Shape of a Drum, a Constructive Approach

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Abstract: For the classical question, "Can you hear the shape of the drum?", the answer is known to be "yes" for certain convex planar regions with analytic boundaries. The answer is also known to be "no" for some polygons with reentrant corners. A large number of mathematicians over four decades have contributed to the topic from various approaches, theoretical and numerical. In this article, we develop a constructive analytic approach to indicate how a preknowledge of the eigenvalues lead to the determination of the parameters of the boundary. This approach is applied to a general boundary and in particular to a circle, an ellipse, and a square. In the case of a square, we obtain an insight into why the analytical procedure does not, as expected, yield an answer. For the Mathieu equation with a parameter, we demonstrate the determination of the parameter when the eigenvalues are known.

Key–Words: Helmholtz equation; Eigenvalues; Mathieu equation.

1 Introduction

The pursuit of a "complete" solution to the question "Can you hear the shape of a drum?", originally posed by Lipman Bers and used as a title in 1966 by Mark Kac [9], has been a fascinating journey and work can only be described as "work in progress". The research has involved many mathematicians and many tools including Asymptotics, Probability Theory, Operator Theory, infinite algebraic systems and inevitably intense computational work involving approximate methods. Mathematically, the problem is, whether a preknowledge of the eigenvalues of the Laplacian in a region leads to the definition of the closed boundary Γ. Specifically, we have

$$u_{xx} + u_{yy} + \lambda^2 u = 0 \ \text{ in } \Omega, \tag{1}$$

$$u = 0, \ \text{ on } \Gamma. \tag{2}$$

The equation (1) is also well known as the Telegraph equation or the Helmholtz equation. In our analysis, we restrict Ω to be a simply connected convex region with an analytic boundary. According to the maximum principle for linear elliptic partial differential equations [6], the infinite eigenvalues λ_n^2, $n = 1, 2, 3, \cdots, \infty$ are positive, real, ordered and satisfy

$$0 < \lambda_1^2 < \lambda_2^2 < \lambda_3^2 < \cdots < \lambda_n^2 < \cdots \infty\,. \tag{3}$$

It is well established that the answer to the question is both "yes" and "no" depending on the nature of the boundary Γ. In this paper, we give a constructive approach for the determination of Γ when the answer is "yes". In such classical problems, theoretical results like existence, uniqueness and even justification of truncation of resulting infinite systems, approximations with error analysis, asymptotic, etc., need to be complemented with constructive approaches for practical considerations. In Section 2, we give a brief account of work so far and give a glimpse of various approaches and results. The account is not claimed to be complete or exhaustive. Section 3 introduces the fully integrated solution of (1) in terms of complex variables z and $\bar{z}$ as well as some identities. A general parameterized analytical boundary Γ in (2) having a biaxial symmetry is introduced in Section 4 and a complete formal solution to (1) and (2) is given. A constructive process to derive equations to determine the parameters of the boundary in terms of eigenvalues of the boundary value problem, is described. The processes defined in Section 4 are used in Section 5 when Γ is a circle. Section 5 also discusses in detail when the boundary is an ellipse. It is shown in Section 6, the failure of the analytic processes when the boundary is a square (as expected), thus giving some insight to the "no" answer to the original question. For the Mathieu equation with a parameter q, we demonstrate how a preknowledge of the eigenvalues leads to the determination of the value of q. Section 8 gives some conclusions and suggests a number of avenues for future research work.

2 Earlier Work

In this section, we trace briefly, various results, mainly theoretical and some approximations. The results are mainly for polygonal drums and drums with analytical boundaries. Connections are made to other topics including quantum mechanics, black body radiation, diffusion, geometrical optics, antenna theory etc. A main tool in the present paper is the role of infinite linear systems. A complete discussion of these systems can be found in a review paper by Shivakumar et al. [17].

(a) In the famous paper, Mark Kac [9] analyzes using only asymptotic properties of large eigenvalues and uses probability theory as the tool to establish that one can hear the area of a polygonal drum and conjectures for multiply connected regions. This paper also establishes that one can hear the perimeter as well.

(b) Since elliptic regions are one of the very few cases for which (1) admits solution by separation of variables, many papers deal with Mathieu equations.

  (i) In 1987, Shivakumar et al [15] give a complete account including location of eigenvalues with precise lower and upper bounds to any required degree of accuracy using a powerful and simple algorithm. This paper lists 18 eigenvalues to eight places of accuracy for the case $q = 1$. Use is made of estimates for truncation of infinite systems when the matrix has a diagonally dominant structure.

  (ii) In 1960, Keller and Rubinov [10] develop asymptotic formulas for large eigenvalues and they apply their techniques to Schrodinger equation. The paper also contains numerical values for some eigenvalues in the case of an elliptic region.

  (iii) In 1984, Chen et al [3] develop numerical packages for the two Mathieu equations and they give a visualization for the eigenfunctions. This paper has valuable information about approximate eigenvalues of the Laplacian.

(c) Based on the fact that there exist nonisometric planar regions that have identical spectra, Driscoll [4] discusses polygons with reentrant corners. The paper also describes algorithms which yield values accurate to 12 digits using finite elements. These polygons clearly imply the answer 'no' to the question.

(d) A major contribution in 2000 is by Zelditch [19], where a positive answer "yes" is given for certain regions with analytic boundaries. Because of its relevance to the present paper, we quote the following abstract:
Let D_L denote the class of bounded, simply connected real analytic plain domains with reflection symmetries across two orthogonal axes, of which one has length L. Under generic conditions, we prove that if $\Omega_1, \Omega_2 \in D_L$ and if the Dirichlet spectra coincide, $\text{Spec}(\Omega_1) = \text{Spec}(\Omega_2)$, then $\Omega_1 = \Omega_2$ up to rigid motion.

(e) In 2008, Yan Wu and Shivakumar [16] use complex variable solution to (1) and derive an infinite system of linear algebraic equations for the ellipse with some numerical computations.

(f) Other results include a paper in 2000 by Ashbaugh et al [2] in proving results of Polya and Weinberger giving a new sharp bound for the ratio of the first two eigenvalues in the form

$$\frac{\lambda_1^2}{\lambda_2^2} = \left(\frac{j_{2,1}}{j_{1,1}}\right)^2,$$

where $j_{p,k}$ denotes the kth positive zero of the Bessel function $J_p(x)$. In addition, there are innumerable papers using numerical techniques for various regions in the antenna theory.

(g) In 1998, Ramm and Shivakumar [14] discuss the inequalities for the minimal eigenvalue of the Laplacian in an annulus as the inner circle moves towards the outer circle.

3 Preliminaries

We express (1) in terms of complex variables $z = x + iy$, $\bar{z} = x - iy$ and the system (1) and (2) becomes

$$\begin{aligned} u_{z\bar{z}} + \frac{\lambda^2}{4} u = 0 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \Gamma. \end{aligned} \tag{4}$$

We start the analysis using the completely integrated form of the solution to (1) as given on page 58 of Vekua [18] by

$$u = \left\{ f_0(z) - \int_0^z f_0(t) \frac{\partial}{\partial t} J_0 \left(\lambda \sqrt{\bar{z}(z-t)} \right) dt \right\} + \text{conjugate}, \quad (5)$$

where $f_0(z)$ is an arbitrary holomorphic function in Ω which can be formally expressed as

$$f_0(z) = \sum_{n=0}^{\infty} a_n z^n \quad (6)$$

and J_0 represents the Bessel function of first kind of order 0 given by

$$J_0 \left(\lambda \sqrt{\bar{z}(z-t)} \right) = \sum_{q=0}^{\infty} \left(-\frac{\lambda^2}{4} \right)^q \frac{\bar{z}^q (z-t)^q}{q! q!} . \quad (7)$$

Incidentally, (5) can also be derived by using

$$u = \sum_{n=0}^{\infty} z^n \, \overline{f_n(z)} + \text{conjugate}, \quad (8)$$

yielding

$$f_n(z) = \left(-\frac{\lambda^2}{4} \right)^n \frac{1}{(n-1)! n!} \int_0^z f_0(t)(z-t)^{n-1} dt . \quad (9)$$

Equations (8) and (9) will be of value in evaluating approximations for u by suitably truncating the series in (8).

Substituting for $f_0(z)$ from (6) in (5) the general solution for (4) yields [16]

$$u = 2a_0 J_0 \left(\lambda \sqrt{z\bar{z}} \right) + \sum_{n=1}^{\infty} a_n \sum_{k=0}^{\infty} \left(-\frac{\lambda^2}{4} \right)^k A_{n\,k} \left(z^n + \bar{z}^n \right) (z\bar{z})^k \quad (10)$$

where

$$A_{n\,k} = \frac{n!}{k!(n+k)!}, \quad n = 1, 2, \cdots, \quad k = 0, 1, \cdots, \quad (11)$$

From an identity given in Abramowitz and Stegun [1], we have, when n is even,

$$z^n + \bar{z}^n = \sum_{m=0}^{\frac{n}{2}} c_{mn} (z+\bar{z})^{n-2m} (z\bar{z})^m, \quad n = 0, 1, \cdots \quad (12)$$

where

$$c_{mn} = (-1)^m \frac{n}{m!} \frac{(n-m-1)!}{(n-2m)!}, \quad m = 0, 1, \cdots, n, \qquad n = 1, 2, 3, \cdots .$$

4 Boundary Γ

In this section we introduce a general parameterized analytical boundary Γ in (2) having a biaxial symmetry, including several cases used in Section 5.

(a) A general boundary

Based on the requirements stated in Section 2(d), as given in Zelditch [19], we consider the parameterized analytical boundary Γ with biaxial symmetry to be given by

$$(z+\bar{z})^2 = \sum_{n=0}^{\infty} d_{n1} (z\bar{z})^n \quad (13)$$

which yields, on using Cauchy products for infinite series,

$$(z+\bar{z})^{2p} = \sum_{n=0}^{\infty} d_{np} (z\bar{z})^n \quad (14)$$

where

$$d_{np} = \sum_{l=0}^{n} d_{l\,p-1} d_{n-l\,1}, \quad p = 1, 2, 3, \cdots . \quad (15)$$

We remark here that knowing the eigenvalues for (1) and (2), we need to find $d_{n,1}$, $n = 0, 1, 2, \cdots$. It may be noted that d_{np} contains $d_{01}, d_{02}, \cdots, d_{0n}$. In fact

$$d_{0p} = d_{01}^p, \quad d_{1p} = p d_{01} d_{11}, \quad \text{etc.} \quad (16)$$

(b) Circular boundary

We consider the circular boundary given by

$$x^2 + y^2 = a^2 \quad \text{or} \quad z\bar{z} = a^2 \quad (17)$$

(c) An elliptic boundary

Here we consider the elliptic boundary Γ given by

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \ \text{ or } \ (z+\bar{z})^2 = a + b z\bar{z}, \quad \alpha > \beta, \quad (18)$$

$$a = \frac{4\alpha^2\beta^2}{\beta^2 - \alpha^2}, \quad b = \frac{4\alpha^2}{\alpha^2 - \beta^2}$$

and

$$\alpha^2 = \frac{a}{4-b}, \quad \beta^2 = -\frac{a}{b}, \quad a < 0, \ b > 0 . \quad (19)$$

(d) **A square boundary**

We give this example to demonstrate why our analytical approach does not yield information of the boundary with sharp corners from a preknowledge of eigenvalues. Consider a square boundary given by $x = \pm a$, $y = \pm a$ or

$$z^4 + \bar{z}^4 = 2(z\bar{z})^2 - 16a^2(z\bar{z}) + 16a^4 \tag{20}$$

or

$$z^2 + \bar{z}^2 = 4(z\bar{z} - 2a)^2 . \tag{21}$$

5 Solutions for (1) and (2)

We assume

$$f_0(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n} \tag{22}$$

instead of (6) and the solution (10) now becomes

$$u = 2a_0 J_0\left(\lambda\sqrt{z\bar{z}}\right) + \sum_{n=1}^{\infty} a_{2n} \sum_{k=0}^{\infty} \left(-\frac{\lambda^2}{4}\right)^k A_{2n\,k} \left(z^{2n} + \bar{z}^{2n}\right)(z\bar{z})^k \tag{23}$$

where

$$A_{2n\,k} = \frac{(2n)!}{k!(2n+k)!}, \qquad n = 1, 2, \cdots \quad \text{and} \quad k = 0, 1, 2, \cdots \tag{24}$$

Also (12) yields

$$z^{2n} + \bar{z}^{2n} = \sum_{m=0}^{n} \left[(-1)^m \frac{2n}{m!} \frac{(2n-m-1)!}{(2n-2m)!} \times (z+\bar{z})^{2(n-m)} (z\bar{z})^m\right], \tag{25}$$

$$n = 1, 2, 3 \cdots ,$$

which on substitution in (23) gives

$$u = 2a_0 J_0(\lambda\sqrt{z\bar{z}}) + \sum_{n=1}^{\infty} a_{2n} \sum_{k=0}^{\infty} \left(\frac{-\lambda^2}{4}\right)^k A_{2nk} \times \sum_{m=0}^{n} (-1)^m \frac{2n(2n-m-1)!}{m!(2n-2m)!} \times (z+\bar{z})^{2(n-m)} (z\bar{z})^m \tag{26}$$

(a) General boundary Γ

From Section 4 (a), we can substitute (14) in (26) to get the solution u to (1) on the boundary (13) as

$$u = 2a_0 J_0\left(\lambda\sqrt{z\bar{z}}\right) + \sum_{n=1}^{\infty} a_{2n} \left\{ \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} \sum_{m=0}^{n} D_{n\,kqm} (z\bar{z})^{k+q+m} \right\} \tag{27}$$

where

$$D_{n\,kqm} = \left(-\frac{\lambda^2}{4}\right)^k A_{2n\,k} b_{mn} d_{q\,n-m} , \tag{28}$$

$$b_{mn} = \frac{(-1)^m\, 2n}{m!} \frac{(2n-m-1)!}{(2n-2m)!} \qquad m = 0, 1, 2, \cdots, \infty \quad n = 1, 2, \cdots \infty \tag{29}$$

and

$$d_{00} = 1, \quad d_{i0} = 0, \quad i = 1, 2, \cdots .$$

After rearrangement of summations, (27) becomes

$$\begin{aligned} u = {} & 2a_0 + \sum_{n=1}^{\infty} a_{2n} D_{n000} + \left[2a_0\left(-\frac{\lambda^2}{4}\right) + \sum_{n=1}^{\infty} a_{2n} \left\{ \sum_{i=0}^{1} \sum_{p=0}^{1-i} D_{np\,1-i-p\,i} \right\}\right] z\bar{z} \\ & + \sum_{q=2}^{\infty} \left\{ 2a_0 \left(-\frac{\lambda^2}{4}\right)^q \frac{1}{q!q!} + \sum_{n=1}^{q-1} a_{2n} \left[\sum_{i=0}^{n} \sum_{p=0}^{q-1} D_{np\,q-i-p\,i}\right] + \sum_{n=q}^{\infty} a_{2n} \left[\sum_{i=0}^{q} \sum_{p=0}^{q-1} D_{np\,q-i-p\,i}\right] \right\} (z\bar{z})^q . \end{aligned} \tag{30}$$

To satisfy (2), now given by (13), we equate the coefficients of $(z\bar{z})^q$, $q = 0, 1, 2, \cdots$ to zero to get the infinite linear algebraic system.

$$2a_0 + \sum_{n=1}^{\infty} a_{2n} d_{0n} = 0\,,$$

$$\left(\frac{-\lambda^2}{4}\right) 2a_0\mu + \sum_{n=1}^{\infty} a_{2n}\left[d_{1n}b_{0n} + d_{0n-1}b_{1n}\right] = 0\,,$$

$$\left(\frac{-\lambda^2}{4}\right) a_{2n}\left\{ \left(\sum_{i=0}^{n} + \sum_{i=0}^{q}\right) \times \sum_{p=0}^{q-1} \mu^p A_{2np} d_{q-p-i\,n-i} b_{in} \right\} = 0,$$

where

$$\mu = \left(\frac{-\lambda^2}{4}\right)\,. \tag{31}$$

Writing the above system as

$$\sum_{n=0}^{\infty} f_{qn} a_{2n}, \quad q = 0, 1, 2, \cdots, \tag{32}$$

we note that f_{qn} is a polynomial of degree q in μ and contains only d_{01}, d_{11}, d_{21}, $\cdots$. The values μ are determined by formally setting

$$D = \det(f_{qn}) = 0 \tag{33}$$

which can be expressed as an infinite series in μ given by

$$D = \sum_{i=0}^{\infty} D_i \mu^i \tag{34}$$

and μ_i's are given by $D = 0$. We note that we can rewrite (34) as

$$D = D_0 \prod_{i=1}^{\infty} \left(1 - \frac{\mu}{\mu_i}\right) \tag{35}$$

formally suggesting that $\dfrac{D_i}{D_0}$ is the sum of the inverse products of $\mu_1,\ \mu_2,\ \cdots$ taken i at a time. In particular,

$$\frac{D_1}{D_0} = -\sum_{i=1}^{\infty} \frac{1}{\mu_i}, \qquad \frac{D_2}{D_0} = \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \frac{1}{\mu_i \mu_j}\,.$$

Also

$$D_0 = \det[f_{qn}]\Big|_{q=0}\,,$$

$$D_1 = \frac{\partial}{\partial q}\det[f_{qn}]\Big|_{q=0}\,, \quad \text{and} \tag{36}$$

$$D_i = \frac{1}{i!}\,\frac{\partial^i}{\partial q^i}\det[f_{qn}]\Big|_{q=0}\,, \quad i = 2, 3, \cdots.$$

It should be noted that the determinants D_i contain only d_{ij}, which in turn, contain d_{i1}'s only. Thus the equations (36) provide a formal approach to the determination of d_{i1}'s, $i = 0, 1, 2, \cdots$ once the eigenvalues are known. In the next three subsections, we apply the above approach to individual cases; circle, ellipse and square.

(b) A circle

It is well known that the solution to (1) and (2) is given by

$$u = a_0 J_0\left(\lambda\sqrt{z\bar{z}}\right) \tag{37}$$

and the eigenvalues are given by the zeros of $J_0(\lambda a) = 0$ namely

$$\lambda_i = \frac{j_{0i}}{a}, \qquad i = 1, 2, \cdots, \infty\,. \tag{38}$$

We can write using (37),

$$\sum_{q=0}^{\infty}\left(-\frac{\lambda^2}{4}\right)^q \frac{1}{q!q!} a^{2q} = \prod_{j=1}^{\infty}\left(1 - \frac{\lambda^2}{\lambda_j^2}\right)\,.$$

As in Section 5 (a), comparing the coefficients of λ^2, we get

$$a^2 = 4\sum_{j=1}^{\infty}\frac{1}{\lambda_j^2}\,,$$

thus establishing uniquely the value of a if all the eigenvalues $\lambda_1, \lambda_2, \cdots$ are known. In fact, substituting for λ_j from (38) we get

$$\frac{a^2}{4} = \sum_{j=1}^{\infty}\frac{a^2}{j_{0j}^2}$$

which checks with the well known fact that

$$\sum_{j=1}^{\infty}\frac{1}{j_{0j}^2} = 4\,.$$

(c) An Ellipse

Here we develop the solution to (1.1) in the elliptic domain described in Section 4(c). The derivation parallels the derivation described in Section 5(a) and (4.1) is now given by

$$d_{01} = a, \;\; d_{11} = b, \;\; d_{i1} = 0, \qquad i = 2, 3, \cdots \quad (39)$$

After considerable calculations, we get an infinite system of equations in a_{2n}, $n = 0, 1, 2, \cdots$ as given by

$$2a_0 + \sum_{n=1}^{\infty} a_{2n} a^n = 0$$

$$\begin{aligned} 2a_0 \frac{\mu^k}{k!k!} &+ \sum_{n=1}^{\infty} \mu^k \frac{(2n)!}{k!(2n+k)!} a^n \\ &+ \sum_{n=1}^{k} \left[\sum_{l=1}^{n} \frac{(2n)! \mu^{k-l} b_{ln}}{(k-l)!(2n+k-l)!} \right] a_{2n} \\ &+ \sum_{n=k+1}^{\infty} \sum_{l=1}^{k} \frac{(2n)! \mu^{k-l} b_{ln}}{(k-l)!(2n+k-l)!} a_{2n} = 0 \\ & k = 1, 2, 3, \cdots, \infty \end{aligned} \quad (40)$$

where $\mu = -\dfrac{\lambda^2}{4}$.
Writing (40) as

$$\sum_{n=0}^{\infty} f_{nq} a_{2n} = 0, \quad q = 0, 1, 2, \cdots,$$

we can formally express

$$\begin{aligned} \det [f_{nq}] &= \sum_{q=0}^{\infty} D_q \mu^q = D_0 \sum_{q=0}^{\infty} \frac{D_q}{D_0} \mu^q \\ &= D_0 \prod_{j=1}^{\infty} \left(1 - \frac{\mu}{\mu_j}\right) \end{aligned}$$

to get

$$\sum_{j=1}^{\infty} \frac{1}{\mu_j} = -\frac{D_1}{D_0}. \quad (41)$$

We note as before, infinite determinants are given by

$$D_i = \frac{1}{i!} \left. \frac{\partial^i f}{\partial \mu^i} \right|_{\mu=0} \qquad i = 1, 2, \cdots, \quad (42)$$

and $\det [f_{nq}] = 0$ gives the eigenvalues.
We will express $\dfrac{D_1}{D_0}$ entirely in terms of a and b thus ensuring a direct relationship between a, b and the eigenvalues. In this article we will assume a value for b which then determines the value for a. To facilitate calculations, we denote

$$\begin{aligned} \beta_{kn} &= \frac{(2n)!}{k!(2n+k)!} a^n, \\ \alpha_{knl} &= \frac{(2n)! b_{ln}}{(k-l)!(2n+k-l)!} \end{aligned} \quad (43)$$

which enables us to write (40) as

$$2a_0 + \sum_{n=1}^{\infty} a_n a^n = 0$$

$$\begin{aligned} \frac{2a_0 \mu^k}{k!k!} &+ \mu^k \sum_{n=1}^{\infty} \beta_{kn} a_n + \sum_{l=1}^{k} \sum_{n=l}^{k} \mu^{k-l} \alpha_{knl} a_{2n} \\ &+ \sum_{n=k+1}^{\infty} \sum_{l=1}^{k} \mu^{k-l} \alpha_{knl} a_{2n} = 0 \\ & k = 1, 2, \cdots, \end{aligned} \quad (44)$$

We note that f_{nq} is a polynomial of degree q in μ. The determinant D_1 is the sum of an infinite determinants derived from differentiating D with respect to μ and setting $\mu = 0$. After considerable algebra, we get, using (41)

$$D_0 = \prod_{i=1}^{\infty} \alpha_{iii}$$

$$\begin{aligned} D_1 = \left(\prod_{i=1}^{\infty} \alpha_{iii} \right) \Bigg\{ & \frac{\beta_{11} - a}{\alpha_{111}} \\ &+ \sum_{n=2}^{\infty} \frac{\alpha_{nnn-1} \alpha_{n-1n-1n-1} - \alpha_{nn-1n-1} \alpha_{n-1nn-1}}{\alpha_{n-1n-1n-1} \alpha_{nnn}} \Bigg\}. \end{aligned}$$

Substituting in (42), using (43) we get

$$\frac{D_1}{D_0} = -\frac{2}{3}\frac{a}{b-2} - \sum_{n=2}^{\infty}\frac{2a}{4n^2-1} \times \left[\frac{\sum\limits_{m=0}^{n-1}\frac{(-1)^m(2n-m-1)!(n-m)}{m!(2n-2m)!b^{m+1}}}{\sum\limits_{m=0}^{n}\frac{(-1)^m(2n-m-1)!}{m!(2n-2m)!b^m}}\right] \tag{45}$$

$$= -\sum_{j=1}^{\infty}\frac{1}{\mu_j} = \sum_{j=1}^{\infty}\frac{4}{\lambda_j^2}.$$

Writing

$$g(n,b) = \frac{1}{4n^2-1}\left\{\frac{\sum\limits_{m=0}^{n-1}\frac{(-1)^m(2n-m-1)!(n-m)}{m!(2n-2m)!b^{m+1}}}{\sum\limits_{m=0}^{n}\frac{(-1)^m(2n-m-1)!}{m!(2n-2m)!b^m}}\right\} \tag{46}$$

and

$$h(k,b) = \sum_{n=2}^{k} g(n,b)\,, \tag{47}$$

we can express (45) by

$$a = -\frac{\sum\limits_{j=1}^{\infty}\frac{4}{\lambda_j^2}}{\frac{2}{3b-6} + 2h(\infty,b)}\,. \tag{48}$$

α β	a b	p	$\sum\limits_{j=1}^{p}\frac{1}{\lambda_j^2}$ $1<\lambda<1000$	k	$g(k,b)$ (5.25)	$h(k,b)$ (5.26)
1	-0.0404	13	0.0274	50	0.0124	1.7709
0.1	4.0404			100	0.0062	2.1967
				200	0.0031	2.6240
1	-0.1667	38	0.1147	50	0.0060	0.9877
0.2	4.1667			100	0.0030	1.1937
				200	0.0015	1.4009
1	-0.3956	63	0.2255	50	0.0038	0.6537
0.3	4.3956			100	0.0019	0.7842
				200	0.0009	0.0038

Table 1

To demonstrate our constructive approach, we assume a value for b which in turn gives the value of a if $\sum\limits_{j=1}^{\infty}\frac{1}{\lambda_j^2}$ is known. It is well known that for a given ellipse, there are no known results which give all the eigenvalues. It is interesting to note that for a given a and b of an ellipse we can find the infinite sum $\sum\limits_{j=1}^{\infty}\frac{1}{\lambda_j^2}$ since $h(\infty,b)$ contains only b. Naturally, to get a precise value of a for a given b we need precise values for $\sum\limits_{j=1}^{\infty}\frac{1}{\lambda_j^2}$ and for $h(\infty,b)$.

For numerical work, we can extract from (48) an approximate value of a for a known b as given by

$$a \approx -\frac{\sum\limits_{j=1}^{p}\frac{4}{\lambda_j^2}}{\frac{2}{3b-6} + 2h(k,b)}. \tag{49}$$

Table points to the numerical convergence of the series given by $h(\infty,b)$ by evaluating $h(k,b)$ for various values of k. Further, the sequence $\{g(n,b)\}$, $n = 1,2,\cdots$ yields a decreasing sequence in n. The calculations were done in MATLAB using a fine mesh for the finite differences method. In the case $\alpha = 1$, $\beta = 0.1$, we can calculate $\sum\limits_{j=1}^{\infty}\frac{1}{\lambda_j^2}$ from (48) when $h(\infty,b)$ is replaced by $h(200,b)$ giving $\sum\limits_{j=1}^{\infty}\frac{1}{\lambda_j^2} = 0.0563$ in comparison to $\sum\limits_{j=1}^{13}\frac{1}{\lambda_j^2} = 0.0274$.

6 A square boundary

We give this example to demonstrate how our approach does not yield information of the boundary

from a preknowledge of the eigenvalues in the case of a boundary containing sharp corners.
Consider the square boundary given by (21), namely,

$$z^2 + \bar{z}^2 = 4(z\bar{z} - 2a)^2 \,. \tag{50}$$

Instead of using (6), (10) and (12), we use

$$f_0(z) = \sum_{n=0}^{\infty} a_{4n} z^{4n} \tag{51}$$

to get

$$u = 2a_0 J_0\left(\lambda\sqrt{z\bar{z}}\right) + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(-\tfrac{\lambda^2}{4}\right)^k A_{4n\,k}\left(z^{4n}+\bar{z}^{4n}\right) a_{4n}\left(z\bar{z}\right)^k , \tag{52}$$

and

$$z^{4n} + \bar{z}^{4n} = \sum_{m=0}^{n} b_{mn}\left(z^2+\bar{z}^2\right)^{2n-2m}\left(z\bar{z}\right)^{2m} \ , \tag{53}$$

which gives on using (50)

$$z^{4n} + \bar{z}^{4n} = \sum_{m=0}^{n} b_{mn} 4^{n-m}\left(z\bar{z}-2a\right)^{n-m} \ . \tag{54}$$

Substituting (54) in (52), we get the solution u of (1) and (50) as given by

$$\begin{aligned} u &= 2a_0 J_0\left(\lambda\sqrt{z\bar{z}}\right) \\ &+ \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{n} A_{4n\,k} a_{4n}\left(z\bar{z}-2a\right)^{2(n-m)}\left(z\bar{z}\right)^{k+2m} \\ &= 2a_0 J_0\left(\lambda\sqrt{z\bar{z}}\right) \\ &\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{n}\sum_{q=0}^{2(n-m)} A_{mk}\binom{2(n-m)}{q} \\ &\quad\quad \times (-2a)^{2(n-m)-q} a_{4n}\left(z\bar{z}\right)^{k+q+2m} . \end{aligned}$$

Now u can be expressed as a power series in $(z\bar{z})$ and setting powers of $(z\bar{z})$ to zero, we get an infinite system of the form

$$\sum_{n=0}^{\infty} g_{qn} a_{4n} = 0, \qquad q = 0, 1, 2, \cdots . \tag{55}$$

As before, we formally write

$$D = \det\left(g_{nq}\right) = \sum_{i=0}^{\infty} D_i \mu^i = D_0 \sum_{i=0}^{\infty}\frac{D_i}{D_0}\mu^i$$

$$= D_0 \prod_{i=1}^{\infty}\left(\frac{D_i}{D_0} q^i\right) = D_0 \prod_{i=1}^{\infty}\left(1 - \frac{\mu}{\mu_i}\right) .$$

It is well known that for a square, the eigenvalues λ^2 are given by $\dfrac{m^2\pi^2}{a^2} + \dfrac{n^2\pi^2}{a^2}$ and using $\mu = -\dfrac{\lambda^2}{4}$, we get

$$\frac{D_1}{D_0} = -\sum_{i=1}^{\infty}\frac{1}{\mu_i} = \frac{4a^2}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\frac{1}{m^2+n^2} .$$

The double series is clearly divergent and hence the failure of the analytic approach, although $\det[g_{qn}] = 0$ yields the eigenvalues.

7 Mathieu equation

In this section, we consider the Mathieu equation

$$\frac{d^2y}{d\theta^2} + \left(\mu - 2q\cos 2\theta\right) y = 0 \tag{56}$$

where q is a parameter. Equation (56) is derived from (1) using separation of varibles in elliptical coordinates (p.721, [1]). See Shivakumar, Williams, and Rudraiah [15] for a detailed discussion of the solution to (56) and where the first 18 eigenvalues μ_1, μ_2, $\cdots$, μ_{18} are precisely calculated with upper and lower bounds. A simple but powerful algorithm is used which can calculate the eigenvalues for any required degree of accuracy. The eigenvalues were derived for $q = 1$. Our aim here is to use the 18 eigenvalues for (56) to lead to establishing the value of q, which is $q = 1$ in the present case.
Using

$$y(\theta) = \sum_{k=1}^{\infty} x_k \cos 2(k-1)\theta, \tag{57}$$

(56) yields the infinite system

$$A\underset{\sim}{x} = \mu\underset{\sim}{x} \tag{58}$$

where $\underset{\sim}{x} = (x_1, x_2, \cdots)^t$ and A is given by

$$A = (a_{ij})$$

where $a_{11} = 0,\ a_{12} = q,\ a_{21} = 2q,\ a_{23} = q,$

$$a_{ij} = \begin{cases} 4(i-1)^2 - \mu, & i = j, \quad j = 2, 3, \cdots \\ q, & j = i-1,\ i+1 \quad i = 3, 4, \cdots \end{cases} \tag{59}$$

Denoting by $D(\mu)$, the determinants of $(A-\mu I)$, we can write as before

$$D(\mu) = \sum_{i=1}^{\infty} D_i \mu^i = D_0 \prod_{i=1}^{\infty}\left(1-\frac{\mu}{\mu_i}\right) \quad (60)$$

giving

$$-\frac{D_1}{D_0} = \sum_{i=1}^{\infty}\frac{1}{\mu_i} \quad (61)$$

and

$$D_i = \frac{1}{i!}\frac{\partial}{\partial \mu_i} D\bigg|_{\mu=0}.$$

For convenience, we will define the determinants $B^{i,j}$ by

$$B^{i,j}(q) = \begin{vmatrix} 4i^2 & q & 0 & \cdots & \cdots \\ q & 4(i+1)^2 & q & \cdots & \cdots \\ & q & 4(i+2)^2 & q & \cdots \\ & & & \ddots & \\ & & & & 4j^2 \end{vmatrix}$$

where $i \geq 2$ and $j > i$, i and j being positive integers. After considerable calculations, we get

$$D(0) = D_0 = -2q^2 B^{2,\infty}(q),$$

$$D_1 = D'(0) = -B^{1,\infty}(q) + 2q^2 B^{3,\infty}(q) + 2q^2 \sum_{i=3}^{\infty} B^{2,\,i-1}(q) B^{i+1,\,\infty}(q)$$

giving

$$-\frac{D'(0)}{D(0)} = \frac{B^{1,\infty}(q)}{2q^2\, B^{2,\infty}(q)} - \frac{1}{B^{2,\infty}(q)} \times \left\{ B^{3,\infty}(q) + \sum_{i=3}^{\infty} B^{2,\,i-1}(q) B^{i+1,\,\infty}(q) \right\} \quad (62)$$

and from (61),

$$-\frac{D'(0)}{D(0)} = \sum_{i=1}^{\infty}\frac{1}{\mu_i} \approx \sum_{i=1}^{18}\frac{1}{\mu_i} = -1.8246\,. \quad (63)$$

We can use (62) to numerically evaluate the right side for various values of q and truncating the infinite matrices to various sizes. We then match that value of q which is closest to (63).

As an alternate approximate method and since all the determinants are strictly diagonally dominant $(0 < 2q < 4\,i^2)$ we can use the error estimates (see [15])

$$\frac{1}{|b_{jj}|(1-\mu_j)} \leq \frac{B_{jj}(\mu)}{\det B} \leq \frac{1}{|b_{jj}|(1-\mu_j)}\,,$$

where for the matrix $B = (b_{ij})_{n\times n}$, B_{jj} represents co-factors of b_{jj} and

$$\mu_j |b_{jj}| = -\sum_{k\neq j} b_{jk}.$$

These approximations yield for (63)

$$S(q,n) = \frac{4\cdot 1^2 - q}{2q^2} - \sum_{i=3}^{n}\frac{1}{4i^2-2q} \leq -\frac{D'(0)}{D(0)} \leq \frac{4\cdot 1^2 + q}{2q^2} - \sum_{i=3}^{n}\frac{1}{4i^2+2q} = R(q,n) \quad (64)$$

Since the two series in the inequality in (64) are convergent, it is easily seen that the series in (62) is also convergent. In the table below, we give the values for $-\frac{D'(0)}{D(0)}$ for various values of q (near $q=1$) and the determinants actually evaluated for various sizes of the truncated matrices of size $k\times k$.

q	k	n	$-\frac{D'(0)}{D(0)}$
0.98	10	20	1.9135
		30	1.9135
		40	1.9135
0.99	10	20	1.8717
		30	1.8717
		40	1.8717
1.00	10	20	1.8311
		30	1.8311
		40	1.8311
1.01	10	20	1.7917
		30	1.7917
		40	1.7917
1.02	10	20	1.7534
		30	1.7534
		40	1.7534

Table 2: Values of $-\frac{D'(0)}{D(0)}$ for various values of q

From Table 2, the value of q is determined by knowing the first nineteen eigenvalues. For $q = 1.0$, equation (62) gives an approximate value of 1.8311, while the sum of the inverses of first 19 eigenvalues is 1.8246. In Table 3 values of $R(q,n)$ and $S(q,n)$ are given. We give rough upper and lower bounds $R(q,n)$ and $S(q,n)$ for various values of q and n.

Table 3: Values of $R(q,n)$ and $S(q,n)$

q	$S(q,n)$	$R(q,n)$
0.9	1.9023	2.9403
1.0	1.4887	2.4158
1.1	1.1871	2.0235

Clearly 1.8246 lies between $S(1,20)$ and $R(1,20)$.

We note that (62) and (63) expresses q in terms of the eigenvalues. In fact the infinite determinants can be evaluated to any required degree of accuracy since all the determinants are strictly diagonally dominant [15].

8 Conclusions

In this article, we have demonstrated a constructive approach when the answer is 'yes'. We give an interpretation to 'a preknowledge of eigenvalues' as knowing the sums of the inverse products of the eigenvalues μ_i, $i = 1, 2, \cdots$, taken i at a time. Equation (46) in the case of an ellipse expresses the parameters of the ellipse in terms of the eigenvalues. Similarly, equations (62) and (63) express the parameter q in terms of the eigenvalues. Our conjecture is that for the answer to be 'yes', it is necessary for the sums of the inverse products of the eigenvalues μ_i, $i = 1, 2, \cdots$, taken i at a time to be convergent series. Another conjecture is that all analytic curves (curvilinear polygons for example) yield the answer 'yes'. Future work consists of dealing with doubly connected regions (circular annulus, elliptic ring, elliptic region with a circular hole, etc.) and infinite region with a hole using conformal mapping when needed.
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