A Generalized Volterra-Fredholm Type Integral Inequality For Discontinuous Functions

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Abstract: In this paper, a new generalized Volterra-Fredholm type nonlinear integral inequality for discontinuous functions is established, which can be used in analysis for the boundedness of solutions of certain Volterra-Fredholm type integral equations. Our results generalize the main results in [18, 19].

Key–Words: Integral inequality; discontinuous function; Integral equation; Differential equation; Bounded

1 Introduction

In recent years many integral inequalities for continuous and discontinuous functions have been established, which provide handy tools for investigating the quantitative and qualitative properties of solutions of integral and differential equations [1-17]. In the investigations for integral inequalities, the idea of generalizing known integral inequalities have gained extensive attention.

In [18], Pachpatte established the following integral inequality

$$(a):\quad u(t)\le k+\int_{h(\alpha)}^{h(t)}a(t,s)[f(s)u(s)+\int_{h(\alpha)}^{s}c(s,\sigma)u(\sigma)d\sigma]ds+\int_{h(\alpha)}^{h(\beta)}b(t,s)u(s)ds,$$

where $u(t)$ is unknown function with $u(t)\in C(I,R_+)$, $I=[\alpha,\beta]$, $a(t,s)$, $b(t,s)$, $c(t,s)\in C(D,R_+)$, $D=\{(t,s)\in I^2:\alpha\le s\le t\le\beta\}$, $a(t,s), b(t,s)$ are nondecreasing in t for each $s\in I$, $h(t)\in C^1(I,I)$ is nondecreasing with $h(t)\le t$ on I, $k\ge 0$ is a constant.

Recently, in [19, Theorem 2.1], the author presented the following Volterra-Fredholm type integral inequality that generalized the inequality (a).

$$(b):\ u(t)\le k+\int_{\alpha(t_0)}^{\alpha(t)}\sigma_1(s)[f(s)\omega(u(s))+\int_{\alpha(t_0)}^{s}\sigma_2(\tau)\omega(u(\tau))d\tau]ds+\int_{\alpha(t_0)}^{\alpha(T)}\sigma_1(s)[f(s)\omega(u(s))+\int_{\alpha(t_0)}^{s}\sigma_2(\tau)\omega(u(\tau))d\tau]ds,$$

where $u(t)$ is unknown function and $u(t)$, $f(t)$, $\sigma_1(t)$, $\sigma_2(t)\in C(I,R_+)$, $\alpha\in C^1(I,I)$ is nondecreasing with $\alpha(t)\le t$ on I, $\omega\in C(R_+,R_+)$ is nondecreasing with $\omega(u)>0$ for $u>0$.

In the present paper, we will establish a new more generalized integral inequality for discontinuous functions than the inequalities mentioned above, also we will present one application for it, and will obtain new bounds for solutions of certain Volterra-Fredholm type integral equations.

2 Main Results

In the rest of the paper, we denote the set of real numbers as R, and $R_+=[0,\infty)$. I, $\widehat{I}$, $\widetilde{I}$ denote intervals in R, and $I=[t_0,T]$, $\widehat{I}=[x_0,A]$, $\widetilde{I}=[y_0,B]$ respectively, where $T>t_0,\ A>x_0,\ B>y_0$ are three fixed numbers.

Theorem 2.1 Suppose $u(t)$ is a nonnegative continuous function defined on I with the first kind of discontinuities in the points $t_i,\ i=1,2,...,n$, and $t_0<t_1<t_2<...<t_n<t_{n+1}=T$. $h_i(t)\in C(I,R_+),\ i=1,2$, and $f_i(s,t)$, $g_i(s,t)$, $\frac{\partial f_i(s,t)}{\partial t}$, $\frac{\partial g_i(s,t)}{\partial t}\in C(I\times I,R_+),\ i=1,2$. $p>0$ is a constant. $\tau(t)\in C^1(I,I)$ is nondecreasing with $\tau(t)\le t$, and $\tau(t)>t_i$ for $\forall t\in(t_i,t_{i+1}],\ i=$

$0,1,...,n$. $\omega \in C(R_+, R_+)$, and ω is nondecreasing with $\omega(u) > 0$ for $u > 0$. $C \geq 0$, $\beta_i > 0$, $i = 1,2,...,n$, and C, β_i are constants. If for $t \in I$, $u(t)$ satisfies the following inequality

$$u^p(t) \leq C+$$

$$\int_{\tau(t_0)}^{\tau(t)} [f_1(s,t)\omega(u(s)) + g_1(s,t)\int_{\tau(t_0)}^{s} h_1(\xi)\omega(u(\xi))d\xi]ds$$

$$+\int_{\tau(t_0)}^{\tau(t_i)} [f_1(s,t)\omega(u(s)) + g_1(s,t)\int_{\tau(t_0)}^{s} h_1(\xi)\omega(u(\xi))d\xi]ds$$

$$+\int_{t_0}^{t} [f_2(s,t)\omega(u(s)) + g_2(s,t)\int_{t_0}^{s} h_2(\xi)\omega(u(\xi))d\xi]ds$$

$$+\int_{t_0}^{t_i} [f_2(s,t)\omega(u(s)) + g_2(s,t)\int_{t_0}^{s} h_2(\xi)\omega(u(\xi))d\xi]ds$$

$$+\sum_{t_0<t_j<t} \beta_j u(t_j - 0), \tag{1}$$

then

$$u(t) \leq \{G_{i-1}^{-1}\{G_{i-1}(m_i) + \int_{\tau(t_{i-1})}^{\tau(t)} [f_1(s,t)$$

$$+g_1(s,t)\int_{\tau(t_0)}^{s} h_1(\xi)d\xi]ds$$

$$+\int_{t_{i-1}}^{t} [f_2(s,t) + g_2(s,t)\int_{t_0}^{s} h_2(\xi)d\xi]ds\}\}^{\frac{1}{p}} \tag{2}$$

holds for $\forall t \in (t_{i-1}, t_i]$, $i = 1,...,n+1$, where

$$G_i(v) = \int_{b_i}^{v} \frac{1}{\omega(s^{\frac{1}{p}}(t))} ds, \ i = 0,1,...,n, \tag{3}$$

$$H_i(t) = G_i(2t - b_i) - G_i(t), \tag{4}$$

and H_i are nondecreasing on $t \geq b_i$, $i = 0,1,...,n$,

$$b_0 = C, \ b_i = a_i + \beta_i a_i^{\frac{1}{p}}, \ i = 1,2,...,n, \tag{5}$$

$$a_i = G_{i-1}^{-1}\{G_{i-1}(m_i) + \int_{\tau(t_{i-1})}^{\tau(t_i)} [f_1(s,t_i)$$

$$+g_1(s,t_i)\int_{\tau(t_0)}^{s} h_1(\xi)d\xi]ds$$

$$+\int_{t_{i-1}}^{t_i} [f_2(s,t_i) + g_2(s,t_i)\int_{t_0}^{s} h_2(\xi)d\xi]ds\}, \tag{6}$$

for $i = 1,2,...,n$,

$$m_i = H_{i-1}^{-1}\{\int_{\tau(t_{i-1})}^{\tau(t_i)} [f_1(s,t_i) + g_1(s,t_i)\int_{\tau(t_0)}^{s} h_1(\xi)d\xi]ds$$

$$+\int_{t_{i-1}}^{t_i} [f_2(s,t_i) + g_2(s,t_i)\int_{t_0}^{s} h_2(\xi)d\xi]ds\}, \tag{7}$$

for $i = 1,2,...,n+1$.

Proof: Firstly we assume $C > 0$. Let the right side of (1) be $v(t)$, then

$$u(t) \leq v^{\frac{1}{p}}(t), \ t \in I. \tag{8}$$

Define

$$v_i(t) = b_i+$$

$$\int_{\tau(t_i)}^{\tau(t)} [f_1(s,t)\omega(u(s)) + g_1(s,t)\int_{\tau(t_0)}^{s} h_1(\xi)\omega(u(\xi))d\xi]ds$$

$$+\int_{\tau(t_i)}^{\tau(t_{i+1})} [f_1(s,t)\omega(u(s)) + g_1(s,t)\int_{\tau(t_0)}^{s} h_1(\xi)\omega(u(\xi))d\xi]ds$$

$$+\int_{t_i}^{t} [f_2(s,t)\omega(u(s)) + g_2(s,t)\int_{t_0}^{s} h_2(\xi)\omega(u(\xi))d\xi]ds$$

$$+\int_{t_i}^{t_{i+1}} [f_2(s,t)\omega(u(s)) + g_2(s,t)\int_{t_0}^{s} h_2(\xi)\omega(u(\xi))d\xi]ds,$$

$$i = 0,1,...,n.$$

Case 1: If $t \in (t_0, t_1]$, considering $b_0 = C$, then

$$v(t) = C+$$

$$\int_{\tau(t_0)}^{\tau(t)} [f_1(s,t)\omega(u(s)) + g_1(s,t)\int_{\tau(t_0)}^{s} h_1(\xi)\omega(u(\xi))d\xi]ds$$

$$+\int_{\tau(t_0)}^{\tau(t_1)} [f_1(s,t)\omega(u(s)) + g_1(s,t)\int_{\tau(t_0)}^{s} h_1(\xi)\omega(u(\xi))d\xi]ds$$

$$+\int_{t_0}^{t} [f_2(s,t)\omega(u(s)) + g_2(s,t)\int_{t_0}^{s} h_2(\xi)\omega(u(\xi))d\xi]ds$$

$$+\int_{t_0}^{t_1} [f_2(s,t)\omega(u(s)) + g_2(s,t)\int_{t_0}^{s} h_2(\xi)\omega(u(\xi))d\xi]ds$$

$$= v_0(t). \tag{9}$$

From (8), obviously we have

$$u(t) \leq v_0^{\frac{1}{p}}(t). \tag{10}$$

According to the assumption of $\tau(t)$ we have $t_0 < \tau(t) \leq t$ for $t \in (t_0, t_1]$. So $\tau(t) \in (t_0, t_1]$, and then

$$u(\tau(t)) \leq v_0^{\frac{1}{p}}(\tau(t)) \leq v_0^{\frac{1}{p}}(t).$$

Furthermore

$$v_0'(t) =$$

$$\int_{\tau(t_0)}^{\tau(t)}[\frac{\partial f_1(s,t)}{\partial t}\omega(u(s))+\frac{\partial g_1(s,t)}{\partial t}\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$

$$+[f_1(\tau(t),t)\omega(u(\tau(t)))$$

$$+g_1(\tau(t),t)\int_{\tau(t_0)}^{\tau(t)}h_1(\xi)\omega(u(\xi))d\xi]\tau'(t)$$

$$+\int_{t_0}^{t}[\frac{\partial f_2(s,t)}{\partial t}\omega(u(s))+\frac{\partial g_2(s,t)}{\partial t}\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$

$$+[f_2(t,t)\omega(u(t))+g_2(t,t)\int_{t_0}^{t}h_2(\xi)\omega(u(\xi))d\xi]$$

$$\le \{\int_{\tau(t_0)}^{\tau(t)}[\frac{\partial f_1(s,t)}{\partial t}+\frac{\partial g_1(s,t)}{\partial t}\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$

$$+[f_1(\tau(t),t)+g_1(\tau(t),t)\int_{\tau(t_0)}^{\tau(t)}h_1(\xi)d\xi]\tau'(t)$$

$$+\int_{t_0}^{t}[\frac{\partial f_2(s,t)}{\partial t}+\frac{\partial g_2(s,t)}{\partial t}\int_{t_0}^{s}h_2(\xi)d\xi]ds$$

$$+[f_2(t,t)+g_2(t,t)\int_{t_0}^{t}h_2(\xi)d\xi]\}\omega(v_0^{\frac{1}{p}}(t)).$$

Then

$$\frac{v_0'(t)}{\omega(v_0^{\frac{1}{p}}(t))}\le$$

$$\{\int_{\tau(t_0)}^{\tau(t)}[\frac{\partial f_1(s,t)}{\partial t}+\frac{\partial g_1(s,t)}{\partial t}\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$

$$+[f_1(\tau(t),t)+g_1(\tau(t),t)\int_{\tau(t_0)}^{\tau(t)}h_1(\xi)d\xi]\tau'(t)$$

$$+\int_{t_0}^{t}[\frac{\partial f_2(s,t)}{\partial t}+\frac{\partial g_2(s,t)}{\partial t}\int_{t_0}^{s}h_2(\xi)d\xi]ds$$

$$+[f_2(t,t)+g_2(t,t)\int_{t_0}^{t}h_2(\xi)d\xi]\}. \qquad (11)$$

An integration for (6) with respect to t from t_0 to t yields

$$G_0(v_0(t))-G_0(v_0(t_0))\le$$

$$\int_{\tau(t_0)}^{\tau(t)}[f_1(s,t)+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$

$$+\int_{t_0}^{t}[f_2(s,t)+g_2(s,t)\int_{t_0}^{s}h_2(\xi)d\xi]ds, \qquad (12)$$

where G_0 is defined in (3).

Considering G_0 is nondecreasing, then it follows

$$v_0(t)\le G_0^{-1}\{G_0(v_0(t_0))+$$

$$\int_{\tau(t_0)}^{\tau(t)}[f_1(s,t)+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$

$$+\int_{t_0}^{t}[f_2(s,t)+g_2(s,t)\int_{t_0}^{s}h_2(\xi)d\xi]ds\}. \qquad (13)$$

Take $t=t_1$ in (13) we can obtain

$$v_0(t_1)\le G_0^{-1}\{G_0(v_0(t_0))+$$

$$\int_{\tau(t_0)}^{\tau(t_1)}[f_1(s,t_1)+g_1(s,t_1)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$

$$+\int_{t_0}^{t_1}[f_2(s,t_1)+g_2(s,t_1)\int_{t_0}^{s}h_2(\xi)d\xi]ds\}. \qquad (14)$$

From (9) and (14) we have

$$2v_0(t_0)-C=v_0(t_1)\le G_0^{-1}\{G_0(v_0(t_0))+$$

$$\int_{\tau(t_0)}^{\tau(t_1)}[f_1(s,t_1)+g_1(s,t_1)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$

$$+\int_{t_0}^{t_1}[f_2(s,t_1)+g_2(s,t_1)\int_{t_0}^{s}h_2(\xi)d\xi]ds\}, \qquad (15)$$

that is,

$$G_0(2v_0(t_0)-C)-G_0(v_0(t_0))\le$$

$$\int_{\tau(t_0)}^{\tau(t_1)}[f_1(s,t_1)+g_1(s,t_1)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$

$$+\int_{t_0}^{t_1}[f_2(s,t_1)+g_2(s,t_1)\int_{t_0}^{s}h_2(\xi)d\xi]ds. \qquad (16)$$

Since $H_0(t)=G_0(2t-b_0)-G_0(t)$ is nondecreasing on $t\ge b_0$, and $b_0=C,\ v_0(t_0)\ge C$, then from (16) it follows

$$v_0(t_0)\le$$

$$H_0^{-1}\{\int_{\tau(t_0)}^{\tau(t_1)}[f_1(s,t_1)+g_1(s,t_1)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$

$$+\int_{t_0}^{t_1}[f_2(s,t_1)+g_2(s,t_1)\int_{t_0}^{s}h_2(\xi)d\xi]ds\}=m_1, \qquad (17)$$

where m_1 is defined in (7).

Combining (10), (13) and (17) we can obtain

$$u(t)\le v_0^{\frac{1}{p}}(t)\le$$

$$\{G_0^{-1}\{G_0(m_1)+\int_{\tau(t_0)}^{\tau(t)}[f_1(s,t)+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$
$$+\int_{t_0}^{t}[f_2(s,t)+g_2(s,t)\int_{t_0}^{s}h_2(\xi)d\xi]ds\}\}^{\frac{1}{p}}. \quad (18)$$

Especially, if we take $t=t_1$, then we have

$$u(t_1)\le v_0^{\frac{1}{p}}(t_1)\le$$

$$\{G_0^{-1}\{G_0(m_1)+\int_{\tau(t_0)}^{\tau(t_1)}[f_1(s,t_1)+g_1(s,t_1)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$
$$+\int_{t_0}^{t_1}[f_2(s,t_1)+g_2(s,t_1)\int_{t_0}^{s}h_2(\xi)d\xi]ds\}\}^{\frac{1}{p}}$$
$$=a_1, \quad (19)$$

where a_1 is defined in (6).

Case 2: If $t\in(t_1,t_2]$, then from the definition of a_1 and (18), (19) we have

$$u(t_1-0)\le v_0^{\frac{1}{p}}(t_1-0)\le v_0^{\frac{1}{p}}(t_1)\le a_1^{\frac{1}{p}}.$$

So

$$v(t)=C+$$
$$\int_{\tau(t_0)}^{\tau(t)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{\tau(t_0)}^{\tau(t_2)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_0}^{t}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_0}^{t_2}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\beta_1 u(t_1-0)$$
$$=C+$$
$$2\int_{\tau(t_0)}^{\tau(t_1)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+2\int_{t_0}^{t_1}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{\tau(t_1)}^{\tau(t)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{\tau(t_1)}^{\tau(t_2)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_1}^{t}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_1}^{t_2}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\beta_1 u(t_1-0)$$
$$=v_0(t_1)+$$
$$\int_{\tau(t_1)}^{\tau(t)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{\tau(t_1)}^{\tau(t_2)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_1}^{t}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_1}^{t_2}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\beta_1 u(t_1-0)$$
$$\le a_1+\beta_1 a_1^{\frac{1}{p}}+$$
$$\int_{\tau(t_1)}^{\tau(t)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{\tau(t_1)}^{\tau(t_2)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_1}^{t}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_1}^{t_2}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$=b_1+$$
$$\int_{\tau(t_1)}^{\tau(t)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{\tau(t_1)}^{\tau(t_2)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_1}^{t}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_1}^{t_2}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$=v_1(t) \quad (20)$$

where b_1 is defined in (5).

Then following in a similar process with (9)-(18), we can obtain

$$u(t)\le v_1^{\frac{1}{p}}(t)\le$$
$$\{G_1^{-1}\{G_1(m_2)+$$
$$\int_{\tau(t_0)}^{\tau(t)}[f_1(s,t)+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$
$$+\int_{t_0}^{t}[f_2(s,t)+g_2(s,t)\int_{t_0}^{s}h_2(\xi)d\xi]ds\}\}^{\frac{1}{p}}. \quad (21)$$

where m_2 is defined in (7).

Especially, if we take $t=t_2$, then we have

$$u(t_2)\le v_1^{\frac{1}{p}}(t_2)\le$$

$$\{G_1^{-1}\{G_1(m_2)+$$
$$\int_{\tau(t_0)}^{\tau(t_2)}[f_1(s,t_2)+g_1(s,t_2)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$
$$+\int_{t_0}^{t_2}[f_2(s,t_2)+g_2(s,t_2)\int_{t_0}^{s}h_2(\xi)d\xi]ds\}\}^{\frac{1}{p}}$$
$$=a_2, \tag{22}$$

where a_2 is defined in (6).

Case 3: If for $t\in(t_{i-1},t_i]$,

$$u(t)\le v_{i-1}^{\frac{1}{p}}(t)$$
$$\le\{G_{i-1}^{-1}\{G_{i-1}(m_i)+$$
$$\int_{\tau(t_{i-1})}^{\tau(t)}[f_1(s,t)+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$
$$+\int_{t_{i-1}}^{t}[f_2(s,t)+g_2(s,t)\int_{t_0}^{s}h_2(\xi)d\xi]ds\}\}^{\frac{1}{p}},$$

and especially

$$u(t_i)\le v_{i-1}^{\frac{1}{p}}(t_i)\le$$
$$\{G_{i-1}^{-1}\{G_{i-1}(m_i)+$$
$$\int_{\tau(t_{i-1})}^{\tau(t_i)}[f_1(s,t_i)+g_1(s,t_i)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$
$$+\int_{t_{i-1}}^{t_i}[f_2(s,t_i)+g_2(s,t_i)\int_{t_0}^{s}h_2(\xi)d\xi]ds\}\}^{\frac{1}{p}}$$
$$=a_i,$$

then for $t\in(t_i,t_{i+1}]$,

$$v(t)=C+$$
$$\int_{\tau(t_0)}^{\tau(t)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{\tau(t_0)}^{\tau(t_2)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_0}^{t}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_0}^{t_2}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\sum_{t_0<t_j<t}\beta_j u(t_j-0)$$
$$=C+$$
$$2\int_{\tau(t_0)}^{\tau(t_i)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+2\int_{t_0}^{t_i}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{\tau(t_i)}^{\tau(t)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{\tau(t_i)}^{\tau(t_{i+1})}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_i}^{t}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_i}^{t_{i+1}}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\sum_{t_0<t_j<t}\beta_j u(t_j-0)$$
$$\le v_{i-1}(t_i)$$
$$+\int_{\tau(t_i)}^{\tau(t)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{\tau(t_i)}^{\tau(t_{i+1})}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_i}^{t}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_i}^{t_{i+1}}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\beta_i u(t_i-0)$$
$$\le a_i+\beta_i a_i^{\frac{1}{p}}+$$
$$\int_{\tau(t_i)}^{\tau(t)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds+$$
$$\int_{\tau(t_i)}^{\tau(t_{i+1})}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_i}^{t}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_i}^{t_{i+1}}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$=b_i+$$
$$\int_{\tau(t_i)}^{\tau(t)}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{\tau(t_i)}^{\tau(t_{i+1})}[f_1(s,t)\omega(u(s))+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_i}^{t}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$+\int_{t_i}^{t_{i+1}}[f_2(s,t)\omega(u(s))+g_2(s,t)\int_{t_0}^{s}h_2(\xi)\omega(u(\xi))d\xi]ds$$
$$=v_i(t). \tag{23}$$

where b_i is defined in (5).

Similar to (9)-(18), we can obtain

$$u(t) \le v_i^{\frac{1}{p}}(t) \le$$

$$\{G_i^{-1}\{G_i(m_{i+1})+$$

$$\int_{\tau(t_i)}^{\tau(t)}[f_1(s,t)+g_1(s,t)\int_{\tau(t_0)}^{s}h_1(\xi)d\xi]ds$$

$$+\int_{t_i}^{t}[f_2(s,t)+g_2(s,t)\int_{t_0}^{s}h_2(\xi)d\xi]ds\}\}^{\frac{1}{p}},$$

and the proof is complete for $C>0$.

If $C=0$, we substitute C with ε in the above process, and then let $\varepsilon \to 0$, then we can obtain the desired inequality (2).

Corollary 2.2 Suppose $u,\ p,\ C,\ \omega,\ \tau,\ G_i,\ H_i,\ a_i,\ b_i,\ m_i,\ t_i,\ \beta_j$ are the same as in Theorem 2.1, $h(t) \in C(I,R_+),\ f(s,t),\ g(s,t),\ \frac{\partial f(s,t)}{\partial t},\ \frac{\partial g(s,t)}{\partial t} \in C(I\times I,R_+)$. If for $t\in I$, $u(t)$ satisfies the following inequality

$$u^p(t) \le C+$$

$$\int_{\tau(t_0)}^{\tau(t)}[f(s,t)\omega(u(s))+g(s,t)\int_{\tau(t_0)}^{s}h(\xi)\omega(u(\xi))d\xi]ds$$

$$+\int_{\tau(t_0)}^{\tau(t_i)}[f(s,t)\omega(u(s))+g(s,t)\int_{\tau(t_0)}^{s}h(\xi)\omega(u(\xi))d\xi]ds$$

$$+\sum_{t_0<t_j<t}\beta_j u(t_j-0), \qquad (24)$$

then

$$u(t) \le \{G_{i-1}^{-1}\{G_{i-1}(m_i)+$$

$$\int_{\tau(t_{i-1})}^{\tau(t)}[f(s,t)+g(s,t)\int_{\tau(t_0)}^{s}h(\xi)d\xi]ds \qquad (25)$$

holds for $\forall t\in(t_{i-1},t_i],\ i=1,2,...,n+1$, where

$$G_i(v)=\int_{b_i}^{v}\frac{1}{\omega(s^{\frac{1}{p}}(t))}ds,\ i=0,1,...,n,$$

$$H_i(t)=G_i(2t-b_i)-G_i(t),$$

and H_i are nondecreasing on $t\ge b_i,\ i=0,1,...,n,$

$$b_0=C,\ b_i=a_i+\beta_i a_i^{\frac{1}{p}},\ i=1,2,...,n,$$

$$a_i=G_{i-1}^{-1}\{G_{i-1}(m_i)+$$

$$\int_{\tau(t_{i-1})}^{\tau(t_i)}[f(s,t_i)+g(s,t_i)\int_{\tau(t_0)}^{s}h(\xi)d\xi]ds\},\ i=1,2,...,n,$$

$$m_i=H_{i-1}^{-1}\{\int_{\tau(t_{i-1})}^{\tau(t_i)}[f_1(s,t_i)$$

$$+g(s,t_i)\int_{\tau(t_0)}^{s}h(\xi)d\xi]ds,\ i=1,2,...,n+1.$$

Corollary 2.3 Suppose $u,\ p,\ C,\ \omega,\ \tau,\ G_i,\ H_i,\ a_i,\ b_i,\ m_i,\ t_i,\ \beta_j$ are the same as in Theorem 2.1, and $h(t),\ f(t),\ g(t),\ n(t) \in C(I,R_+)$. If for $t\in I$, $u(t)$ satisfies the following inequality

$$u^p(t) \le C+$$

$$n(t)\int_{\tau(t_0)}^{\tau(t)}[f(s)\omega(u(s))+g(s)\int_{\tau(t_0)}^{s}h(\xi)\omega(u(\xi))d\xi]ds$$

$$+n(t)\int_{\tau(t_0)}^{\tau(t_i)}[f(s)\omega(u(s))+g(s)\int_{\tau(t_0)}^{s}h(\xi)\omega(u(\xi))d\xi]ds$$

$$+\sum_{t_0<t_j<t}\beta_j u(t_j-0), \qquad (26)$$

then

$$u(t) \le \{G_{i-1}^{-1}\{G_{i-1}(m_i)+$$

$$n(t)\int_{\tau(t_{i-1})}^{\tau(t)}[f(s)+g(s)\int_{\tau(t_0)}^{s}h(\xi)d\xi]ds \qquad (27)$$

holds for $\forall t\in(t_{i-1},t_i],\ i=1,2,...,n+1$, where

$$G_i(v)=\int_{b_i}^{v}\frac{1}{\omega(s^{\frac{1}{p}}(t))}ds,\ i=0,1,...,n,$$

$$H_i(t)=G_i(2t-b_i)-G_i(t),$$

and H_i are nondecreasing on $t\ge b_i,\ i=0,1,...,n,$

$$b_0=C,\ b_i=a_i+\beta_i a_i^{\frac{1}{p}},\ i=1,2,...,n,$$

$$a_i=G_{i-1}^{-1}\{G_{i-1}(m_i)$$

$$+n(t_i)\int_{\tau(t_{i-1})}^{\tau(t_i)}[f(s)+g(s)\int_{\tau(t_0)}^{s}h(\xi)d\xi]ds\},$$

$$i=1,2,...,n,$$

$$m_i=H_{i-1}^{-1}\{n(t_i)\int_{\tau(t_{i-1})}^{\tau(t_i)}[f_1(s)$$

$$+g(s)\int_{\tau(t_0)}^{s}h(\xi)d\xi]ds,\ i=1,2,...,n+1.$$

Remark If we take $n(t)\equiv 1$, and $u(t)$ is continuous on $[t_0,T]$, then Corollary 2.2 reduces to [19, Theorem 2.1]. Furthermore, if we take $\omega(u)=u$, then Corollary 2.2 reduces Pachpatte's result in [18] with slight difference.

Corollary 2.4 Suppose $u,\ p,\ C,\ \omega,\ \tau,\ G_i,\ H_i,$ $a_i,\ b_i,\ m_i,\ t_i,\ \beta_j$ are the same as in Theorem 2.1, $h(t)\in C(I,R_+),\ f(s,t),\ g(s,t),\ \frac{\partial f(s,t)}{\partial t},\ \frac{\partial g(s,t)}{\partial t}\in C(I\times I,R_+)$. If for $t\in I$, $u(t)$ satisfies the following inequality

$$u^p(t)\le C+$$

$$\int_{t_0}^{t}[f(s,t)\omega(u(s))+g(s,t)\int_{t_0}^{s}h(\xi)\omega(u(\xi))d\xi]ds$$

$$+\int_{t_0}^{t_i}[f(s,t)\omega(u(s))+g(s,t)\int_{t_0}^{s}h(\xi)\omega(u(\xi))d\xi]ds$$

$$+\sum_{t_0<t_j<t}\beta_j u(t_j-0), \tag{28}$$

then

$$u(t)\le\{G_{i-1}^{-1}\{G_{i-1}(m_i)+\int_{t_{i-1}}^{t}[f(s,t)$$

$$+g(s,t)\int_{t_0}^{s}h(\xi)d\xi]ds \tag{29}$$

holds for $\forall t\in(t_{i-1},t_i],\ i=1,2,...,n+1$, where

$$G_i(v)=\int_{b_i}^{v}\frac{1}{\omega(s^{\frac{1}{p}}(t))}ds,\ i=0,1,...,n,$$

$$H_i(t)=G_i(2t-b_i)-G_i(t),$$

and H_i are nondecreasing on $t\ge b_i,\ i=0,1,...,n,$

$$b_0=C,\ b_i=a_i+\beta_i a_i^{\frac{1}{p}},\ i=1,2,...,n,$$

$$a_i=G_{i-1}^{-1}\{G_{i-1}(m_i)+\int_{t_{i-1}}^{t_i}[f(s,t_i)$$

$$+g(s,t_i)\int_{t_0}^{s}h(\xi)d\xi]ds\},\ i=1,2,...,n,$$

$$m_i=H_{i-1}^{-1}\{\int_{t_{i-1}}^{t_i}[f_1(s,t_i)$$

$$+g(s,t_i)\int_{t_0}^{s}h(\xi)d\xi]ds,\ i=1,2,...,n+1.$$

Corollary 2.5 Suppose $u,\ p,\ C,\omega,\ \tau,\ G_i,\ H_i,\ a_i,$ $b_i,\ m_i,\ t_i,\ \beta_j$ are the same as in Theorem 2.1, and $h(t),\ f(t),\ g(t),\ n(t)\in C(I,R_+)$. If for $t\in I$, $u(t)$ satisfies the following inequality

$$u^p(t)\le C+$$

$$n(t)\int_{t_0}^{t}[f(s)\omega(u(s))+g(s)\int_{t_0}^{s}h(\xi)\omega(u(\xi))d\xi]ds$$

$$+n(t)\int_{t_0}^{t_i}[f(s)\omega(u(s))+g(s)\int_{t_0}^{s}h(\xi)\omega(u(\xi))d\xi]ds$$

$$+\sum_{t_0<t_j<t}\beta_j u(t_j-0), \tag{30}$$

then

$$u(t)\le\{G_{i-1}^{-1}\{G_{i-1}(m_i)+$$

$$n(t)\int_{t_{i-1}}^{t}[f(s)+g(s)\int_{t_0}^{s}h(\xi)d\xi]ds \tag{31}$$

holds for $\forall t\in(t_{i-1},t_i],\ i=1,2,...,n+1$, where

$$G_i(v)=\int_{b_i}^{v}\frac{1}{\omega(s^{\frac{1}{p}}(t))}ds,\ i=0,1,...,n,$$

$$H_i(t)=G_i(2t-b_i)-G_i(t),$$

and H_i are nondecreasing on $t\ge b_i,\ i=0,1,...,n,$

$$b_0=C,\ b_i=a_i+\beta_i a_i^{\frac{1}{p}},\ i=1,2,...,n,$$

$$a_i=G_{i-1}^{-1}\{G_{i-1}(m_i)+$$

$$n(t_i)\int_{t_{i-1}}^{t_i}[f(s)+g(s)\int_{t_0}^{s}h(\xi)d\xi]ds\},\ i=1,2,...,n,$$

$$m_i=H_{i-1}^{-1}\{n(t_i)\int_{t_{i-1}}^{t_i}[f_1(s)$$

$$+g(s)\int_{t_0}^{s}h(\xi)d\xi]ds,\ i=1,2,...,n+1.$$

If we take $n(t)\equiv 1$ in Corollary 2.3 and 2.5, then we we can obtain another two corollaries, which can be left to the readers.

3 Applications

In this section, we will present some examples to illustrate the validity of our results in making estimates for the bounds of the solutions of integral equations.

Example: Consider the retarded Volterra-Fredholm integral equation of the form

$$u^p(t)=a(t)+$$

$$\int_{t_0}^{t}M_1[s,t,u(\tau(s)),\int_{t_0}^{s}N_1(\xi,u(\tau(\xi)))d\xi]ds$$

$$+\int_{t_0}^{t_i}M_2[s,t,u(\tau(s)),\int_{t_0}^{s}N_2(\xi,u(\tau(\xi)))d\xi]ds$$

$$+\sum_{t_0<t_j<t}\beta_j u(t_j-0),\ t\in I, \tag{32}$$

where u is a continuous function defined on I with the first kind of discontinuities in the points $t_i,\ i=1,2,...,n$, and $t_0<t_1<t_2<...<t_n<t_{n+1}=T$, $p>0$ is a constant, $a\in C(I,I)$, $\tau\in C^1(I,R)$ is strictly increasing with $0\le\tau(t)\le t$ and $\tau(t)>t_i$ for $\forall t\in(t_i,t_{i+1}],\ i=0,1,...,n$. $\beta_j\ge 0,\ j=1,2,...,n$, and $M\in C(I^2\times R^2,R)$.

Theorem 3.1 Assume that $u(t)$ is a solution of Eq. (32), and the following conditions satisfies

$$\begin{cases} |a(t)|\le C \\ |M_i(s,t,x,y)|\le f(s,t)|x|^q+g(s,t)|y|,\ i=1,2 \\ |N_i(x,y)|\le h(x)|y|^q,\ i=1,2 \\ \omega(v)=v^q \end{cases} \tag{33}$$

where $f(s,t)$,$g(s,t)$,$\frac{\partial f(s,t)}{\partial t}$, $\frac{\partial g(s,t)}{\partial t}\in C(I\times I,R_+)$, $\omega\in C(R_+,R_+)$, q is a constant with $0<q<p$, and $C>0$ is a constant, then we have

$$u(t)\le\{G_{i-1}^{-1}\{G_{i-1}(m_i)+\int_{\tau(t_{i-1})}^{\tau(t)}[\widetilde{f}(s,t)+\widetilde{g}(s,t)\int_{\tau(t_0)}^{s}\widetilde{h}(\xi)d\xi]ds\}\}^{\frac{1}{p}},$$
$$\forall t\in(t_{i-1},t_i],\ i=1,2,...,n+1, \tag{34}$$

where

$$\widetilde{f}(s,t)=f(\tau^{-1}(s),t),$$
$$\widetilde{g}(s,t)=g(\tau^{-1}(s),t),$$
$$\widetilde{h}(t)=h(\tau^{-1}(t)),$$
$$G_i(u)=\int_{b_i}^{u}\frac{1}{s^{\frac{q}{p}}(t)}ds,\ i=0,1,...,n,$$
$$H_i(t)=G_i(2t-b_i)-G_i(t),$$

and H_i are nondecreasing on $t\ge b_i,\ i=0,1,...,n$,

$$b_0=C,\ b_i=a_i+\beta_i a_i^{\frac{1}{p}},\ i=1,2,...,n,$$
$$a_i=G_{i-1}^{-1}\{G_{i-1}(m_i)+\int_{\tau(t_{i-1})}^{\tau(t_i)}[\widetilde{f}(s,t_i)+\widetilde{g}(s,t_i)\int_{\tau(t_0)}^{s}\widetilde{h}(\xi)d\xi]ds\},$$
$$i=1,2,...,n,$$
$$m_i=H_{i-1}^{-1}\{\int_{\tau(t_{i-1})}^{\tau(t_i)}[\widetilde{f}(s,t_i)+\widetilde{g}(s,t_i)\int_{\tau(t_0)}^{s}\widetilde{h}(\xi)d\xi]ds\},\quad i=1,2,...,n+1.$$

Proof: From (32) and (33), for $\forall t\in(t_{i-1},t_i],\ i=1,2,...,n+1$, we have

$$|u(t)|\le C+\int_{t_0}^{t}|M_1[s,t,u(\tau(s)),\int_{t_0}^{s}N_1(\xi,u(\tau(\xi)))d\xi]|ds$$
$$+\int_{t_0}^{t_i}|M_2[s,t,u(\tau(s)),\int_{t_0}^{s}N_2(\xi,u(\tau(\xi)))d\xi]|ds$$
$$+\sum_{t_0<t_j<t}\beta_j|u(t_j-0)|$$
$$\le C+\int_{t_0}^{t}[f(s,t)|u(\tau(s))|^q+g(s,t)|\int_{t_0}^{s}N_1(\xi,u(\tau(\xi)))d\xi|]ds$$
$$+\int_{t_0}^{t_i}[f(s,t)|u(\tau(s))|^q+g(s,t)|\int_{t_0}^{s}N_2(\xi,u(\tau(\xi)))d\xi|]ds$$
$$+\sum_{t_0<t_j<t}\beta_j|u(t_j-0)|$$
$$\le C+\int_{t_0}^{t}[f(s,t)|u(\tau(s))|^q+g(s,t)\int_{t_0}^{s}h(\xi)|u(\tau(\xi))|^q d\xi]ds$$
$$+\int_{t_0}^{t_i}[f(s,t)|u(\tau(s))|^q+g(s,t)\int_{t_0}^{s}h(\xi)|u(\tau(\xi))|^q d\xi]ds$$
$$+\sum_{t_0<t_j<t}\beta_j|u(t_j-0)|$$
$$=C+\int_{t_0}^{t}[f(s,t)|u(\tau(s))|^q+g(s,t)\int_{\tau(t_0)}^{\tau(s)}h(\tau^{-1}(\eta))|u(\eta)|^q d\eta]ds$$
$$+\int_{t_0}^{t_i}[f(s,t)|u(\tau(s))|^q+g(s,t)\int_{\tau(t_0)}^{\tau(s)}h(\tau^{-1}(\eta))|u(\eta)|^q d\eta]ds$$
$$+\sum_{t_0<t_j<t}\beta_j|u(t_j-0)|$$
$$=C+\int_{\tau(t_0)}^{\tau(t)}[f(\tau^{-1}(\lambda),t)|u(\lambda)|^q$$
$$+g(\tau^{-1}(\lambda),t)\int_{\tau(t_0)}^{\lambda}h(\tau^{-1}(\eta))|u(\eta)|^q d\eta]d\lambda$$
$$+\int_{\tau(t_0)}^{\tau(t_i)}[f(\tau^{-1}(\lambda),t)|u(\lambda)|^q$$
$$+g(\tau^{-1}(\lambda),t)\int_{\tau(t_0)}^{\lambda}h(\tau^{-1}(\eta))|u(\eta)|^q d\eta]d\lambda$$
$$+\sum_{t_0<t_j<t}\beta_j|u(t_j-0)|$$
$$=C+\int_{\tau(t_0)}^{\tau(t)}[\widetilde{f}(\lambda,t)|u(\lambda)|^q+\widetilde{g}(\lambda,t)\int_{\tau(t_0)}^{\lambda}\widetilde{h}(\eta)|u(\eta)|^q d\eta]d\lambda$$

$$+\int_{\tau(t_0)}^{\tau(t_i)}[\widetilde{f}(\lambda,t)|u(\lambda)|^q+\widetilde{g}(\lambda,t)\int_{\tau(t_0)}^{\lambda}\widetilde{h}(\eta)|u(\eta)|^q d\eta]d\lambda$$

$$+\sum_{t_0<t_j<t}\beta_j|u(t_j-0)|$$

$$=C+$$

$$\int_{\tau(t_0)}^{\tau(t)}[\widetilde{f}(s,t)\omega(|u(s)|)+\widetilde{g}(s,t)\int_{\tau(t_0)}^{s}\widetilde{h}(\xi)\omega(|u(\xi)|)d\xi]ds$$

$$+\int_{\tau(t_0)}^{\tau(t_i)}[\widetilde{f}(s,t)\omega(|u(s)|)+\widetilde{g}(s,t)\int_{\tau(t_0)}^{s}\widetilde{h}(\xi)\omega(|u(\xi)|)d\xi]ds$$

$$+\sum_{t_0<t_j<t}\beta_j|u(t_j-0)|. \tag{35}$$

From the definition of H_i we have

$$H_i'(t)=[G_i(2t-b_i)-G_(t)]'=\frac{2}{(2t-b_i)^{\frac{q}{p}}}-\frac{1}{t^{\frac{q}{p}}}$$

$$=\frac{2t^{\frac{q}{p}}-(2t-b_i)^{\frac{q}{p}}}{t^{\frac{q}{p}}(2t-b_i)^{\frac{q}{p}}}\geq 0 \tag{36}$$

for $t\geq b_i,\ i=0,1,...,n$.

So $H_i(t)$ are nondecreasing on $t\geq b_i,\ i=0,1,...,n$. Then by an suitable application of Corollary 2.1 we can obtain the desired inequality (34).

Theorem 3.2: Under the conditions of Theorem 3.1, we have

$$u(t)\leq\{\frac{p-q}{p}\{\frac{p}{p-q}(m_i^{\frac{p-q}{p}}-b_{i-1}^{\frac{p-q}{p}})+$$

$$\int_{\tau(t_{i-1})}^{\tau(t)}[\widetilde{f}(s,t)+\widetilde{g}(s,t)\int_{\tau(t_0)}^{s}\widetilde{h}(\xi)d\xi]ds\}+b_{i-1}^{\frac{p-q}{p}}\}^{\frac{1}{p-q}}, \tag{37}$$

where $m_i,\ b_i$ are the same as in Theorem 3.1.

Proof: As long as we notice

$$G_i(v)=\int_{b_i}^{v}\frac{1}{s^{\frac{q}{p}}(t)}ds=$$

$$\frac{p}{p-q}(v^{\frac{p-q}{p}}-b_i^{\frac{p-q}{p}}),\ i=0,1,...,n, \tag{38}$$

then combining (38) and Theorem 3.1 we can easily deduce the desired result.

4 Conclusions

In this paper, we have established a new generalized Voterra-Fredholm type integral inequality, which provide a handy tool in the investigation of making estimates for bounds of solutions of certain integral equations. From the example one can see new explicit bounds are derived by the presented inequality. The process of establishing the inequality in Theorem 2.1 can be applied to the situation with two independent variables, which can be left to the further research.

References:

[1] W. N. Li, M. A. Han, F.W. Meng, Some new delay integral inequalities and their applications, J. Comput. Appl. Math. 180 (2005) 191-200.

[2] O. Lipovan, A retarded integral inequality and its applications, J. Math. Anal. Appl. 285 (2003) 436-443.

[3] Q.H. Ma, E.H.Yang, Some new Gronwall-CBellmanCBihari type integral inequalities with delay, Period. Math. Hungar. 44 (2) (2002) 225-238.

[4] Z. L. Yuan, X.W. Yuan, F.W. Meng, Some new delay integral inequalities and their applications, Appl. Math. Comput. 208 (2009) 231-237.

[5] O. Lipovan, Integral inequalities for retarded Volterra equations, J. Math. Anal. Appl. 322 (2006) 349-358.

[6] B. G. Pachpatte, Explicit bounds on certain integral inequalities, J. Math. Anal. Appl. 267 (2002) 48-61.

[7] B. G. Pachpatte, A note on certain integral inequalities with delay, Period. Math. Hungar. 31 (1995) 234-299.

[8] B. G. Pachpatte, On some new nonlinear retarded integral inequalities, J. Inequal. Pure Appl. Math. 5 (2004) (Article 80).

[9] Y. G. Sun, On retarded integral inequalities and their applications, J. Math. Anal. Appl. 301 (2005) 265-275.

[10] R. A. C. Ferreira, D. F. M. Torres, Generalized retarded integral inequalities, Appl. Math. Letters 22 (2009) 876-881.

[11] R. Xu, Y. G. Sun, On retarded integral inequalities in two independent variables and their applications, Appl. Math. Comput. 182 (2006) 1260-1266.

[12] F. C. Jiang, F. W. Meng, Explicit bounds on some new nonlinear integral inequality with delay, J. Comput. Appl. Math. 205 (2007) 479-486.

[13] L. Z. Li, F. W. Meng, L. L. He, Some generalized integral inequalities and their applications, J. Math. Anal. Appl. 372 (2010) 339-349.

[14] A. Gallo, A. M. Piccirillo, About some new generalizations of Bellman-Bihari results for integro-functional inequalities with discontinuous functions and applications, Nonlinear Anal. 71 (2009) e2276-e2287.

[15] G. Iovane, Some new integral inequalities of Bellman-Bihari type with delay for discontinuous functions, Nonlinear Anal. 66 (2007) 498-508.

[16] S. Borysenko, G. Iovane, About some new integral inequalities of Wendroff type for discontinuous functions, Nonlinear Anal. 66 (2007) 2190-2203.

[17] D. R. Meng, F. W. Meng, some new integral inequalities for discontinuous function of two independent variables, J. Sys. Sci. & Math. Scis. 29 (2009) 440-450.

[18] B. G. Pachpatte, Explicit bound on a retarded integral inequality, Math. Inequal. Appl. 7 (2004) 7-11.

[19] Q. H. Ma, J. Pečarić, Estimates on solutions of some new nonlinear retarded Volterra-Fredholm type integral inequalities, Nonlinear Anal. 69 (2008) 393-407.