} \Box H$ is just β_0-excellent if and only if H is just β_0-excellent.*

Theorem 4.29. *If every vertex of H belongs to an union of disjoint independent sets of H of maximum cardinality, then $K_n \Box H$ is not just β_0-excellent*

Proof. Suppose every vertex of H belongs to an union of disjoint independent sets of H of maximum cardinality. Then by theorem 2.22, $K_n \Box H$ is β_0-excellent.

Suppose $\{S_1, S_2, \ldots, S_n\}$ and $\{X_1, X_2, \ldots, X_n\}$ are collections of disjoint independent sets of H with union having maximum cardinality. If $S_i \cap X_j \neq \phi$, for some i, j, then as seen in theorem 2.22, any element of $S_i \cap X_j$ is contained in two maximum independent sets and hence $K_n \Box H$ is not just β_0-excellent. Suppose $S_i \cap X_j = \phi$, for every i, j.

Claim: For some order of $\{S_1, S_2, \ldots, S_n\}$, $|S_i| = |X_i|, 1 \leq i \leq n$.

Let $\sum_{i=1}^{n}|S_i| = t$. Then $\sum_{i=1}^{n}|X_i| = t$. Suppose $|S_i| < |X_i|$. Then $|S_1| + |S_2| + \ldots + |S_{i-1}| + |S_{i+1}| + \ldots + |S_n| > |X_1| + |X_2| + \ldots + |X_{i-1}| + |X_{i+1}| + \ldots + |X_n|$. Therefore $|X_i| + |S_1| + \ldots + |S_{i-1}| + |S_{i+1}| + \cdots + |S_n| > t$. Since $S_i \cap X_j = \phi$, we have disjoint independent sets of H, $X_i, S_1, S_2, \ldots, S_{i-1}, S_{i+1}, \ldots S_n$ such that $|X_i| + |S_1| + \ldots + |S_{i-1}| + |S_{i+1}| + \ldots + |S_n| > t$, a contradiction. Similarly, if $|S_i| > |X_i|$, we get a contradiction. Therefore $|S_i| = |X_i|, 1 \leq i \leq n$. Let $v \in S_1$. Then as seen in theorem 2.22, $S_1, S_2, \ldots, S_n$ as well as $S_1, X_2, \ldots, X_n$ give rise to β_0-set of $K_n \Box H$ and (u_i, v) belongs to at least two β_0-sets of $K_n \Box H$. Therefore $K_n \Box H$ is not just β_0-excellent.

Theorem 4.30. *Let G be a bipartite graph. $G \Box C_{2m}$ is just β_0-excellent and $G \Box C_{2m+1}$ is not just β_0-excellent.*

Proof follows from theorem 2.46

Theorem 4.31. *The following are just β_0-excellent graphs.*

(i) $P_{2n} \Box P_{2k+1}$ is just β_0-excellent.
(ii) $P_{2n} \Box C_{2k}$ is just β_0-excellent.
(iii) $P_{2n+1} \Box C_{2k}$ is just β_0-excellent.

Proof follows from remark 2.30 and theorem 2.31.

Theorem 4.32. *The following are not just β_0-excellent graphs.*

(i) $P_{2n} \Box C_{2k+1}$
(ii) $P_{2n+1} \Box C_{2k+1}$

4.4 Just β_0-excellence in Generalized Petersen graphs $P(n,k)$

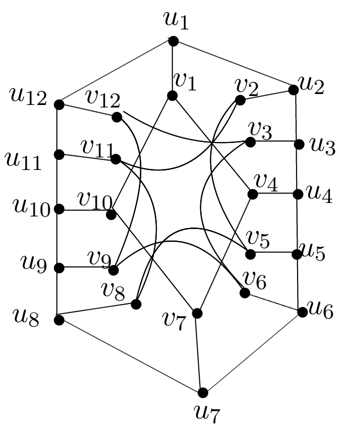
Definition 4.33. *Generalised Petersen Graphs $P(n,k)$: For each $n \geq 3$ and $0 < k < n$, $P(n,k)$ denotes the generalised Petersen graph with vertex set $V(G) = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and the edge set $E(G) = \{u_i u_{i+1(mod\ n)}, u_i v_i, v_i v_{i+k(mod\ n)}\}, 1 \leq i \leq n$.*

Theorem 4.34. *$P(2n,k)$ is just β_0-excellent if k is odd.*

Proof. Let $\{u_1, u_2, \ldots, u_n\}$ be the vertices in the outer circle and $\{v_1, v_2, \ldots, v_n\}$ be the remaining vertices. Let $S_1 = \{v_1, u_2, v_3, u_4, \ldots, v_{2n-1}, u_{2n}\}$ and $S_2 = \{u_1, v_2, u_3, v_4, \ldots, u_{2n-1}, v_{2n}\}$. Then S_1, S_2 are disjoint β_0-sets of $P(2n,k)$. Clearly for any nonempty proper subset A of S_1 or S_2, $|N(A)| > |A|$. Therefore $P(2n,k)$ is just β_0-excellent.

Illustration 4.35.

$P(12,3)$:



The β_0-sets are
$\{u_1, v_2, u_3, v_4, u_5, v_6, u_7, v_8, u_9, v_{10}, u_{11}, v_{12}\}$,
$\{v_1, u_2, v_3, u_4, v_5, u_6, v_7, u_8, v_9, u_{10}, v_{11}, u_{12}\}$.
Hence $P(12,3)$ is just β_0-excellent.

Theorem 4.36. *$P(n,1)$, n odd is β_0-excellent but not just β_0-excellent.*

Proof. Let

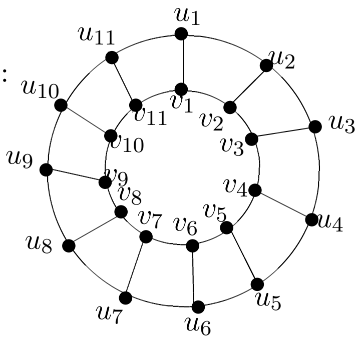
$V(P(n,1) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$.

$E(P(n,1)) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_i, (mod\, n)\}$, where $1 \leq i \leq n$,

$\beta_0(P(n,1)) = n-1$. The following are β_0-sets $\{u_1, v_2, u_3, v_4, \ldots, u_{n-2}, v_{n-1}\}$, $\{v_1, u_2, v_3, u_4, \ldots, v_{n-2}, u_{n-1}\}$, $\{u_n, v_1, u_2, v_3, u_4, \ldots, u_{n-3}, v_{n-2}\}$ and $\{v_n, u_1, v_2, \ldots, v_{n-3}, u_{n-2}\}$. Therefore $P(n,1)$, n is odd is β_0-excellent. Clearly, it is not just β_0-excellent.

Illustration 4.37.

$P(11,1)$:



The β_0-sets of $P(11,1)$ are
$\{u_1, v_2, u_3, v_4, u_5, v_6, u_7, v_8, u_9, v_{10}\}$,
$\{v_1, u_2, v_3, u_4, v_5, u_6, v_7, u_8, v_9, u_{10}\}$,
$\{u_{11}, v_1, u_2, v_3, u_4, v_5, u_6, v_7, u_8, v_9\}$ and
$\{u_1, v_2, u_3, v_4, u_5, v_6, u_7, v_8, u_9, v_{11}\}$. Hence all the vertices are in at least one β_0-set. Hence $P(11,1)$ is β_0-excellent, but clearly $P(11,1)$ is not just β_0-excellent.

Theorem 4.38. *$P(n,3)$, n odd is β_0-excellent but not just β_0-excellent*

Proof. Let

$V(P(n,3) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$,

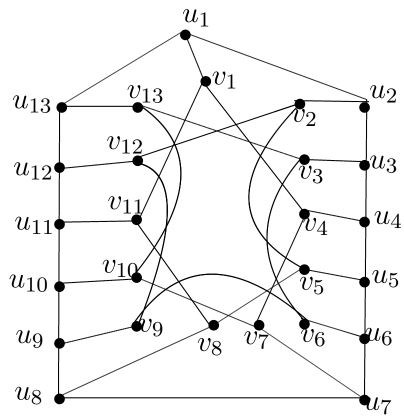
$E(P(n,3)) = \{u_i u_{i+1}, v_i v_{i+3}, u_i v_i, (mod\, n)\}$ where $1 \leq i \leq n$. $\beta_0(P(n,3)) = n-2$.

The following are β_0-sets $\{u_1, v_2, u_3, v_4, \ldots, u_{n-2}, v_{n-2}\}$, $\{v_1, u_2, v_3, u_4, \ldots, v_{n-2}\}$, $\{u_{n-1}, v_n, u_1, v_2, \ldots, u_{n-3}\}$ and $\{v_{n-1}, u_n, v_1, u_2, \ldots, v_{n-3}\}$, $\{u_n, v_1, u_2, \ldots, u_{n-3}\}, \{v_n, u_1, v_2, \ldots, v_{n-3}\}$.

Therefore $P(n,3)$, n is odd is β_0-excellent. Clearly, it is not just β_0-excellent. Hence the result.

Illustration 4.39.

$P_{13,3}$:



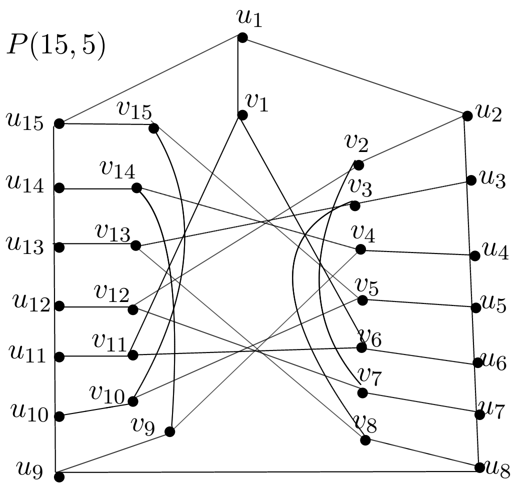
The β_0-sets are
$\{u_1, v_2, u_3, v_4, u_5, v_6, u_7, v_8, u_9, v_{10}, u_{11}\}$,
$\{v_1, u_2, v_3, u_4, v_5, u_6, v_7, u_8, v_9, u_{10}, v_{11}\}$,
$\{u_{12}, v_{13}, u_1, v_2, u_3, v_4, u_5, v_6, u_7, v_8, u_9\}$,
$\{u_{13}, v_1, u_2, v_3, u_4, v_5, u_6, v_7, u_8, v_9, u_{10}\}$.

Therefore $P_{13,3}$ is β_0-excellent. Clearly, $P_{13,3}$ is not just β_0-excellent.

Theorem 4.40. *$P(n,5)$, n odd is β_0-excellent but not just β_0-excellent*

Proof. Let
$V(P(n,5)) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$, $E(P(n,5)) = \{u_i u_{i+1}, v_i v_{i+5}, u_i v_i, 1 \le i \le n\ (mod\, n)\}$. $\beta_0(P(n,3)) = n-3$. The following are β_0-sets $\{u_1, v_2, u_3, v_4, \ldots, v_{n-5}, u_{n-4}, u_{n-2}\}$, $\{v_1, u_2, v_3, u_4, \ldots, u_{n-5}, v_{n-4}, v_{n-2}\}$. Similar β_0-sets can be written starting with $u_{n-1}; v_{n-1}; u_n; v_n; u_{n-3}; v_{n-3}$. Therefore $P(n,5)$, n odd is β_0-excellent. Clearly, it is not just β_0-excellent.

Illustration 4.41.



The β_0-sets of $P(15,5)$ are
$\{u_1, v_2, u_3, v_4, u_5, v_6, u_7, v_8, u_9, v_{10}, u_{11}, u_{13}\}$,
$\{v_1, u_2, v_3, u_4, v_5, u_6, v_7, u_8, v_9, u_{10}, v_{11}, v_{13}\}$,
$\{u_{15}, v_{14}, u_{13}, v_{12}, u_{10}, v_9, u_8, v_7, u_6, v_5, v_4, u_3, \}$.
Therefore $P(15,5)$ is β_0-excellent and it is not just β_0-excellent.

Theorem 4.42. *$P(n,2)$, n odd is β_0-excellent but not just β_0-excellent*

Proof. Let
$V(P(n,2)) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$,
$E(P(n,2)) = \{u_i u_{i+1}, v_i v_{i+2}, u_i v_i, (mod\, n)\}$
where $1 \le i \le n$. $\beta_0(P(n,2)) = [\frac{4n}{5}]$. The following are β_0-sets $\{u_1, v_2, u_3, v_4, \ldots, v_{n-4}, u_{n-3}, u_{n-1}\}$, $\{v_1, u_2, v_3, u_4, \ldots, u_{n-4}, v_{n-3}, v_{n-1}\}$. Similar β_0-sets can be written starting with the remaining vertices of $P(n,2)$. Therefore $P(n,2)$, n is odd is β_0-excellent. Clearly, it is not just β_0-excellent.

4.5 Just β_0-excellence of Harary graphs

The β_0-excellence of Harary graph has been discussed in the third section. Based on the results in that section, the just β_0-excellence of Harary graphs are discussed here.

Observation 4.43. *The condition $\left\lfloor \frac{n-r}{r+1} \right\rfloor \neq \frac{j-i}{r+1}$ is not sufficient to ensure that $H_{2r,n}$ is not just β_0- excellent.*

Consider $H_{5,9}$. Here $r = 2$, $n = 9$,$n - r = 7$. $t = \frac{n-r}{r+1} = \frac{7}{3} = 2$. Let $j = 3, i = 0$. $\frac{j-i}{r+1} = \frac{3}{3} = 1 \neq \left\lfloor \frac{n-r}{r+1} \right\rfloor$.

But $S_3 = \{3,6,0\}, S_0 = \{0,3,6\}$ are not distinct. Here $3 \in S_0$ and $0 \in S_3$.

Remark 4.44. *The condition $j \notin S_i$ (or) S'_i implies that $\left\lfloor \frac{n-r}{r+1} \right\rfloor \neq \frac{j-i}{r+1}$*

Suppose $\left\lfloor \frac{n-r}{r+1} \right\rfloor = \frac{j-i}{r+1}$. Then $\frac{j-i}{r+1} = t$, where $t = \left\lfloor \frac{n-r}{r+1} \right\rfloor$.

Therefore $j - i = t(r+1)$. Therefore $j = t(r+1) + i$. Therefore $j \in S_i$, a contradiction.

Remark 4.45. *The condition that $\left\lfloor \frac{n-r}{r+1} \right\rfloor \neq \frac{j-i}{r+1}$ need not imply that $j \notin S_i$ and S'_i.*

Consider $H_{5,9}$. Here $r = 2$, $n = 9$,$n - r = 7$. $t = \frac{n-r}{r+1} = \frac{7}{3} = 2$.

Let $j = 3, i = 0$. $S_0 = \{0,3,6\}, S'_0 = \{0,3,6\}$. $3 \in S_0$ and S'_0.

$\frac{j-i}{r+1} = \frac{3-0}{3} = 1 \neq \frac{n-r}{r+1}$. But $j \in S_i$ and $j \in S'_i$.

Theorem 4.46. *Let $j - i = q(r+1)$, $q > 0$. Then $q \le 2t$ and q can be written as $t - m$, where $m \ge -t$, $t = \left\lfloor \frac{n-r}{r+1} \right\rfloor$.*

Proof. Suppose $r+1$ divides $j-i$. Let $j-i = (r+1)q$. Write $q = l - m$, $l \le t$, where $t = \left\lfloor \frac{n-r}{r+1} \right\rfloor$.Suppose $q > 2t$ and $n - r = q_1(r+1) + \alpha_1$, $0 \le \alpha_1 < r+1$. $2t(r+1) = 2\left\lfloor \frac{n-r}{r+1} \right\rfloor (r+1) = 2\left\lfloor \frac{q_1(r+1)+\alpha_1}{r+1} \right\rfloor (r+1) = 2q_1(r+1) = 2n - 2r - 2\alpha_1$.

$2t(r+1) = n + (n - 2r - \alpha_1) - \alpha_1$. $q > 2t$ implies that $q \ge 2t+1$.

Therefore,

$$\begin{aligned} q(r+1) &\ge (2t+1)(r+1) \\ &= 2t(r+1) + (r+1) \\ &= n + (n - 2r - \alpha_1) + (r + 1 - \alpha_1)). \end{aligned}$$

Since $n - r = q_1(r+1) + \alpha_1, n - 2r - \alpha_1 = q_1(r+1) - r$ ($q_1 = 0$ implies that $n - r = \alpha_1 < r+1$ implies that $n < 2r+1$, a contradiction, since $n \ge 2r+1$).

Therefore, $q_1 \ge 1$. Therefore, $n - 2r - \alpha_1 > 0$. Also, $\alpha_1 < r+1$. Therefore, $j - i = q(r+1) > n$, a contradiction. Thus, $q \le 2t$. Suppose $q \le t$. Then $q = l - m$, where $l = t, m \ge 0$. Suppose

$t < q \leq 2t$. Then $q = t - m$, where $m \geq -t$. Thus for $j - i = q(r+1)$ with $q > 0$, we can always write $q = l - m, l = t, m \geq -t$.

Theorem 4.47. *$H_{2r,n}$ is not just β_0-excellent if and only if there exist $i, j (i < j), 0 \leq i, j \leq n-1$ such that $r+1$ divides $j-i$ (or) $j-i-n$ and j does not belong to S_i or S_i', where $S_i = \{i, r+1+i, \ldots, t(r+1)+i\}$ and $S_i' = \{i, i-(r+1), \ldots, i-t(r+1)\}$, where $t = \left\lfloor \frac{n-r}{r+1} \right\rfloor$.*

Proof. Suppose $(r+1)$ divides $j-i$. Then by the theorem 4.46, $j-i = q(r+1)$ and $q = t-m$, where $m \geq -t$. Therefore $(t-m)(r+1) = j-i$. Therefore $t(r+1)+i = m(r+1)+j$. The two β_0-sets $S_i = \{i, r+1+i, \ldots, t(r+1)+i\}$, $S_j' = \{j, j-(r+1), \ldots, m(r+1)+j, \ldots, j-t(r+1)\}$ (or) $S_i = \{i, r+1+i, \ldots, t(r+1)+i\}$, and $S_j = \{j, j+(r+1), \ldots, j+t(r+1)\}$ have a common element namely $t(r+1)+i$ according as $m < 0$ (or) $m \geq 0$. $S_i = S_j'$ or $S_i = S_j$ implies $j \in S_i$, a contradiction. Therefore $S_i \neq S_j'$ or $S_i \neq S_j$. Therefore $H_{2r,n}$ is not just β_0-excellent.

A similar proof holds when $r+1$ divides $j-i-n$. Conversely, Suppose $H_{2r,n}$ is not just β_0-excellent. Then there exist distinct β_0-sets S_1, S_2 such that $S_1 \cap S_2 \neq \phi$. Without loss of generality, let $S_1 = \{i, r+1+i, \ldots, t(r+1)+i\}, S_2 = \{j, j+(r+1), \ldots, j+t(r+1)\}$ (or) $S_1 = \{i, r+1+i, \ldots, t(r+1)+i\}$, $S_2 = \{j, j-(r+1), \ldots, m(r+1)+j, \ldots, j-t(r+1)\}$. Since S_1 and S_2 are distinct, i does not belong to S_2 and j does not belong to S_1. Let $l(r+1)+i = m(r+1)+j$ (or) $l(r+1)+i = m(r+1)+j-n$. Then $l \neq 0$, $m \neq 0$. That is $(r+1)(l-m) = j-i$ (or) $(r+1)(l-m) = j-i-n$. Therefore $(r+1)$ divides $j-i$ (or)$j-i-n$.

Observation 4.48. *If $r+1$ divides $n-r$ and $\frac{n-r}{r+1}$ does not divide n, then $H_{2r,n}$ is not just β_0-excellent.*

Proof. Suppose $r+1$ divides $n-r$, then $\beta_0(H_{2r,n}) = \frac{n-r}{r+1}$. If $H_{2r,n}$ is just β_0-excellent, then β_0 divides n. But by hypothesis, $\frac{n-r}{r+1} = \beta_0$ does not divide n. Therefore $H_{2r,n}$ is not just β_0-excellent.

Illustration 4.49. *Consider $H_{5,11}$. Here $r = 2, n = 11, n-r = 9, r+1 = 3$. $t = \frac{n-r}{r+1} - 1 = 3 - 1 = 2$. $2(r+1) = 6$ divides $n+1 = 12$. $\beta_0(G) = \frac{n-r}{r+1} = 3$. So $S_0 = \{0,3,7\}$, $S_1 = \{1,4,8\}$, $S_2 = \{2,5,9\}$, $S_3 = \{3,6,10\}$, $S_4 = \{4,7,0\}$, $S_5 = \{5,8,1\}$, $S_6 = \{6,9,2\}$, $S_7 = \{7,10,3\}$, $S_8 = \{8,0,4\}$, $S_9 = \{9,1,5\}$, $S_{10} = \{10,2,6\}$. $j = 10, i = 1$. $r+1$ divides $j-i$. $S_i = S_1 = \{1,9,5\}$. $S_i' = S_1' = \{1,9,5\}$. $10 \notin S_1$ and S_1'. Therefore $H_{5,11}$ is β_0-excellent but not just β_0-excellent.*

References:

[1] C. Berge, *Some common properties for regularizable graphs, edge-critical graphs and B-graphs*, Lecture Notes in Computer Science, Graph Theory and Alogrithms, Proc. Symp. Res. Inst. Electr. Comm. Tohoku Univ.; Sendi, 1980, **Vol. 108**, 108–123, Berlin, 1987, Springer.

[2] T. W. Haynes, S. T. Hedetniemi and P. J. Slater,*Fundamentals of Domination in Graphs*, Marcel Dekker. Inc., New York, 1998.

[3] G. H. Fricke, T. W. Haynes, S. T. Hetniemi, S. M. Hedetniemi and R. C. Laskar, Excellent trees, *Bull. Inst. Combin. Appl.*, **Vol. 34** (2002), pp 27–38.

[4] N. Sridharan and K. Subramanian, γ-graph of a graph, *Bulletin of Kerala Mathematics Association*, **Vol.5**, 1,(2008, June),17–34.

[5] N. Sridharan and M. Yamuna, Excellent- Just Excellent -Very Excellent Graphs, *Journal of Math. Phy. Sci.*, **Vol.14**, No.5, 1980, 471-475.

[6] N. Sridharan and M. Yamuna, A Note on Excellent graphs, *ARS Combinatoria*, 78(2006), pp. 267–276.

[7] F. Harary, *Graph Theory*, Addison Wesley, Reading Mass (1972).

[8] M. A. Henning anf T. W. Haynes, Total domination excellent trees, *Discrete Math.*, **Vol .263**, (2003), 93-104.

[9] O. Ore, *Theory of Graphs*, Amer. Math. Soc. Colloq. Publication, 38, Providence, (1962).

[10] M. Yamuna, *Excellent Just Excellent Very Excellent Graphs*, Ph.D Thesis, Alagappa University, 2003.