Transient Temperature Solutions of a Cylindrical Fin

G.-C. Kuo[a] , Y.-H. Hu[a], W.-L. Liaw[a], K.-J Wang[b], K.-Y. Kung[a,*]

[a,*]Department of Mechanical Engineering

Nanya Institute of Technology

* NO.414, Sec.3, Jhongshan E. Rd., Jhongli City, 32091

Taiwan.

[b] Department of Industrial Management

National Taiwan University of Science and Technology

Taiwan

Email:kykoung@nanya.edu.tw, kykoung@gmail.com

Phone: 886-3-4361070-6407, FAX: 886-3-4388198.

Abstract: - Analytical temperature solutions to the transient heat conduction for a two dimensional cylindrical fin with arbitrary convective effects on lateral surface is obtained by the method of superposition and separation variables. The temperature distributions are generalized for a linear combination of the product of Bessel function, Fourier series and exponential type for nine different cases. Relevant connections with some other closely-related recent works are also indicated.

Key-Words: - Bessel function, Fourier series, Heat conduction, Temperature distribution, Separation variables, superposition.

1 Introduction

A systematic procedure for determining the separation of variables for a given partial differential equation can be found in [1-2]. However, a cylindrical heat problem involved two dimensions with convective effects on lateral surface in present study is presumably not solved in the existing literature on this subject. The superposition and the separation method are used in this study to get the analytical solutions of the temperature distribution.

The separation variables method is applied in this study. The partial differential equations are transferred into ordinary differential equations by separating the independent variables involved in the problem. The temperature distributions of fins under transient condition are important for proper prediction and control of the fin performance. Closed-form analytical solution for the transient temperature distribution would provide continuous physical insight which is much better than discrete numbers from a numerical computation.

The main purpose of this study is to investigate the analytical transient solutions by using the method of separation of variables. A group of theory, being a systematic procedure of determining the separation variables can be found in [3-8]. The methods of superposition are widely used [9-15]. Munteanu et.al. [9] studied the motion of two pendulums coupled by an elastic spring and presented analytical solutions for a nonlinear coupled pendulum. Rashidi [10] experimentally determined the temperature distribution of piston and cylinder head, and showed that by increasing the engine speed, the temperature of oil reduced, but both the angular velocity of fan and the heat transfer coefficient of air increased. Arruda [11] empirically choice the period and number of frequency lines of the Fourier series in the regressive discrete Fourier series and studied numerical instability. It showed that the regressive discrete Fourier series were useful for data smoothing, extrapolation, and computation of derivatives from noisy signals. Wen and Khonsari [12] present the Fourier series as the working function involving a body subjected to oscillating heat flux on one of its boundaries.

Meshii and Watanabe [13] indicated a surface axisymmetric circumferential crack inside a hollow cylinder shows tendency toward crack arrest. The transient temperature distribution of the cylinder under thermal striping was analytically obtained by Bessel function. Gordeliy, Crouch, and Mogilevskaya [14] depicted the semi-analytical temperature solution of a truncated Fourier series for the transient heat conduction in a medium with two circular cavities. V. Jovanovic [15] investigated the problem of the transverse vibration of a beam with a viscous boundary and generalized Fourier series solution. H. M. Srivastava, K. Y. Kung and K-J Wang [16] present analytic solutions of a two-dimensional rectangular heat equation with a heat source. A heat problem involving two dimensions and boundary conditions of the type considered in our present study is presumably not solved. The superposition and the separation method are used in this study to get the analytical solutions for a two dimensional cylindrical fin cases.

2 Problems

Fins provide a considerable increase in the surface area available for heat transfer rate. The transient behavior is related to many natural phenomena and industrial applications, such as the cooling of electronic components, power generators, semiconductors, cylinders of air-cooled aircraft, automatic control mechanism and many other equipments in which heat is generated and must be disposed.

It is important to understand the temperature distribution of fins under transient or unsteady thermal condition for proper prediction and control of the fin performance. Analytic solutions are particularly useful, since they may be used in an on-line computer. A number of restrictive assumptions are introduced before studying the transient analysis of two dimensional cylinder fin, some of which are due to Kung [3]. The fundamental assumptions are as follows:

a. The pin fin is equipped with homogeneous material properties and thickness.

b. The convective heat transfer coefficient, the ambient temperature and the fin properties are assumed to be constant.

c. The temperature distribution in the fin is axial symmetric.

d. There is no heat source in the inner part of the fin,

e. Thermal radiation effects are negligible.

The present research investigates the transient analysis of two- dimensional cylindrical fin with finite length. The method of separable variables is used in this study to get an analytical solution for the transient two-dimensional heat conduction. One can apply Fourier's law and energy conservation law to form a set of dimensionless governing differential equation, the initial condition and the boundary conditions as following.

$$\frac{\partial u(r,z,t)}{\partial t}=\frac{\partial^2 u(r,z,t)}{\partial z^2}+\frac{\partial^2 u(r,z,t)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,z,t)}{\partial r}, \tag{1}$$

$$t=0,\quad u(r,z,0)=0, \tag{2}$$

$t>0$,

$$r=0,\ u(0,z,t)\to\infty, \tag{3}$$

$$r=1,\ u_r(1,z,t)+Bi_r u(1,z,t)=0, \tag{4}$$

$$z=0,-u_z(r,0,t)+Biu(r,0,t)=Bi+q\cos\omega t \tag{5}$$

$$z=L,\ u_z(r,L,t)+Bi_L u(r,L,t)=0. \tag{6}$$

Where $u(r,z,t)$ denotes the temperature, L fin length. $\mathrm{B}i, \mathrm{Bi}_\mathrm{L}, \mathrm{Bi}_\mathrm{r}$ are the base Biot number; tip Biot number and lateral Boit number respectively. The governing partial differential equations are converted to ordinary differential equations by reducing the number of independent variables. Note that a Bessel's equation satisfied a cylindrical coordinate in r-direction can be made orthogonal, one can separate the temperature distribution as follows:

$$u(r,z,t)=\sum_{n=1}^{\infty}u_n(z,t)J_0(\beta_n r). \tag{7}$$

where β_n are the positive characteristic values of the transcendent equation (8),

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0 \text{ ,(where } r = 1\text{)}, \tag{8}$$

Because the boundary conditions in equations (3) and (4) are satisfied automatically, the partial differential equation with initial and boundary conditions can be simplified as:

$$\frac{\partial u_n(z,t)}{\partial t} = \frac{\partial^2 u_n(z,t)}{\partial z^2} - \beta_n u_n(z,t), \tag{9}$$

I.C. $t = 0$,

$$u_n(z,0) = 0, \tag{10}$$

B. Cs. $t > 0$,

$$z = 0, \text{-} u_{nz}(0,t) + Bi_r u_n(0,z,t) = g_n, \tag{11}$$

$$z = L, \; u_{nz}(L,t) + B_{iL} u_n(L,t) = 0, \tag{12}$$

where

$$g_n = \frac{2\beta_n (B_i + q\cos\omega t) J_1(\beta_n)}{(Bi_r^2 + \beta_n^2) J_0^2(\beta_n)}. \tag{13}$$

Revised the boundary conditions to a homogeneous one, which lead to the required coefficients are

$$A = \frac{g_n}{(Bi + Bi_L + LBiBi_L)L},$$

$$B = \frac{(LBi_L + 1) g_n}{(Bi + Bi_L + LBiBi_L)L}. \tag{14}$$

The differential equation (9) and initial condition and boundary condition could be written as

$$\bar{u}_n(z,0) = -[zA + (L - z)B]. \tag{15}$$

$$\frac{\partial \bar{u}_n(z,t)}{\partial t} = \frac{\partial^2 \bar{u}_n(z,t)}{\partial z^2} - \beta_n^2 [zA + (L - z)B + \bar{u}_n(z,t)], \tag{16}$$

and the boundary conditions are

B.Cs. $t > 0$,

$$z = 0, \; -\bar{u}_{nz}(0,t) + Bi\bar{u}_n(0,t) = 0, \tag{17}$$

$$z = L, \; \bar{u}_{nz}(L,t) + Bi_L \bar{u}_n(L,t) = 0. \tag{18}$$

Then let the dependent variable be written as

$$\bar{u}_n(z,t) = \sum_{m=1}^{\infty} u_{nm}(t)(\cos\alpha_m z + \frac{Bi}{\alpha_m}\sin\alpha_m z). \tag{19}$$

where α_m are the characteristic values of the transcendent equation (20).

$$\tan\alpha_m L = \frac{\alpha_m (Bi + Bi_L)}{\alpha_m - BiBi_L}. \tag{20}$$

Substitute Equation (19) into equation (16), initial condition and do necessary calculation. Thus only time variable is left and the ordinary differential equation could be written as

$$\frac{\partial u_{nm}(t)}{\partial t} = -(\beta_n^2 + \alpha_m^2) u_{nm}(t) + \beta_n^2 C_{nm}, \tag{21}$$

$$u_{nm}(0) = C_{nm}. \tag{22}$$

In the last equation, C_{nm} are the values of the equation (23),

$$C_{nm}=\frac{E}{F},$$
$$E=-2[(A-B-ABiL)\alpha_m\cos\alpha_m L$$
$$+(A\alpha_m^2 L+ABi-BBi)\sin\alpha_m L-(A-B-BBiL)\alpha_m],$$
$$F=(\alpha_m^2-Bi^2)[\cos\alpha_m L\sin\alpha_m L$$
$$-2Bi\alpha_m\cos^2\alpha_m L+(\alpha_m^2 L+Bi^2 L+2Bi)\alpha_m] \tag{23}$$

The analytic solution of equation (21) combined with the initial condition of equation (22) is shown as

$$u_{nm}(t)=\frac{C_{nm}}{\beta_n^2+\alpha_m^2}[\beta_n^2+\alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}]. \tag{24}$$

By combining the relating solutions, we get the analytic temperature solution for the partial differential euqation (1) together with the relating I.C. and B.Cs.

2 Results and Discussion

The analytical temperature profiles for above governing equation with the initial condition and the boundary conditions obtained by the method of separation variables and superposition are shown below. The temperature of lateral surface is natural convection, and heat flux can dissipate through circumferential and tip surface. The temperature profile involved nine different boundary conditions are presented.

Case 1: $Bi=$ constant, $Bi_L=$ constant

While Bi_L is constant, heat convection condition on the tip of the fin, the larger Bi_L will result in the faster heat dissipation though the fin. The analytical temperature and heat transfer rate solutions are

$$u(r,z,t)=\sum_{n=1}^{\infty}\{zA+(L-z)B$$
$$+\sum_{m=1}^{\infty}u_{nm}(t)(\cos\alpha_m z+\frac{Bi}{\alpha_m}\sin\alpha_m z)\}J_0(\beta_n r), \tag{25}$$

where

$$u_{nm}(t)=\frac{C_{nm}}{\beta_n^2+\alpha_m^2}[\beta_n^2+\alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}] . \tag{26}$$

The relating coefficients are shown as following:

$$g_n=\frac{2\beta_n(B_i+q\cos\omega t)J_1(\beta_n)}{(Bi_r^2+\beta_n^2)J_0^2(\beta_n)}, \tag{27}$$

$$A=\frac{g_n}{(Bi+Bi_L+LBiBi_L)L},$$
$$B=\frac{(LBi_L+1)g_n}{(Bi+Bi_L+LBiBi_L)L}, \tag{28}$$

where

$$C_m=\frac{D}{E},$$
$$D=-2[(A-B-ABi_L)\alpha_m\cos\alpha_m L$$
$$+(A\alpha_m^2 L+ABi-BBi)\sin\alpha_m L+(A-B-BBiL)\alpha_m],$$
$$E=(\alpha_m^2-Bi^2)\sin(2\alpha_m L)/2$$
$$-2Bi\alpha_m\cos^2\alpha_m L+(\alpha_m^2 L+Bi^2 L+2Bi)\alpha_m. \tag{29}$$

α_m and β_n are the positive roots of the characteristic equations (30) and (31), respectively.

$$\tan\alpha_m L=\frac{\alpha_m(Bi+Bi_L)}{\alpha_m^2-BiBi_L}, \tag{30}$$

$$\beta_n\frac{dJ_0(\beta_n r)}{dr}+Bi_r J_0(\beta_n)=0 \quad \text{, (where } r=1\text{)}. \tag{31}$$

Case 2: $Bi=0$, $Bi_L=0$

As $B_i=0$, the fin roots is constraint to constant heat flux and a constant heat flux conduct into fin through fin roots. While $B_{iL}=0$, the tip of the fin is adiabatic, heat cannot dissipate though the fin and the lateral surface also adiabatic. All energy will be stored in the fin, and the boundary conditions are revised to be as following:

$$z=0,\ -u_z(r,0,t)=q\cos\omega t, \tag{32}$$

$$z=L,\ u_z(r,L,t)=0. \tag{33}$$

The analytical temperature profile is

$$u(r,z,t)=\sum_{n=1}^{\infty}\{[Lz-\frac{z^2}{2}]B+C_{n0} \\ +\sum_{m=1}^{\infty}u_{nm}(t)\cos\alpha_m z\}J_0(\beta_n r) , \tag{34}$$

where

$$u_{nm}(t)=\frac{C_{nm}}{\beta_n^2+\alpha_m^2}[\beta_n^2+\alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}] . \tag{35}$$

The relating coefficients are as following:

$$g_n=\frac{2q\cos\omega t J_1(\beta_n)}{(Bi_r^2+\beta_n^2)J_0^2(\beta_n)}, \tag{36}$$

$$\begin{cases} C_{n0}=\dfrac{-BL^2}{3}, m=0 \\ C_{nm}=\dfrac{2BL^2}{(m\pi)^2}, m>0 \end{cases} , \tag{37}$$

$$B=-\frac{g_n}{L},$$

α_m and β_n are the positive roots of the characteristic equations (38) and (39), respectively.

$$\alpha_m=\frac{m\pi}{L}; m,n=1,2,3,... \tag{38}$$

$$\beta_n\frac{dJ_0(\beta_n r)}{dr}+Bi_r J_0(\beta_n)=0 \quad \text{, (where } r=1\text{)}. \tag{39}$$

Case 3: $Bi=0$, $Bi_L\to\infty$

While $B_{iL}\to\infty$, the tip of the fin is isothermal to environment, the lateral surface also isothermal and heat dissipates fast though the fin. The temperature of the fin, left sitting on the base temperature, will fall until it reaches the surrounding temperature, and the boundary conditions are revised to be as following:

$$z=0,\ -u_z(r,0,t)=q\cos\omega t, \tag{40}$$

$$z=L,\ u(r,L,t)=0 . \tag{41}$$

The analytical temperature is

$$u(r,z,t)=\sum_{n=1}^{\infty}\{(L-z)B \\ +\sum_{m=1}^{\infty}u_{nm}(t)\cos\alpha_m z\}J_0(\beta_n r) , \tag{42}$$

where

$$u_{nm}(t)=\frac{C_{nm}}{\beta_n^2+\alpha_m^2}[\beta_n^2+\alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}] . \tag{43}$$

where the relating coefficients are as following:

$$g_n=\frac{2\beta_n q\cos\omega t J_1(\beta_n)}{(Bi_r^2+\beta_n^2)J_0^2(\beta_n)}, \tag{44}$$

$$C_{nm}=\frac{-8BL}{[(2m-1)\pi]^2}, \tag{45}$$

$$B=g_n \tag{46}$$

$$\alpha_m=\frac{(2m-1)\pi}{2L}; n,m=1,2,3,... \tag{47}$$

$$\beta_n\frac{dJ_0(\beta_n r)}{dr}+Bi_r J_0(\beta_n)=0 \quad \text{, (where } r=1\text{)}. \tag{48}$$

Case 4: $Bi=0$, $Bi_L=$**constant**

While $B_{iL}=$constant, heat convection condition on the tip of the fin, the larger B_{iL} will result in the faster heat dissipation though the fin tip surface. , and the boundary conditions are revised to be as following:

$$z=0,\ -u_z(r,0,t)=q\cos\omega t, \tag{49}$$

$$z=L,\ u_z(r,L,t)+Bi_L u(r,L,t)=0 . \tag{50}$$

Then the analytical temperature and heat transfer rate profile are

$$u(r,z,t)=\sum_{n=1}^{\infty}\{zA+(L-z)B \\ +\sum_{m=1}^{\infty}u_{nm}(t)\cos\alpha_m z\}J_0(\sqrt{\beta_n}r) , \tag{51}$$

where

$$u_{nm}(t)=\frac{C_{nm}}{\beta_n^2+\alpha_m^2}[\beta_n^2+\alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}] \ . \tag{52}$$

The relating coefficients are shown as following:

$$g_n=\frac{2\beta_n q\cos\omega t J_1(\beta_n)}{(Bi_r^2+\beta_n^2)J_0^2(\beta_n)}, \tag{53}$$

$$C_{nm}=\frac{-2[(A-B)(\cos\alpha_m L-1)+A\alpha_m L\sin\alpha_m L]}{\alpha_m(\cos\alpha_m L\sin\alpha_m L+\alpha_m L)}, \tag{54}$$

$$A=\frac{g_n}{LBi_L},\ B=\frac{(LBi_L+1)g_n}{LBi_L}, \tag{55}$$

α_m and β_n are the positive roots of the characteristic equations (56) and (57), respectively.

$$\alpha_m\tan\alpha_m L=Bi_L \ . \tag{56}$$

$$\beta_n\frac{dJ_0(\beta_n r)}{dr}+Bi_r J_0(\beta_n)=0 \quad ,(\text{ where } r=1) \tag{57}$$

Case 5: $Bi\to\infty$, $Bi_L=0$

As $B_i\to\infty$, the fin root is constraint to isothermal conductivity, the root interface temperature is kept constant. While $B_{iL}=0$, the tip of the fin is adiabatic, heat cannot dissipate though the fin and the lateral surface also isothermal. All energy will be dissipated by lateral surface. The boundary conditions are revised to

$$u(r,0,t)=1, \tag{58}$$

$$u_z(r,L,t)=0. \tag{59}$$

The analytical temperature and heat transfer rate profile are

$$u(r,z,t)=\sum_{n=1}^{\infty}\{zA+(L-z)B+\sum_{m=1}^{\infty}u_{nm}(t)\sin\alpha_m z\}J_0(\sqrt{\beta_n}r), \tag{60}$$

where

$$u_{nm}(t)=\frac{C_{nm}}{\beta_n^2+\alpha_m^2}[\beta_n^2+\alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}] \ . \tag{61}$$

The relating coefficients are shown as following:

$$g_n=\frac{2\beta_n J_1(\beta_n)}{(Bi_r^2+\beta_n^2)J_0^2(\beta_n)}, \tag{62}$$

$$C_{nm}=\frac{4L[2(A-B)(-1)^m-(2m-1)B\pi]}{[(2m-1)\pi]^2}, \tag{63}$$

$$A=\frac{g_n}{L},\ B=\frac{g_n}{L}, \tag{64}$$

α_m and β_n are the positive roots of the characteristic equations (65) and (66), respectively.

$$\alpha_m=\frac{(2m-1)\pi}{2L},\ m=1,2,3,... \tag{65}$$

$$\beta_n\frac{dJ_0(\beta_n r)}{dr}+Bi_r J_0(\beta_n)=0 \quad (\text{ where } r=1). \tag{66}$$

Case 6: $Bi\to\infty$, $Bi_L\to\infty$

While $B_{iL}\to\infty$, the tip and root of the fin are isothermal, heat dissipates fast though the tip of the fin. The boundary conditions are

$$u(r,0,t)=1, \tag{67}$$

$$u(r,L,t)=0, \tag{68}$$

The analytical temperature and heat transfer rate profile are

$$u(r,z,t)=\sum_{n=1}^{\infty}\{(L-z)B+\sum_{m=1}^{\infty}u_{nm}(t)\sin\alpha_m z\}J_0(\beta_n r), \tag{69}$$

where

$$u_{nm}(t)=\frac{C_{nm}}{\beta_n^2+\alpha_m^2}[\beta_n^2+\alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}] \ . \tag{70}$$

The relating coefficients are shown as following:

$$g_n = \frac{2\beta_n J_1(\beta_n)}{(Bi_r^2 + \beta_n^2) J_0^2(\beta_n)}, \tag{71}$$

$$C_{nm} = \frac{-2BL}{m\pi}, \tag{72}$$

$$B = \frac{g_n}{L}. \tag{73}$$

α_m and β_n are the positive roots of the characteristic equations (74) and (75), respectively.

$$\alpha_m = \frac{m\pi}{L}, m = 1,2,3,... \tag{74}$$

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0 \quad (\text{ where } r = 1). \tag{75}$$

Case 7: $Bi \to \infty$, $B_{iL} =$ constant

While $B_{iL} =$ constant, heat convection condition on the tip of the fin, the larger B_{iL} will result in the faster heat dissipation though the fin. The analytical solution is

$$u(r,z,t) = \sum_{n=1}^{\infty} \{zA + (L-z)B + \sum_{m=1}^{\infty} u_{nm}(t) \sin \alpha_m z\} J_0(\sqrt{\beta_n} r), \tag{76}$$

where

$$u_{nm}(t) = \frac{C_{nm}}{\beta_n^2 + \alpha_m^2} [\beta_n^2 + \alpha_m^2 e^{-(\alpha_m^2 + \beta_n^2)t}] . \tag{77}$$

The relating coefficients are shown as following:

$$g_n = \frac{2\beta_n J_1(\beta_n)}{(Bi_r^2 + \beta_n^2) J_0^2(\beta_n)}, \tag{78}$$

$$C_{nm} = \frac{2[(A-B)\sin\alpha_m L - A\alpha_m L\cos\alpha_m L + B\alpha_m L]}{\alpha_m(\cos\alpha_m L\sin\alpha_m L - \alpha_m L)}, \tag{79}$$

$$A = \frac{g_n}{(LBi_L + 1)L}, B = \frac{g_n}{L}. \tag{80}$$

α_m and β_n are the positive roots of equations (81) and (82), respectively.

$$\alpha_m \cot \alpha_m L = -Bi_L, \tag{81}$$

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0 \quad (\text{ where } r = 1). \tag{82}$$

Case 8: $Bi =$ constant, $Bi_L = 0$

As $B_i =$ constant, it exist heat convection effects between the root of the fin and heat source. The root of the fin approaches to isothermal as the value of B_i is large enough, but it come near to constant heat flux for infinitesimal B_i value. While $B_{iL} = 0$, the tip of the fin is adiabatic, heat cannot dissipate though the fin and the lateral surface also adiabatic. All heat energy will be reserved in the fin. The analytical solution is

$$u(r,z,t) = \sum_{n=1}^{\infty} \{zA + (L-z)B + \sum_{m=1}^{\infty} u_{nm}(t) \frac{\cos\alpha_m (L-z)}{\cos\alpha_m L}\} J_0(\sqrt{\beta_n} r), \tag{83}$$

where

$$u_{nm}(t) = \frac{C_{nm}}{\beta_n^2 + \alpha_m^2} [\beta_n^2 + \alpha_m^2 e^{-(\alpha_m^2 + \beta_n^2)t}] . \tag{84}$$

The relating coefficients are shown as following:

$$g_n = \frac{2\beta_n (Bi + q\cos\omega t) J_1(\beta_n)}{(Bi_r^2 + \beta_n^2) J_0^2(\beta_n)} \tag{85}$$

$$C_{nm} = \frac{2[(A-B)(\cos\alpha_m L - 1) - B\alpha_m L\sin\alpha_m L]\cos\alpha_m L}{\alpha_m(\cos\alpha_m L\sin\alpha_m L + \alpha_m L)} \tag{86}$$

$$A = \frac{g_n}{LBi}, B = \frac{g_n}{LBi} \tag{87}$$

α_m and β_n are the positive roots of equations (88) and (89), respectively.

$$\alpha_m \tan \alpha_m L = Bi, \tag{88}$$

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0 \quad (\text{ where } r = 1). \quad (89)$$

Case 9: $Bi = \text{constant}$, $Bi_L \to \infty$

While $B_{iL} \to \infty$, the tip of the fin is isothermal to environment, the lateral surface also adiabatic and heat dissipates fast though the fin tip. The boundary conditions are

$$-u_z(r,0,t) + Biu(r,0,t) = Bi + q\cos(\omega t)\,, \quad (90)$$

$$u(r,L,t) = 0\,. \quad (91)$$

The analytical solution is

$$u(r,z,t) = \sum_{n=1}^{\infty}\{(L-z)B + \sum_{m=1}^{\infty} u_{nm}(t)\frac{\sin\alpha_m(L-z)}{\sin\alpha_m L}\}J_0(\sqrt{\beta_n}r)\,, \quad (92)$$

where

$$u_{nm}(t) = \frac{C_{nm}}{\beta_n^2 + \alpha_m^2}[\beta_n^2 + \alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}]\;. \quad (93)$$

The relating coefficients are shown as following:

$$g_n = \frac{2\beta_n(B_i + q\cos\omega t)J_1(\beta_n)}{(Bi_r^2 + \beta_n^2)J_0^2(\beta_n)}\,, \quad (94)$$

$$C_{nm} = \frac{-2B[\cos^2\alpha_m L + \alpha_m L\cos\alpha_m L\sin\alpha_m L - 1]}{\alpha_m(\cos\alpha_m L\sin\alpha_m L - \alpha_m L)}\,, \quad (95)$$

$$B = \frac{g_n}{LBi+1}\,, \quad (96)$$

α_m and β_n are the positive roots of equations (97) and (98), respectively.

$$\alpha_m \cot\alpha_m L = -Bi\,, \quad (97)$$

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0 \quad (\text{ where } r = 1). \quad (98)$$

4 Conclusion

The principle of superposition and separable variables are applied to the transient heat conduction in a cylindrical fin subjected to convective lateral surface to provide a simplified formulation that can be used to identify the temperature distribution. The temperature distributions are formed in a Fourier Bessel series and exponential type and are given by nine different cases.

Nomenclature

Bi	base Biot number
Bi_L	tip Biot number
Bi_r	lateral Boit number
J_0	First kind Bessel function of zero order
β_n	Characteristic values of Bessel function
L	pin fin length
m, n	positive integral values
t	dimensionless time
$u(r,z,t)$	dimensionless transient temperature
α_m; β_n	eigenvalues

Acknowledgement
The present investigation was supported by the Teacher Research Support Scheme of Nanya Institute of Technology under Grant 099P-039.

References:

[1] H. S. Carslaw, and J. C. Jaeger., Conduction of heat in Solids, Oxford University Press, New York, 1959.
[2] S. A. Vedat, Conduction Heat Transfer, Addison --Wesley, U.S.A. 1966.
[3] K. Y. Kung, Transient Analysis of Two-Dimensional Rectangular Fin With Various

Surface Heat Effects, *WSEAS Transactions on Mathematics,* Vol.3, 2004, pp.198-203.

[4] K. Y. Kung, Shih-Ching Lo, Transient Analysis of Two Dimensional Cylindrial Fin With Various Surface Heat Effects, *WSEAS Transactions on Mathematics,* Vol.3, 2004, pp.120-125.

[5] C. H. Hsu, K. Y. Kung, Analytic Transient Solutions of a Two Dimensional Rectangular Heat Equation, *WSEAS Transactions on Mathematics,* Vol.5, 2006, pp.1347-1349.

[6] Ko-Ta Chiang, G. C. Kuo, K.-J Wang, Y.F. Hsiao, K.-Y. Kung, Transient Temperature Analysis of a cylindrical heat equation, *WSEAS Transactions on Mathematics*, Vol. 8, Issue 7, 2009, pp.309 -319

[7] Ko-Ta Chiang, K.Y. Kung and H. M. Srivastava, Analytic transient Solutions of a Cylindrical Heat Equation with a heat source, *Applied Mathematics and Computation*, Vol. 215, 2009, pp.2877-2885.

[8] K. Y. Kung, H.M. Srivastava, Analytic transient solutions of a cylindrical heat equation with oscillating heat flux, *Applied Mathematics and Computation*, Vol.195, No. 2, 2008, pp. 736-744.

[9] L. Munteanu, V. Chiroiu, Ş. Donescu, Analytical solutions for a nonlinear coupled pendulum, *Proceedings of the 10th WSEAS int. Conf. on Mathematical and Computational Methods in Science and Engineering*, 2010, pp.274-279.

[10] M. H. M. Rashidi, Determination of piston and cylinder head temperature distribution in a 4-cylinder gasoline engine at actual process, *Proceedings of the 4th WSEAS Int. Conf. on Heat Transfer, Thermal* Engineering *and Environment*, 2006, pp153-158.

[11] Jose´ Roberto deFranc-a Arruda, A robust one-dimensional regressive discrete Fourier series, *Mechanical Systems and Signal Processing*, Vol.24, 2010, pp.835-840.

[12] J. Wen and M. M. Khonsari1, Analytical Formulation for the Temperature Profile by Duhamel's Theorem in Bodies Subjected to an Oscillatory Heat Source, *Transactions of the ASME,* Vol.129, 2007, pp.236-240.

[13] T. Meshii and K. Watanabe, Stress intensity factor of a circumferential crack in a thick-walled cylinder under thermal striping, *Journal of Pressure Vessel Technology*, Vol. 126, 2004, pp.157-162.

[14] E. Gordeliy, S. L. Crouch and S. G. Mogilevskaya, Transient heat conduction in a medium with two circular cavities: Semi-analytical solution, *International Journal of Heat and Mass Transfer,* Vol.51, 2008, pp.3556-3570.

[15] V. Jovanovic, A Fourier series solution for the transverse vibration response of a beam with a viscous boundary, *Journal of Sound and Vibration*, Vol. 330, Issue 7, 2011, pp.1504-1515.

[16] H. M. Srivastava, K. Y. Kung , and K. J. Wang, Analytical solutions of a two-dimensional rectangular heat equation with a heat source, *Russian Journal of Mathematical Physics*, Vol. 15, No. 4, 2008, pp.542-547