A further improved $(\frac{G'}{G})$- expansion method and the extended tanh- method for finding exact solutions of nonlinear PDEs

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Abstract: In the present article, we construct the exact traveling wave solutions of nonlinear PDEs in mathematical physics via the $(1+1)$ dimensional modified Kawahara equation by using the following two methods: (i) A further improved $(\frac{G'}{G})$- expansion method, where $G = G(\xi)$ satisfies the auxiliary ordinary differential equation $[G'(\xi)]^2 = aG^2(\xi) + bG^4(\xi) + cG^6(\xi)$, where $\xi = x - Vt$ while a, b, c and V are constants. (ii) The well known extended tanh- function method. We show that the exact solutions obtained by these two methods are equivalent. Note that the first method (i) has not been used by anyone before.

Key–Words: $(\frac{G'}{G})$- expansion method, auxiliary equation, extended tanh- function method, traveling wave solutions, modified Kawahara equation.

1 Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors(see for example [1]–[49]) who are interested in nonlinear physical phenomena. Many powerful different methods have been presented by those authors. For integrable nonlinear differential equations, the inverse scattering transform method [2], the Hirota method [10], the truncated Painleve expansion method [43], the Backlund transform method [19]–[21] and the exp-function method [4, 9, 36, 44, 45] are used in looking for the exact solutions. Among non-integrable nonlinear differential equations there is a wide class of the equations that referred to as the partially integrable, because these equations become integrable for some values of their parameters. There are many different methods to look for the exact solutions of these equations. The most famous algorithms are the truncated Painleve expansion method [14], the Weierstrass elliptic function method [13], the tanh- function method [1, 7, 8, 32, 34, 39, 46] and the Jacobi elliptic function expansion method [6, 16, 18, 30, 37, 38, 40]. There are other methods which can be found in [11],[22],[23]-[29],[33].

Wang et al [30] have introduced a simple method which is called the $(\frac{G'}{G})$-expansion method to look for traveling wave solutions of nonlinear PDEs, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = x - Vt$ while V, λ and μ are arbitrary constants. For further references see the articles [3, 5, 7, 20, 41, 42, 48, 49]. Recently El-Wakil et al [7] and Parkes [20] have shown that the extended tanh- function method proposed by Fan [8] and the basic $(\frac{G'}{G})$- expansion method proposed by Wang et al [30] are entirely equivalent in as much as they deliver exactly the same set of solutions to a given evolution equation. This observation has been pointed out recently by Kudryashov [15]. In this article, we introduce an alternative approach which is called a further improved $(\frac{G'}{G})$- expansion method to find the exact traveling wave solutions of some nonlinear PDEs, where $G = G(\xi)$ satisfies the auxiliary ordinary differential equation $[G'(\xi)]^2 = aG^2(\xi) + bG^4(\xi) + cG^6(\xi)$, where a, b and c are constants. This approach has not been used by anyone before. It will play an important role in constructing many exact traveling wave solutions for the nonlinear PDEs via the $(1+1)$ dimensional modified Kawahara equation. The objective of this article is to show that the exact solutions of these two equation obtained by using the further improved $(\frac{G'}{G})$- expansion method and the well known extended tanh- function method are equivalent.

2 Description of a further improved $(\frac{G'}{G})$- expansion method

Suppose we have the following nonlinear partial differential equation

$$F(u, u_t, u_x, u_y, u_{tt}, u_{xt}, ...) = 0, \quad (2.1)$$

where $u = u(x,t)$ is an unknown function, F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. In the following we give the main steps of a further improved $(\frac{G'}{G})$- expansion method:

Step 1. The traveling wave variable

$$u(x,t) = u(\xi), \quad \xi = x - V\,t, \quad (2.2)$$

where V is a constant, permits us reducing Eq. (2.1) to an ODE for $u = u(\xi)$ in the form

$$P(u, u', u'', u''', ...) = 0 \quad (2.3)$$

where $\prime = \frac{d}{d\xi}$.

Step 2. Suppose the solution of Eq.(2.3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows

$$u(\xi) = \sum_{i=0}^{n} \alpha_i \left(\frac{G'}{G}\right)^i, \quad (2.4)$$

where $G = G(\xi)$ satisfies the following auxiliary equation

$$[G'(\xi)]^2 = aG^2(\xi) + bG^4(\xi) + cG^6(\xi), \quad (2.5)$$

where α_i, a, b, c and V are arbitrary constants to be determined provided $\alpha_n \neq 0$. The positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq (2.1) or (2.3).

More precisely, we define the degree of $u(\xi)$ as $D[u(\xi)] = n$ which gives rise to the degree of other expressions as follows

$$\begin{cases} D[\frac{d^q u}{d\xi^q}] = n + q, \\ D[u^p(\frac{d^q u}{d\xi^q})^s] = np + s(q+n). \end{cases} \quad (2.6)$$

Therefore, we can get the value of n in (2.4).

Step 3. Substituting (2.4) into (2.3) and using Eq (2.5), we obtain polynomials in $G^j(\xi)$, $G'(\xi)G^j(\xi)$ $(j = 0, \pm1, \pm2, \cdots)$. Equating each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for α_i, a, b, c and V which can be solved by Maple or Mathematica.

Step 4. The general solutions of the auxiliary equation (2.5) have been well known (see, for example [35, 47]) which can be written in the form

No	$G(\xi)$
1	$\left[\frac{-a\ b\ \mathrm{sech}^2(\sqrt{a}\xi)}{b^2-ac(1+\varepsilon\tanh(\sqrt{a}\xi))^2}\right]^{1/2}, \quad a>0$
2	$\left[\frac{a\ b\ \mathrm{csch}^2(\sqrt{a}\xi)}{b^2-ac(1+\varepsilon\coth(\sqrt{a}\xi))^2}\right]^{1/2}, \quad a>0$
3	$\left[\frac{2a}{\varepsilon\sqrt{\Delta}\cosh(2\sqrt{a}\xi)-b}\right]^{1/2}, \quad a>0, \Delta>0.$
4	$\left[\frac{2a}{\varepsilon\sqrt{\Delta}\cos(2\sqrt{-a}\xi)-b}\right]^{1/2}, \ a<0, \Delta>0$
5	$\left[\frac{2a}{\varepsilon\sqrt{-\Delta}\sinh(2\sqrt{a}\xi)-b}\right]^{1/2}, \ a>0, \Delta<0$
6	$\left[\frac{2a}{\epsilon\sqrt{\Delta}\sin(2\sqrt{-a}\xi)-b}\right]^{1/2}, a<0, \Delta>0$
7	$\left[\frac{-a\ \mathrm{sech}^2(\sqrt{a}\xi)}{b+2\varepsilon\sqrt{ac}\tanh(\sqrt{a}\xi)}\right]^{1/2}, \ a>0,\ c>0$
8	$\left[\frac{-a\ \sec^2(\sqrt{-a}\xi)}{b+2\varepsilon\sqrt{-ac}\tan(\sqrt{-a}\xi)}\right]^{1/2}, \ a<0,\ c>0$
9	$\left[\frac{a\ \mathrm{csch}^2(\sqrt{a}\xi)}{b+2\varepsilon\sqrt{ac}\coth(\sqrt{a}\xi)}\right]^{1/2}, \ a>0,\ c>0$
10	$\left[\frac{-a\ \csc^2(\sqrt{-a}\xi)}{b+2\varepsilon\sqrt{-ac}\cot(\sqrt{-a}\xi)}\right]^{1/2}, \ a<0,\ c>0$
11	$\left[-\frac{a}{b}(1+\varepsilon\tanh(\frac{1}{2}\sqrt{a}\xi)\right]^{1/2}, \ a>0, \Delta=0$
12	$\left[-\frac{a}{b}(1+\varepsilon\coth(\frac{1}{2}\sqrt{a}\xi)\right]^{1/2}, \ a>0, \Delta=0$
13	$\left[\frac{a a e^{2\varepsilon\sqrt{a}\xi}}{(e^{2\varepsilon\sqrt{a}\xi}-4b)^2-64ac}\right]^{1/2}, \ a>0,$
14	$\left[\frac{\varepsilon a e^{2\varepsilon\sqrt{a}\xi}}{1-64ace^{4\epsilon\sqrt{a}\xi}}\right]^{1/2}, \ a>0,\ b=0$

where $\Delta = b^2 - 4ac$ and $\varepsilon = \pm1$.

Step 5. Substituting α_i, V and the general solution of Eq (2.5) into (2.4) we have many exact traveling wave solutions of the nonlinear partial differential equation (2.1).

3 Some applications

In this section, we apply the further improved $(\frac{G'}{G})$- expansion method to construct the exact traveling wave solutions for one dimensional modified Kawahara equation, which are very important nonlinear evolution equations in the mathematical physics and have been paid attention by many researchers.

3.1 Example 1. On solving the modified Kawahara equation by a further improved $(\frac{G'}{G})$- expansion

It is well known [12] that the modified Kawahara equation has the form:

$$u_t + \alpha u u_x + \beta u_{xxx} - \gamma u_{xxxxx} = 0, \tag{3.1}$$

where α, β and γ are arbitrary constants. This equation has been derived by Kawahara [12] as a model for water waves in the long- wave regime for moderate values of surface tension. The Kawahara equation (3.1) gives an appropriate description of several phenomena observed in the dynamics of the water-wave problem.

Let us now solve Eq. (3.1) by the proposed method. To this end, we see that the traveling wave variable (2.2) permits us converting Eq. (3.1) into the following ODE:

$$C - Vu + \frac{1}{2}\alpha u^2 + \beta u'' - \gamma u^{(4)} = 0, \tag{3.2}$$

where C is a constant of integration. Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.2), we deduce from (2.6) that $D(u^{(4)}) = D(u^2)$. Therefore $n + 4 = 2n$ and hence $n = 4$. Thus, we get

$$\begin{aligned} u(\xi) &= \alpha_4\left(\frac{G'}{G}\right)^4 + \alpha_3\left(\frac{G'}{G}\right)^3 + \alpha_2\left(\frac{G'}{G}\right)^2 \\ &\quad + \alpha_1\left(\frac{G'}{G}\right) + \alpha_0. \end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2), collecting all the terms of powers of $\left(\frac{G'}{G}\right)$ and setting each coefficient to zero, we get the following system of algebraic equations:

$$\begin{aligned} &-V\alpha_3 a + \alpha\alpha_3 a^2\alpha_2 + \alpha\alpha_3 a\alpha_0 \\ &+\alpha\alpha_1\alpha_2 a + \alpha\alpha_3 a^3\alpha_4 + \alpha\alpha_1\alpha_4 a^2 \\ &-V\alpha_1 + \alpha\alpha_1\alpha_0 = 0, \end{aligned}$$

$$\begin{aligned} &2\alpha\alpha_4^2 b^3 c + 3\alpha\alpha_4 bc^2\alpha_2 + 168\beta\alpha_4 c^2 b \\ &-31008\gamma\alpha_4 c^2 ab + \frac{3}{2}\alpha\alpha_3^2 c^2 b - 3072\gamma\alpha_2 c^2 b \\ &-8736\gamma\alpha_4 b^3 c + 6\alpha\alpha_4^2 abc^2 = 0, \end{aligned}$$

$$\begin{aligned} &\frac{1}{2}\alpha\alpha_4^2 b^4 - V\alpha_4 c^2 - 840\gamma\alpha_4 b^4 \\ &+\frac{3}{2}\alpha\alpha_3^2 b^2 c + 3\alpha\alpha_4 b^2\alpha_2 c + \alpha\alpha_4 c^2\alpha_0 \\ &+\frac{1}{2}\alpha\alpha_2^2 c^2 + 3\alpha\alpha_4^2 a^2 c^2 + 112\beta\alpha_4 c^2 a \\ &+24\beta\alpha_2 c^2 + 3\alpha\alpha_4 ac^2\alpha_2 - 1320\gamma\alpha_2 b^2 c \\ &+\alpha\alpha_3 c^2\alpha_1 + 6\alpha\alpha_4^2 ab^2 c - 12960\gamma\alpha_4 b^2 ac \\ &+108\beta\alpha_4 b^2 c - 1920\gamma\alpha_2 c^2 a - 7936\gamma\alpha_4 c^2 a^2 \\ &+\frac{3}{2}\alpha\alpha_3^2 c^2 a = 0, \end{aligned}$$

$$\begin{aligned} &-2V\alpha_4 bc - 1280\gamma\alpha_4 b^3 a + \alpha\alpha_2^2 bc \\ &+2\alpha\alpha_4 bc\alpha_0 + 2\alpha\alpha_3 b\alpha_1 c - 120\gamma\alpha_2 b^3 \\ &+\frac{1}{2}\alpha\alpha_3^2 b^3 + 6\alpha\alpha_4 ab\alpha_2 c - 5312\gamma\alpha_4 a^2 bc \\ &-1360\gamma\alpha_2 bac + 3\alpha\alpha_3^2 abc + 6\alpha\alpha_4^2 a^2 bc \\ &+28\beta\alpha_2 bc + 2\alpha\alpha_4^2 ab^3 + \alpha\alpha_4 b^3\alpha_2 \\ &+128\beta\alpha_4 abc + 20\beta\alpha_4 b^3 = 0, \end{aligned}$$

$$\begin{aligned} &-V\alpha_4 b^2 - V\alpha_2 c + 6\beta\alpha_2 b^2 - 2V\alpha_4 ac \\ &+\alpha\alpha_4 b^2\alpha_0 + 3\alpha\alpha_4^2 a^2 b^2 + \frac{3}{2}\alpha\alpha_3^2 a^2 c \\ &+2\alpha\alpha_4^2 a^3 c + \alpha\alpha_2^2 ca + \alpha\alpha_2 c\alpha_0 + \alpha\alpha_3 b^2\alpha_1 \\ &+\frac{3}{2}\alpha\alpha_3^2 b^2 a + 3\alpha\alpha_4 a^2 c\alpha_2 + 2\alpha\alpha_4 ac\alpha_0 \\ &+\frac{1}{2}\alpha\alpha_2^2 b^2 + 28\beta\alpha_4 b^2 a + 16\beta\alpha_2 ca + 32\beta\alpha_4 a^2 c \\ &-120\gamma\alpha_2 b^2 a - 496\gamma\alpha_4 b^2 a^2 - 256\gamma\alpha_2 ca^2 \\ &-512\gamma\alpha_4 a^3 c + \frac{1}{2}\alpha\alpha_1^2 c + 3\alpha\alpha_4 ab^2\alpha_2 \\ &+2\alpha\alpha_3 a\alpha_1 c = 0, \end{aligned}$$

$$4\beta\alpha_2 ba - 32\gamma\alpha_4 a^3 b + 2\alpha\alpha_4 ab\alpha_0 + 8\beta\alpha_4 a^2 b$$
$$+\alpha\alpha_2^2 ba - 2V\alpha_4 ab - 16\gamma\alpha_2 ba^2 + 2\alpha\alpha_4^2 a^3 b$$
$$+3\alpha\alpha_4 a^2 b\alpha_2 - V\alpha_2 b + \alpha\alpha_2 b\alpha_0 + \frac{1}{2}\alpha\alpha_1^2 b$$
$$+2\alpha\alpha_3 a\alpha_1 b + \frac{3}{2}\alpha\alpha_3^2 a^2 b = 0,$$

$$\alpha\alpha_4 c^3\alpha_3 - 5760\gamma\alpha_3 c^3 = 0,$$

$$3\alpha\alpha_4 bc^2\alpha_3 - 8640\gamma\alpha_3 c^2 b = 0,$$

$$2\alpha\alpha_3 b\alpha_2 c + 2\alpha\alpha_4 bc\alpha_1 - 288\gamma\alpha_1 cb$$
$$-360\gamma\alpha_3 b^3 + 54\beta\alpha_3 bc + \alpha\alpha_4 b^3\alpha_3$$
$$-2808\gamma\alpha_3 abc + 6\alpha\alpha_4 ab\alpha_3 c = 0,$$

$$\alpha\alpha_3 c\alpha_0 + \alpha\alpha_1\alpha_2 c + 8\beta\alpha_1 c - 264\gamma\alpha_3 b^2 a$$
$$+\alpha\alpha_3 b^2\alpha_2 - V\alpha_3 c + 3\alpha\alpha_4 ab^2\alpha_3$$
$$+2\alpha\alpha_4 ac\alpha_1 + 2\alpha\alpha_3 a\alpha_2 c + \alpha\alpha_4 b^2\alpha_1$$
$$+3\alpha\alpha_4 a^2 c\alpha_3 + 24\beta\alpha_3 ac + 12\beta\alpha_3 b^2$$
$$-128\gamma\alpha_1 ca - 24\gamma\alpha_1 b^2 - 384\gamma\alpha_3 a^2 c = 0,$$

$$6\beta\alpha_3 ab - V\alpha_3 b - 8\gamma\alpha_1 ba + 2\beta\alpha_1 b$$
$$+\alpha\alpha_1\alpha_2 b + 2\alpha\alpha_3 a\alpha_2 b + \alpha\alpha_3 b\alpha_0$$
$$+3\alpha\alpha_4 a^2 b\alpha_3 + 2\alpha\alpha_4 ab\alpha_1 - 24\gamma\alpha_3 a^2 b = 0,$$

$$-27120\gamma\alpha_4 b^2 c^2 - 20480\gamma\alpha_4 c^3 a + \frac{1}{2}\alpha\alpha_3^2 c^3$$
$$+2\alpha\alpha_4^2 ac^3 + 80\beta\alpha_4 c^3 - 1920\gamma\alpha_2 c^3$$
$$+\alpha\alpha_4 c^3\alpha_2 + 3\alpha\alpha_4^2 b^2 c^2 = 0,$$

$$-13440\gamma\alpha_4 c^4 + \frac{1}{2}\alpha\alpha_4^2 c^4 = 0,$$

$$-32640\gamma\alpha_4 c^3 b + 2\alpha\alpha_4^2 bc^3 = 0,$$

$$3\alpha\alpha_4 b^2\alpha_3 c - 384\gamma\alpha_1 c^2 + \alpha\alpha_3 c^2\alpha_2$$
$$+48\beta\alpha_3 c^2 + \alpha\alpha_4 c^2\alpha_1 - 4224\gamma\alpha_3 c^2 a$$
$$+3\alpha\alpha_4 ac^2\alpha_3 - 3600\gamma\alpha_3 b^2 c = 0,$$

$$-V\alpha_4 a^2 + \frac{1}{2}\alpha\alpha_3^2 a^3 - V\alpha_0 + C$$
$$+\frac{1}{2}\alpha\alpha_0^2 + \alpha\alpha_4 a^3\alpha_2 + \frac{1}{2}\alpha\alpha_4^2 a^4$$
$$-V\alpha_2 a + \alpha\alpha_2 a\alpha_0 + \frac{1}{2}\alpha\alpha_1^2 a + \alpha\alpha_3 a^2\alpha_1$$
$$+\frac{1}{2}\alpha\alpha_2^2 a^2 + \alpha\alpha_4 a^2\alpha_0 = 0. \tag{3.4}$$

With the aid of Maple or Mathematica we can solve the above system (3.4) to obtain the following sets of solutions:

The set 1.

$$\begin{aligned}
&\alpha_4 = \tfrac{1680\ \beta}{13\alpha a},\\
&\alpha_2 = \tfrac{-3360\ \beta}{13\alpha},\\
&\gamma = \tfrac{\beta}{208a},\\
&V = \alpha\alpha_0 - \tfrac{1104\beta a}{13},\\
&C = \tfrac{1}{338\alpha}\ [-28704\alpha_0\alpha\beta a + 887040\beta^2 a^2\\
&\qquad +169\alpha_0^2\alpha^2],\\
&\alpha_3 = \alpha_1 = 0.
\end{aligned} \tag{3.5}$$

The set 2.

$$\begin{aligned}
&\alpha_4 = \tfrac{168D\ \beta}{13\alpha a},\\
&\alpha_2 = \tfrac{-224\beta(D+5)}{13\alpha},\\
&\gamma = \tfrac{D\beta}{2080a}\\
&V = \tfrac{1}{65D}[65\alpha_0\alpha D + 4368\beta a D + 89280\beta a],\\
&C = \tfrac{1}{1690\alpha}[-7488\alpha_0\alpha\beta a D + 14784\beta^2 a^2 D\\
&\qquad -118560\beta a\alpha_0\alpha - 1196160\beta^2 a^2\\
&\qquad +845\alpha_0^2\alpha^2],\\
&\alpha_3 = \alpha_1 = 0,
\end{aligned} \tag{3.6}$$

where $D = \frac{-31}{2} \pm \frac{3i}{2}\sqrt{31}$.

For the set 1, we have the following solutions:

$$u(\xi) = \frac{1680\ \beta}{13\alpha a}\left(\frac{G'}{G}\right)^4 - \frac{3360\ \beta}{13\alpha}\left(\frac{G'}{G}\right)^2 + \alpha_0, \tag{3.7}$$

where

$$\xi = x - \left[\alpha\alpha_0 - \frac{1104\beta a}{13}\right]t. \tag{3.8}$$

While for the set 2, we have the following solutions:

$$u(\xi) = \frac{168D\ \beta}{13\alpha a}\left(\frac{G'}{G}\right)^4 - \frac{224\beta(D+5)}{13\alpha}\left(\frac{G'}{G}\right)^2 + \alpha_0, \tag{3.9}$$

where

$$\xi = x - \frac{t}{65D}[65\alpha_0\alpha D + 4368\beta a D + 89280\beta a]. \tag{3.10}$$

According to the step 4 of section 2, we have the following families of exact solutions

Family 1. If $a > 0, \Delta > 0$, then the exact solution for the set 1 has the form

$$\begin{aligned} u &= \frac{1680\ \beta a}{13\alpha}\tanh^4(2\sqrt{a}\xi) \\ &-\frac{3360\ \beta a}{13\alpha}\tanh^2(2\sqrt{a}\xi)+\alpha_0, \quad (3.11) \end{aligned}$$

Family 2. If $a < 0,\ \Delta > 0$, then the exact solution for the set 1 has the form

$$\begin{aligned} u &= \frac{1680\beta a}{13\alpha}\tan^4(2\sqrt{-a}\,\xi) \\ &+ \frac{3360\beta a}{13\alpha}\tan^2(2\sqrt{-a}\,\xi)+\alpha_0, \quad (3.12) \end{aligned}$$

or

$$\begin{aligned} u &= \frac{1680\beta a}{13\alpha}\cot^4(2\sqrt{-a}\,\xi) \\ &+ \frac{3360\beta a}{13\alpha}\cot^2(2\sqrt{-a}\,\xi)+\alpha_0. \quad (3.13) \end{aligned}$$

Family 3. If $a > 0, \Delta < 0$, then the exact solution for the set 1 has the form

$$\begin{aligned} u &= \frac{1680\ \beta a}{13\alpha}\coth^4(2\sqrt{a}\ \xi) \\ &- \frac{3360\ \beta a}{13\alpha}\coth^2(2\sqrt{a}\ \xi)+\alpha_0. \quad (3.14) \end{aligned}$$

Family 4. If $a > 0, c > 0$ then the exact solution for the set 1 has the form

$$\begin{aligned} u &= \alpha_0+\frac{105\beta a}{13\alpha}[\tanh(\sqrt{a}\xi)+\coth(\sqrt{a}\,\xi)]^4 \\ &- \frac{840\beta a}{13\alpha}[\tanh(\sqrt{a}\xi)+\coth(\sqrt{a}\xi)]^2, \quad (3.15) \end{aligned}$$

or

$$\begin{aligned} u &= \alpha_0+\frac{105\ \beta a^3}{13\alpha c^2}\mathrm{csch}^8(\sqrt{a}\xi) \\ &-\frac{840\ \beta a^2}{13\alpha c}\mathrm{csch}^4(\sqrt{a}\xi). \quad (3.16) \end{aligned}$$

Family 5. If $a < 0, c > 0$, then the exact solution for the set 1 has the form

$$\begin{aligned} u &= \alpha_0+\frac{105\ \beta a}{13\alpha}[\tan(\sqrt{-a}\xi)-\cot(\sqrt{-a}\,\xi)]^4 \\ &+ \frac{840\beta a}{13\alpha}[\tan(\sqrt{-a}\xi)-\cot(\sqrt{-a}\xi)]^2. \quad (3.17) \end{aligned}$$

Family 6. If $a > 0,\ b = 0$, then the exact solution for the set 1 has the form

$$\begin{aligned} u &= \frac{105\beta}{3328\alpha a c^2}\coth^4\left(\frac{\varepsilon\xi}{4\sqrt{c}}\right) \\ &-\frac{1680\beta}{299\alpha c}\coth^2\left(\frac{\varepsilon\xi}{4\sqrt{c}}\right)+\alpha_0. \quad (3.18) \end{aligned}$$

Similarly, we can find the exact solutions for the set 2, using (3.9) and (3.10) which are omitted here.

3.2 Example 2. On solving the modified Kawahara equation by the extended tanh-function method

With reference to the well known extended tanh-function method [1, 7, 8, 32, 34, 39, 46], the solution of the equation (3.1) can be written in the form:

$$\begin{aligned} u(\xi) &= \alpha_4\phi^4(\xi)+\alpha_3\phi^3(\xi)+\alpha_2\phi^2(\xi) \\ &+\alpha_1\phi(\xi)+\alpha_0, \quad (3.19) \end{aligned}$$

where $\phi(\xi)$ satisfies the Riccati equation

$$\phi'(\xi) = R+\phi^2(\xi) \quad (3.20)$$

The Riccati equation (3.20) have the following solutions: (i) If $R < 0$, then

$$\begin{aligned} &\phi(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi) \\ &\text{or} \\ &\phi(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\xi) \end{aligned} \quad (3.21)$$

(ii) If $R > 0$, then

$$\begin{aligned} &\phi(\xi) = \sqrt{R}\tan(\sqrt{R}\xi), \\ &\text{or} \\ &\phi(\xi) = -\sqrt{R}\cot(\sqrt{R}\xi) \end{aligned} \quad (3.22)$$

(iii) If $R = 0$, then

$$\phi(\xi) = \frac{-1}{\xi}. \quad (3.23)$$

Substituting (3.19) along with (3.20) into (3.1) we get the following polynomial:

$$\begin{aligned} &(-840\gamma\alpha_4+\frac{1}{2}\alpha\alpha_4^2)\phi^8+(\alpha_3\alpha\alpha_4-360\gamma\alpha_3)\phi^7 \\ &+ (20\beta\alpha_4+\frac{1}{2}\alpha\alpha_3^2+\alpha\alpha_2\alpha_4-2080\gamma\alpha_4R-120\gamma\alpha_2)\phi^6 \\ &+ (\alpha\alpha_2\alpha_3-816\gamma\alpha_3R-24\gamma\alpha_1+12\beta\alpha_3+\alpha\alpha_1\alpha_4)\phi^5 \\ &+ [\frac{1}{2}\alpha\alpha_2^2+\alpha\alpha_1\alpha_3+\alpha\alpha_4\alpha_0+6\beta\alpha_2 \\ &\quad -240\gamma\alpha_2R+32\beta\alpha_4R-1696\gamma\alpha_4R^2-V\alpha_4]\phi^4 \\ &+ [-40\gamma\alpha_1R+\alpha\alpha_1\alpha_2+18\beta\alpha_3R-V\alpha_3 \\ &\quad -576\gamma\alpha_3R^2+\alpha\alpha_3\alpha_0+2\beta\alpha_1]\phi^3 \\ &+ [8\beta\alpha_2R+12\beta\alpha_4R^2-480\gamma\alpha_4R^3-136\gamma\alpha_2R^2 \\ &\quad -V\alpha_2+\alpha\alpha_2\alpha_0+\frac{1}{2}\alpha\alpha_1^2]\phi^2 \\ &+ [-V\alpha_1+2\beta\alpha_1R-120\gamma\alpha_3R^3 \\ &\quad -16\gamma\alpha_1R^2+\alpha\alpha_1\alpha_0+6\beta\alpha_3R^2]\phi \\ &- V\alpha_0+C_1-16\gamma\alpha_2R^3+\frac{1}{2}\alpha\alpha_0^2 \end{aligned}$$

$$-24\gamma\alpha_4 R^4 + 2\beta\alpha_2 R^2 = 0 \tag{3.24}$$

Equating the coefficients of this polynomial to zero and solving the algebraic equations by Maple or Mathematica, we have the following two sets of solutions:

The set 3

$$\begin{cases} \alpha_4 = \frac{-420\beta}{13\alpha R}, \\ \alpha_2 = \frac{-840\beta}{13\alpha}, \\ \alpha_3 = \alpha_1 = 0, \\ \gamma = \frac{-\beta}{52R}, \\ V = \alpha\alpha_0 + \frac{276\beta R}{13}, \\ C = \frac{1}{338\alpha}[7176\alpha_0\alpha\beta R + 55440R^2\beta^2 \\ \quad +169\alpha_0^2\alpha^2]. \end{cases} \tag{3.25}$$

The set 4

$$\begin{cases} \alpha_4 = \frac{42D_1\ \beta}{13\alpha R}, \\ \alpha_2 = \frac{56\ \beta(D_1-5)}{13\alpha}, \\ \alpha_3 = \alpha_1 = 0, \\ \gamma = \frac{D_1\beta}{520R}, \\ V = \frac{1}{65D_1}[-1092D_1R\beta + 65\alpha_0\alpha D_1 + 22320\beta R], \\ C = \frac{1}{1690\alpha}[-1872\alpha_0\alpha\beta R D_1 - 924D_1\beta^2R^2 \\ \quad +29640\beta R\alpha_0\alpha - 74760\beta^2R^2 + 845\alpha_0^2\alpha^2], \end{cases} \tag{3.26}$$

where $D_1 = \frac{31}{2} \pm \frac{3i}{2}\sqrt{31}$.

Thus, the exact solutions of the modified Kawahara equation (3.1) have the following forms:

For the set 3 we deduce for $R < 0$ that

$$\begin{aligned} u &= -\frac{420\ \beta R}{13\alpha}\tanh^4(\sqrt{-R}\xi) \\ &+ \frac{840\ \beta R}{13\alpha}\tanh^2(\sqrt{-R}\xi) + \alpha_0, \end{aligned} \tag{3.27}$$

or

$$\begin{aligned} u = & -\frac{420\ \beta R}{13\alpha}\coth^4(\sqrt{-R}\xi) \\ & +\frac{840\ \beta R}{13\alpha}\coth^2(\sqrt{-R}\xi) + \alpha_0, \end{aligned} \tag{3.28}$$

while for $R > 0$ we deduce that

$$\begin{aligned} u = & -\frac{420\ \beta R}{13\alpha}\tan^4(\sqrt{R}\xi) \\ & -\frac{840\ \beta R}{13\alpha}\tan^2(\sqrt{R}\xi) + \alpha_0, \end{aligned} \tag{3.29}$$

or

$$\begin{aligned} u = & -\frac{420\ \beta R}{13\alpha}\cot^4(\sqrt{R}\xi) \\ & -\frac{840\ \beta R}{13\alpha}\cot^2(\sqrt{R}\xi) + \alpha_0, \end{aligned} \tag{3.30}$$

where

$$\xi = x - (\alpha\alpha_0 + \frac{276\beta R}{13})t$$

Similarly, we can write down the exact solutions for the set 4, which are omitted here. From the previous results, we have the following remarks:

Remark 1 *If we put $R = -4a$ where $a > 0$ then the results (3.11) and (3.14) are equivalent to the results (3.27) and (3.28) respectively.*

Remark 2 *If we put $R = -4a$ wherea < 0 then the results (3.12) and (3.13) are equivalent to the results (3.29) and (3.30) respectively. From these remarks we have the following observation:*

The exact solutions of the modified Kawahara equation obtained using the extended tanh- function method are equivalent to its exact solutions obtained using the further improved ($\frac{G'}{G}$)- expansion method.

Remark 3 *These solutions have been checked with Maple by putting them back into the original equation.*

4 Conclusions

In summary, we have found the exact solutions of the $(1+1)$-dimensional modified Kawahara equation (3.1) using two methods via the further improved ($\frac{G'}{G}$)-expansion method and the extended tanh-function method. We have arrived at the observation that these exact solutions are equivalent.

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