Abstract: In this paper, a new generalized Volterra-Fredholm type nonlinear integral inequality for discontinuous functions is established, which can be used in analysis for the boundedness of solutions of certain Volterra-Fredholm type integral equations. Our results generalize the main results in [18, 19].

Key–Words: Integral inequality; discontinuous function; Integral equation; Differential equation; Bounded

1 Introduction

In recent years many integral inequalities for continuous and discontinuous functions have been established, which provide handy tools for investigating the quantitative and qualitative properties of solutions of integral and differential equations [1-17]. In the investigations for integral inequalities, the idea of generalizing known integral inequalities have gained extensive attention.

In [18], Pachpatte established the following integral inequality

\[(a): \quad u(t) \leq k + \int_{h(t)}^{\hat{h}(t)} a(t, s)[f(s)u(s)] ds + \int_{h(t)}^{\hat{h}(t)} c(s, \sigma)u(\sigma)d\sigma ds + \int_{h(t)}^{\beta(t)} b(t, s)u(s) ds,\]

where \(u(t)\) is an unknown function with \(u(t) \in C(I, R_+), I = [\alpha, \beta], a(t, s), b(t, s), c(t, s) \in C(D, R_+), D = \{(t, s) \in I^2 : \alpha \leq s \leq t \leq \beta\}\), \(a(t, s), b(t, s)\) are nondecreasing in \(t\) for each \(s \in I\), \(h(t) \in C^1(I, I)\) is nondecreasing with \(h(t) \leq t\) on \(I\), \(k \geq 0\) is a constant.

Recently, in [19, Theorem 2.1], the author presented the following Volterra-Fredholm type integral inequality that generalized the inequality (a).

\[(b): \quad u(t) \leq k + \int_{a(t_0)}^{\alpha(t)} \sigma_1(s)[f(s)\omega(u(s))] ds + \int_{\alpha(t_0)}^{\beta(t)} \sigma_2(\tau)\omega(u(\tau))d\tau ds + \int_{\alpha(t_0)}^{\beta(t)} \sigma_1(s)[f(s)\omega(u(s))] ds + \int_{\alpha(t_0)}^{\beta(t)} \sigma_2(\tau)\omega(u(\tau))d\tau ds,\]

where \(u(t)\) is an unknown function and \(u(t), f(t), \sigma_1(t), \sigma_2(t) \in C(I, R_+), \alpha \in C^1(I, I)\) is nondecreasing with \(\alpha(t) \leq t\) on \(I\), \(\omega \in C(R_+, R_+)\) is nondecreasing with \(\omega(u) > 0\) for \(u > 0\).

In the present paper, we will establish a new more generalized integral inequality for discontinuous functions than the inequalities mentioned above, also we will present one application for it, and will obtain new bounds for solutions of certain Volterra-Fredholm type integral equations.

2 Main Results

In the rest of the paper, we denote the set of real numbers as \(R\), and \(R_+ = [0, \infty). I, \hat{I}, \hat{I}\), denote intervals in \(R\), and \(I = [t_0, T], \hat{I} = [x_0, A], \hat{I} = [y_0, B]\) respectively, where \(T > t_0, A > x_0, B > y_0\) are three fixed numbers.

Theorem 2.1 Suppose \(u(t)\) is a nonnegative continuous function defined on \(I\) with the first kind of discontinuities in the points \(t_i, i = 1, 2, ..., n\), and \(t_0 < t_1 < t_2 < ... < t_n < t_{n+1} = T\).

\(h_i(t) \in C(I, R_+), i = 1, 2, \text{ and } f_i(s, t), g_i(s, t), \frac{\partial f_i(s, t)}{\partial t}, \frac{\partial g_i(s, t)}{\partial t} \in C(I \times I, R_+), i = 1, 2, p > 0\) is a constant. \(\tau(t) \in C(I, I)\) is nondecreasing with \(\tau(t) \leq t\), and \(\tau(t) > t_i\) for \(\forall t \in (t_i, t_{i+1}), i = 1, 2, ..., n\).
$0, 1, \ldots, n$. $\omega \in C(R_+, R_+)$, and $\omega$ is nondecreasing with $\omega(u) > 0$ for $u > 0$. $C \geq 0$, $\beta_i > 0$, $i = 1, 2, \ldots, n$, and $C$, $\beta_i$ are constants. If for $t \in I$, $u(t)$ satisfies the following inequality

$$u^p(t) \leq C + \int_{\tau(t)}^{\tau(t)} [f_1(s, t)\omega(u(s)) + g_1(s, t) \int_{\tau(t)}^{s} h_1(\xi)\omega(u(\xi))d\xi]ds$$

$$+ \int_{\tau(t)}^{t} [f_2(s, t)\omega(u(s)) + g_2(s, t) \int_{\tau(t)}^{s} h_2(\xi)\omega(u(\xi))d\xi]ds$$

$$+ \int_{t}^{t} \sum_{t_0 < t_j < t} \beta_j u(t_j - 0), \quad (1)$$

then

$$u(t) \leq \{G_i^{-1}(m_i) + \int_{\tau(t)}^{\tau(t)} [f_1(s, t)$$

$$+ g_1(s, t) \int_{\tau(t)}^{s} h_1(\xi)d\xi]ds$$

$$+ \int_{\tau(t)}^{t} [f_2(s, t) + g_2(s, t) \int_{\tau(t)}^{s} h_2(\xi)d\xi]ds \}^{\frac{1}{p}}$$

holds for $\forall t \in (t_{i-1}, t_i]$, $i = 1, \ldots, n + 1$, where

$$G_i(v) = \int_{b_i}^{v} \frac{1}{\omega(s^{\frac{1}{p}})}ds, \quad i = 0, 1, \ldots, n, \quad (3)$$

$$H_i(t) = G_i(2t - b_i) - G_i(t), \quad (4)$$

and $H_i$ are nondecreasing on $t \geq b_i$, $i = 0, 1, \ldots, n$,

$$b_0 = C, \quad b_i = a_i + \beta_i a_i^{\frac{1}{p}}, \quad i = 1, 2, \ldots, n, \quad (5)$$

$$a_i = G_i^{-1}(m_i) + \int_{\tau(t)}^{\tau(t)} [f_1(s, t)$$

$$+ g_1(s, t) \int_{\tau(t)}^{s} h_1(\xi)d\xi]ds$$

$$+ \int_{t}^{t} [f_2(s, t) + g_2(s, t) \int_{\tau(t)}^{s} h_2(\xi)d\xi]ds \}, \quad (6)$$

for $i = 1, 2, \ldots, n$,

$$m_i = H_i^{-1}(\int_{\tau(t)}^{\tau(t)} [f_1(s, t) + g_1(s, t) \int_{\tau(t)}^{s} h_1(\xi)d\xi]ds$$

$$+ \int_{t}^{t} [f_2(s, t) + g_2(s, t) \int_{\tau(t)}^{s} h_2(\xi)d\xi]ds \} \quad (7)$$

for $i = 1, 2, \ldots, n + 1$.

**Proof:** Firstly we assume $C > 0$. Let the right side of (1) be $v(t)$, then

$$u(t) \leq v^p(t), \quad t \in I. \quad (8)$$

Define

$$v_i(t) = b_i + \int_{\tau(t)}^{t} [f_1(s, t)\omega(u(s)) + g_1(s, t) \int_{\tau(t)}^{s} h_1(\xi)\omega(u(\xi))d\xi]ds$$

$$+ \int_{t}^{t} [f_2(s, t)\omega(u(s)) + g_2(s, t) \int_{\tau(t)}^{s} h_2(\xi)\omega(u(\xi))d\xi]ds$$

$$+ \int_{t}^{t} \sum_{t_0 < t_j < t} \beta_j v(t_j - 0), \quad (9)$$

which holds for $\forall t \in (t_{i-1}, t_i]$, $i = 0, 1, \ldots, n$.

**Case 1:** If $t \in (t_0, t_1]$, considering $b_0 = C$, then

$$v(t) = C + \int_{\tau(t)}^{\tau(t)} [f_1(s, t)\omega(u(s)) + g_1(s, t) \int_{\tau(t)}^{s} h_1(\xi)\omega(u(\xi))d\xi]ds$$

$$+ \int_{\tau(t)}^{t} [f_1(s, t)\omega(u(s)) + g_1(s, t) \int_{\tau(t)}^{s} h_1(\xi)\omega(u(\xi))d\xi]ds$$

$$+ \int_{t}^{t} [f_2(s, t)\omega(u(s)) + g_2(s, t) \int_{\tau(t)}^{s} h_2(\xi)\omega(u(\xi))d\xi]ds$$

$$= v_0(t). \quad (10)$$

From (8), obviously we have

$$u(t) \leq v^p_0(t). \quad (11)$$

According to the assumption of $\tau(t)$ we have $t_0 < \tau(t) \leq t$ for $t \in (t_0, t_1]$. So $\tau(t) \in (t_0, t_1]$, and then

$$u(\tau(t)) \leq v^p_0(\tau(t)) \leq v^p_0(t). \quad (12)$$
Furthermore
\[ v'_0(t) = \int_{\tau(t_0)}^{\tau(t)} \left[ \frac{\partial f_1(s, t)}{\partial t} - \omega(u(s)) + \frac{\partial g_1(s, t)}{\partial t} \int_{\tau(t_0)}^{s} h_1(\xi)\omega(u(\xi))d\xi \right] ds \\
+ [f_1(\tau(t), t)\omega(u(\tau(t))) + g_1(\tau(t), t) \int_{\tau(t_0)}^{\tau(t)} h_1(\xi)\omega(u(\xi))d\xi]^{\tau(t)}_0 \\
+ \int_{t_0}^t \left[ \frac{\partial f_2(s, t)}{\partial t} - \omega(u(s)) + \frac{\partial g_2(s, t)}{\partial t} \int_{t_0}^s h_2(\xi)\omega(u(\xi))d\xi \right] ds \\
+ [f_2(t, t)\omega(u(t)) + g_2(t, t) \int_{t_0}^t h_2(\xi)\omega(u(\xi))d\xi]^{t}_0. \]

Then
\[ \frac{v'_0(t)}{\omega(v_0^2(t))} \leq \int_{\tau(t_0)}^{\tau(t)} \left[ \frac{\partial f_1(s, t)}{\partial t} + \frac{\partial g_1(s, t)}{\partial t} \int_{\tau(t_0)}^{s} h_1(\xi)d\xi \right] ds \\
+ [f_1(\tau(t), t) + g_1(\tau(t), t) \int_{\tau(t_0)}^{\tau(t)} h_1(\xi)d\xi]^{\tau(t)}_0 \\
+ \int_{t_0}^t \left[ \frac{\partial f_2(s, t)}{\partial t} + \frac{\partial g_2(s, t)}{\partial t} \int_{t_0}^s h_2(\xi)d\xi \right] ds \\
+ [f_2(t, t) + g_2(t, t) \int_{t_0}^t h_2(\xi)d\xi]^{t}_0. \] (11)

An integration for (6) with respect to \( t \) from \( t_0 \) to \( t \) yields
\[ G_0(v_0(t)) - G_0(v_0(t_0)) \leq \int_{\tau(t_0)}^{\tau(t)} \left[ f_1(s, t) + g_1(s, t) \int_{\tau(t_0)}^{s} h_1(\xi)d\xi \right] ds \\
+ \int_{t_0}^t \left[ f_2(s, t) + g_2(s, t) \int_{t_0}^s h_2(\xi)d\xi \right] ds, \] (12)
where \( G_0 \) is defined in (3).

Considering \( G_0 \) is nondecreasing, then it follows
\[ v_0(t) \leq G_0^{-1}\{G_0(v_0(t_0)) + \int_{\tau(t_0)}^{\tau(t)} f_1(s, t) + g_1(s, t) \int_{\tau(t_0)}^{s} h_1(\xi)d\xi ds \\
+ \int_{t_0}^t f_2(s, t) + g_2(s, t) \int_{t_0}^s h_2(\xi)d\xi ds \}. \] (13)

Take \( t = t_1 \) in (13) we can obtain
\[ v_0(t_1) \leq G_0^{-1}\{G_0(v_0(t_0)) + \int_{\tau(t_0)}^{\tau(t_1)} f_1(s, t_1) + g_1(s, t_1) \int_{\tau(t_0)}^{s} h_1(\xi)d\xi ds \\
+ \int_{t_0}^{t_1} f_2(s, t_1) + g_2(s, t_1) \int_{t_0}^s h_2(\xi)d\xi ds \}. \] (14)

From (9) and (14) we have
\[ 2v_0(t_0) - C = v_0(t_1) \leq G_0^{-1}\{G_0(v_0(t_0)) + \int_{\tau(t_0)}^{\tau(t_1)} f_1(s, t_1) + g_1(s, t_1) \int_{\tau(t_0)}^{s} h_1(\xi)d\xi ds \\
+ \int_{t_0}^{t_1} f_2(s, t_1) + g_2(s, t_1) \int_{t_0}^s h_2(\xi)d\xi ds \}, \] (15)
that is,
\[ G_0(2v_0(t_0) - C) - G_0(v_0(t_0)) \leq \int_{\tau(t_0)}^{\tau(t_1)} f_1(s, t_1) + g_1(s, t_1) \int_{\tau(t_0)}^{s} h_1(\xi)d\xi ds \\
+ \int_{t_0}^{t_1} f_2(s, t_1) + g_2(s, t_1) \int_{t_0}^s h_2(\xi)d\xi ds. \] (16)

Since \( H_0(t) = G_0(2t - b_0) - G_0(t) \) is nondecreasing on \( t \geq b_0 \), and \( b_0 = C \), \( v_0(t_0) \geq C \), then from (16) it follows
\[ v_0(t_0) \leq H_0^{-1}\left\{ \int_{\tau(t_0)}^{\tau(t_1)} f_1(s, t_1) + g_1(s, t_1) \int_{\tau(t_0)}^{s} h_1(\xi)d\xi ds \\
+ \int_{t_0}^{t_1} f_2(s, t_1) + g_2(s, t_1) \int_{t_0}^s h_2(\xi)d\xi ds \right\} = m_1, \] (17)
where \( m_1 \) is defined in (7).

Combining (10), (13) and (17) we can obtain
\[ u(t) \leq \frac{1}{v_0^2(t)} \leq \]
\[
\begin{align*}
\{G_0^{-1}(G_0(m_1) + \int_{\tau(t_0)}^{\tau(t)} [f_1(s, t) + g_1(s, t) \int_{\tau(t_0)}^{s} h_1(\xi) d\xi] ds \\
+ \int_{t_0}^{t} [f_2(s, t) + g_2(s, t) \int_{t_0}^{s} h_2(\xi) d\xi] ds \} \right\}^{\frac{1}{p}}. (18)
\end{align*}
\]

Especially, if we take \( t = t_1 \), then we have
\[
u(t_1) \leq v_0^v(t_1) \leq \frac{1}{p} v_0^v(t_1) \leq a_1, \quad (19)
\]
where \( a_1 \) is defined in (6).

**Case 2:** If \( t \in (t_1, t_2) \), then from the definition of \( a_1 \) and (18), (19) we have
\[
u(t_1 - 0) \leq v_0^v(t_1 - 0) \leq v_0^v(t_1) \leq a_1^v.
\]
So
\[
v(t) = C + \int_{\tau(t_0)}^{\tau(t)} [f_1(s, t) \omega(u(s)) + g_1(s, t) \int_{\tau(t_0)}^{s} h_1(\xi) \omega(u(\xi)) d\xi] ds \\
+ \int_{t_0}^{t} [f_2(s, t) \omega(u(s)) + g_2(s, t) \int_{t_0}^{s} h_2(\xi) \omega(u(\xi)) d\xi] ds \\
+ \beta_1 u(t_1 - 0) = C + \int_{\tau(t_0)}^{\tau(t)} [f_1(s, t) \omega(u(s)) + g_1(s, t) \int_{\tau(t_0)}^{s} h_1(\xi) \omega(u(\xi)) d\xi] ds \\
+ \int_{t_0}^{t} [f_2(s, t) \omega(u(s)) + g_2(s, t) \int_{t_0}^{s} h_2(\xi) \omega(u(\xi)) d\xi] ds \\
+ \beta_1 u(t_1 - 0) = b_1 + \int_{\tau(t_0)}^{\tau(t)} [f_1(s, t) \omega(u(s)) + g_1(s, t) \int_{\tau(t_0)}^{s} h_1(\xi) \omega(u(\xi)) d\xi] ds \\
+ \int_{t_0}^{t} [f_2(s, t) \omega(u(s)) + g_2(s, t) \int_{t_0}^{s} h_2(\xi) \omega(u(\xi)) d\xi] ds \\
+ \beta_1 u(t_1 - 0) = v_1(t) \quad (20)
\]
where \( b_1 \) is defined in (5).

Then following in a similar process with (9)-(18), we can obtain
\[
u(t) \leq v_0^v(t) \leq \frac{1}{p} v_0^v(t) \leq a_1^v.
\]
\[
\begin{align*}
\{G_0^{-1}(G_0(m_2) + \int_{\tau(t_0)}^{\tau(t)} [f_1(s, t) + g_1(s, t) \int_{\tau(t_0)}^{s} h_1(\xi) d\xi] ds \\
+ \int_{t_0}^{t} [f_2(s, t) + g_2(s, t) \int_{t_0}^{s} h_2(\xi) d\xi] ds \} \right\}^{\frac{1}{p}}. (21)
\end{align*}
\]

where \( m_2 \) is defined in (7).

Especially, if we take \( t = t_2 \), then we have
\[
u(t_2) \leq v_0^v(t_2) \leq a_1^v.
\]
\[ \{G_1^{-1}\{G_1(\eta_2) + \int_{\tau(t_0)}^{\tau(t)} [f_1(s, t) + g_1(s, t) \int_{\tau(t_0)}^{s} \eta_1(\xi) d\xi] ds + \int_{t_0}^{t} [f_2(s, t) + g_2(s, t) \int_{t_0}^{s} \eta_2(\xi) d\xi] ds \} \}^{\frac{1}{p}} = a_2, \]  
\text{Case 3:} \text{ If for } t \in (t_{i-1}, t_i],
\[ u(t) \leq v_{i-1}^{\frac{1}{p}}(t) \leq \{G_1^{-1}\{G_1(\eta_i) + \int_{\tau(t_{i-1})}^{\tau(t_i)} [f_1(s, t_i) + g_1(s, t_i) \int_{\tau(t_0)}^{s} \eta_1(\xi) d\xi] ds + \int_{t_{i-1}}^{t} [f_2(s, t_i) + g_2(s, t_i) \int_{t_0}^{s} \eta_2(\xi) d\xi] ds \} \}^{\frac{1}{p}} = a_i, \]
then for \( t \in (t_i, t_{i+1}], \)
\[ v(t) = C + \int_{\tau(t_0)}^{\tau(t)} [f_1(s, t) \omega(u(s)) + g_1(s, t) \int_{\tau(t_0)}^{s} \omega(u(\xi)) d\xi] ds + \int_{t_0}^{t} [f_2(s, t) \omega(u(s)) + g_2(s, t) \int_{t_0}^{s} \omega(u(\xi)) d\xi] ds + \int_{t_0}^{t} \sum_{t_0 < t_j < t} \beta_j u(t_j - 0) \]
\[ = C + 2 \int_{\tau(t_0)}^{\tau(t)} [f_1(s, t) \omega(u(s)) + g_1(s, t) \int_{\tau(t_0)}^{s} \omega(u(\xi)) d\xi] ds + \int_{t_0}^{t} [f_2(s, t) \omega(u(s)) + g_2(s, t) \int_{t_0}^{s} \omega(u(\xi)) d\xi] ds + \int_{t_0}^{t} \sum_{t_0 < t_j < t} \beta_j u(t_j - 0) = v_i(t), \]  
where \( b_i \) is defined in (5).
Similar to (9)-(18), we can obtain

\[ u(t) \leq v^\frac{1}{n}(t) \leq \{G^{-1}_i \{G_i(m_{i+1})\} + \int_{\tau(t_i)}^{\tau(t)} [f_1(s, t) + g_1(s, t)] \int_{\tau(t_0)}^{s} h_1(\xi) d\xi ds \]

\[ + \int_{t_i}^{t} [f_2(s, t) + g_2(s, t)] \int_{t_0}^{s} h_2(\xi) d\xi ds \} \frac{1}{\tau}, \]

and the proof is complete for \( C > 0 \).

If \( C = 0 \), we substitute \( C \) with \( \varepsilon \) in the above process, and then let \( \varepsilon \to 0 \), then we can obtain the desired inequality (2).

**Corollary 2.2** Suppose \( u, p, C, \omega, \tau, G_i, H_i, a_i, b_i, m_i, t_i, \beta_j \) are the same as in Theorem 2.1, \( h(t) \in C(I, R_+), f(s, t), g(s, t), \frac{\partial f(s, t)}{\partial t}, \frac{\partial g(s, t)}{\partial t} \in C(I \times I, R_+) \). If for \( t \in I \), \( u(t) \) satisfies the following inequality

\[ u^p(t) \leq C + \]

\[ \int_{\tau(t_0)}^{\tau(t)} [f(s, t) \omega(u(s)) + g(s, t)] \int_{\tau(t_0)}^{s} h(\xi) \omega(u(\xi)) d\xi ds \]

\[ + \int_{\tau(t_0)}^{\tau(t)} [f(s, t) \omega(u(s)) + g(s, t)] \int_{\tau(t_0)}^{s} h(\xi) \omega(u(\xi)) d\xi ds \]

\[ + \sum_{t_0 < t_j < t} \beta_j u(t_j - 0), \]

then

\[ u(t) \leq \{G^{-1}_{i-1} \{G_{i-1}(m_i)\} + \]

\[ \int_{\tau(t_{i-1})}^{\tau(t)} [f(s, t) + g(s, t)] \int_{\tau(t_0)}^{s} h(\xi) d\xi ds \] \( (24) \)

holds for \( \forall t \in (t_{i-1}, t_i] \), \( i = 1, 2, ..., n + 1 \), where

\[ G_i(v) = \int_{b_i}^{v} \frac{1}{\omega(\frac{1}{\tau}(t))} ds, \]

\[ H_i(t) = G_i(2t - b_i) - G_i(t), \]

and \( H_i \) are nondecreasing on \( t \geq b_i, i = 0, 1, ..., n, \)

\[ b_0 = C, b_i = a_i + \beta t_i, i = 1, 2, ..., n, \]

\[ a_i = G^{-1}_{i-1} \{G_{i-1}(m_i)\} + \]

\[ \int_{\tau(t_{i-1})}^{\tau(t)} [f(s, t_i) + g(s, t_i)] \int_{\tau(t_0)}^{s} h(\xi) d\xi ds, \]

\[ m_i = H^{-1}_{i-1} \{\int_{\tau(t_{i-1})}^{\tau(t_i)} [f_1(s, t)] \int_{\tau(t_0)}^{s} h(\xi) d\xi ds, i = 1, 2, ..., n + 1 \].

**Corollary 2.3** Suppose \( u, p, C, \omega, \tau, G_i, H_i, a_i, b_i, m_i, t_i, \beta_j \) are the same as in Theorem 2.1, and \( h(t), f(s, t), g(s, t), n(t) \in C(I, R_+) \). If for \( t \in I \), \( u(t) \) satisfies the following inequality

\[ u^p(t) \leq C + \]

\[ n(t) \int_{\tau(t_0)}^{\tau(t)} [f(s) \omega(u(s)) + g(s)] \int_{\tau(t_0)}^{s} h(\xi) \omega(u(\xi)) d\xi ds \]

\[ + n(t) \int_{\tau(t_0)}^{\tau(t)} [f(s) \omega(u(s)) + g(s)] \int_{\tau(t_0)}^{s} h(\xi) \omega(u(\xi)) d\xi ds \]

\[ + \sum_{t_0 < t_j < t} \beta_j u(t_j - 0), \]

then

\[ u(t) \leq \{G^{-1}_{i-1} \{G_{i-1}(m_i)\} + \]

\[ \int_{\tau(t_{i-1})}^{\tau(t)} [f(s) + g(s)] \int_{\tau(t_0)}^{s} h(\xi) d\xi ds \] \( (27) \)

holds for \( \forall t \in (t_{i-1}, t_i], i = 1, 2, ..., n + 1 \), where

\[ G_i(v) = \int_{b_i}^{v} \frac{1}{\omega(\frac{1}{\tau}(t))} ds, \]

\[ H_i(t) = G_i(2t - b_i) - G_i(t), \]

and \( H_i \) are nondecreasing on \( t \geq b_i, i = 0, 1, ..., n, \)

\[ b_0 = C, b_i = a_i + \beta t_i, i = 1, 2, ..., n, \]

\[ a_i = G^{-1}_{i-1} \{G_{i-1}(m_i)\} + \]

\[ n(t) \int_{\tau(t_{i-1})}^{\tau(t)} [f(s) + g(s)] \int_{\tau(t_0)}^{s} h(\xi) d\xi ds, i = 1, 2, ..., n + 1 \].

**Remark** If we take \( n(t) \equiv 1 \), and \( u(t) \) is continuous on \([t_0, T]\), then Corollary 2.2 reduces to [19, Theorem 2.1]. Furthermore, if we take \( \omega(u) = u \), then Corollary 2.2 reduces Pachpatte’s result in [18] with slight difference.
Corollary 2.4 Suppose $u$, $p$, $C$, $\omega$, $\tau$, $G_i$, $H_i$, $a_i$, $b_i$, $m_i$, $t_i$, $\beta_j$ are the same as in Theorem 2.1, $h(t) \in C(I, R_+)$, $f(s, t)$, $g(s, t)$, $\frac{\partial f(s, t)}{\partial t}$, $\frac{\partial g(s, t)}{\partial t} \in C(I \times I, R_+)$. If for $t \in I$, $u(t)$ satisfies the following inequality
\[
u^p(t) \leq C + \int_{t_0}^{t} [f(s, t)\omega(u(s)) + g(s, t) \int_{t_0}^{s} h(\xi)\omega(u(\xi))d\xi]ds + \int_{t_0}^{t_i} [f(s, t)\omega(u(s)) + g(s, t) \int_{t_0}^{s} h(\xi)\omega(u(\xi))d\xi]ds + \sum_{t_0 < t_j < t} \beta_j u(t_j - 0), \tag{28}
\]
then
\[u(t) \leq \{G_i^{-1} \{G_i^{-1}(m_i) + t \int_{t_{i-1}}^{t} [f(s, t)]ds + g(s, t) \int_{t_0}^{s} h(\xi)d\xi]ds \tag{29}\]
holds for $\forall t \in (t_{i-1}, t_i)$, $i = 1, 2, ..., n+1$, where
\[G_i(v) = \int_{b_i}^{v} \frac{1}{\omega(s^{\frac{p}{\nu}}(t))}ds, \quad i = 0, 1, ..., n,\]
\[H_i(t) = G_i(2t - b_i) - G_i(t),\]
and $H_i$ are nondecreasing on $t \geq b_i$, $i = 0, 1, ..., n$,
\[b_0 = C, \quad b_i = a_i + \beta_i a_i^{\frac{1}{p}}, \quad i = 1, 2, ..., n,\]
\[a_i = G_i^{-1} \{G_i^{-1}(m_i) + t \int_{t_{i-1}}^{t} [f(s, t)]ds + g(s, t) \int_{t_0}^{s} h(\xi)d\xi]ds \}, \quad i = 1, 2, ..., n,\]
\[m_i = H_i^{-1} \{t \int_{t_{i-1}}^{t_i} [f_1(s, t)]ds + g(s, t) \int_{t_0}^{s} h(\xi)d\xi]ds, \quad i = 1, 2, ..., n+1.\]

Corollary 2.5 Suppose $u$, $p$, $C$, $\omega$, $\tau$, $G_i$, $H_i$, $a_i$, $b_i$, $m_i$, $t_i$, $\beta_j$ are the same as in Theorem 2.1, and $h(t)$, $f(t)$, $g(t)$, $n(t) \in C(I, R_+)$. If for $t \in I$, $u(t)$ satisfies the following inequality
\[
u^p(t) \leq C + \int_{t_0}^{t} [f(s, t)\omega(u(s)) + g(s, t) \int_{t_0}^{s} h(\xi)\omega(u(\xi))d\xi]ds + \int_{t_0}^{t_i} [f(s, t)\omega(u(s)) + g(s, t) \int_{t_0}^{s} h(\xi)\omega(u(\xi))d\xi]ds + \sum_{t_0 < t_j < t} \beta_j u(t_j - 0), \tag{30}
\]
then
\[u(t) \leq \{G_i^{-1} \{G_i^{-1}(m_i) + n(t) \int_{t_{i-1}}^{t} [f(s) + g(s) \int_{t_0}^{s} h(\xi)d\xi]ds \}, \quad i = 1, 2, ..., n,\]
holds for $\forall t \in (t_{i-1}, t_i)$, $i = 1, 2, ..., n+1$, where
\[G_i(v) = \int_{b_i}^{v} \frac{1}{\omega(s^{\frac{p}{\nu}}(t))}ds, \quad i = 0, 1, ..., n,\]
\[H_i(t) = G_i(2t - b_i) - G_i(t),\]
and $H_i$ are nondecreasing on $t \geq b_i$, $i = 0, 1, ..., n$,
\[b_0 = C, \quad b_i = a_i + \beta_i a_i^{\frac{1}{p}}, \quad i = 1, 2, ..., n,\]
\[a_i = G_i^{-1} \{G_i^{-1}(m_i) + n(t_i) \int_{t_{i-1}}^{t_i} [f_1(s, t)]ds + g(s, t) \int_{t_0}^{s} h(\xi)d\xi]ds \}, \quad i = 1, 2, ..., n,\]
\[m_i = H_i^{-1} \{t \int_{t_{i-1}}^{t_i} [f_1(s, t)]ds + g(s, t) \int_{t_0}^{s} h(\xi)d\xi]ds, \quad i = 1, 2, ..., n+1.\]

If we take $n(t) = 1$ in Corollary 2.3 and 2.5, then we can obtain another two corollaries, which can be left to the readers.

3 Applications

In this section, we will present some examples to illustrate the validity of our results in making estimates for the bounds of the solutions of integral equations.

Example: Consider the retarded Volterra-Fredholm integral equation of the form
\[
u^p(t) = a(t) + \int_{t_0}^{t} M_1[s, t, u(\tau(s))] \int_{t_0}^{s} N_1(\xi, u(\tau(\xi)))d\xi]ds + \int_{t_0}^{t_i} M_2[s, t, u(\tau(s))] \int_{t_0}^{s} N_2(\xi, u(\tau(\xi)))d\xi]ds + \sum_{t_0 < t_j < t} \beta_j u(t_j - 0), \quad t \in I, \tag{32}
\]
where $u$ is a continuous function defined on $I$ with the first kind of discontinuities in the points $t_i$, $i = 1, 2, ..., n$, and $t_0 < t_1 < t_2 < ... < t_n < t_{n+1} = T$, $p > 0$ is a constant, $a \in C(I, I)$, and $\tau \in C^1(I, R)$ is strictly increasing with $0 \leq \tau(t) \leq t$ and $\tau(t) > t_i$ for all $t \in (t_i, t_{i+1}]$, $i = 0, 1, ..., n$. $eta_j \geq 0$, $j = 1, 2, ..., n$, and $M \in C(I^2 \times R^3, R)$.

**Theorem 3.1** Assume that $u(t)$ is a solution of Eq. (32), and the following conditions satisfies

$$
\begin{cases}
|a(t)| \leq C \\
|M_i(s, t, x, y)| \leq f(s, t)|x|^q + g(s, t)|y|, \quad i = 1, 2, 3 \\
|N_i(x, y)| \leq h(x)|y|^q, \quad i = 1, 2 \\
\omega(v) = v^q
\end{cases}
$$

(33)

where $f(s, t), g(s, t), \frac{\partial f(s, t)}{\partial t}, \frac{\partial g(s, t)}{\partial t} \in C(I \times I, R)$, $\omega \in C(R^2, R)$, $q$ is a constant with $0 < q < p$, and $C > 0$ is a constant, then we have

$$
u(t) \leq \left\{ G_{t-1}^{-1} \{ G_{t-1}^{-1}(m_i) + \right. \\
\int_{\tau(t)}^{\tau(t)} \left[ \int_{\tau(t)}^{s} \int_{\tau(t)}^{s} \tilde{f}(s, t) + \tilde{g}(s, t) \int_{\tau(t)}^{s} \tilde{h}(s, t) \right] ds \right\}^{\frac{1}{q}}, \quad \forall t \in (t_i, t_{i+1}], \quad i = 1, 2, ..., n + 1,
$$

(34)

where

$$
\tilde{f}(s, t) = f(\tau^{-1}(s), t), \\
\tilde{g}(s, t) = g(\tau^{-1}(s), t), \\
\tilde{h}(s, t) = h(\tau^{-1}(t)),
$$

$$
G_i(u) = \int_{b_i}^{u} \frac{1}{s^\frac{1}{q}} ds, \quad i = 0, 1, ..., n,
$$

$$
H_i(t) = G_i(2t - b_i) - G_i(t),
$$

and $H_i$ is nondecreasing on $t \geq b_i$, $i = 0, 1, ..., n$.

$$
b_0 = C, \quad b_i = a_i + \beta_i a_i^\frac{1}{q}, \quad i = 1, 2, ..., n,
$$

$$
a_i = G_{i-1}^{-1} \{ G_{i-1}(m_i) + \right. \\
\int_{\tau(t)}^{\tau(t)} \left[ \tilde{f}(s, t) + \tilde{g}(s, t) \int_{\tau(t)}^{s} \tilde{h}(s, t) \right] ds \right\}^{\frac{1}{q}}, \quad i = 1, 2, ..., n,
$$

$$
m_i = H_{i-1}^{-1} \{ \int_{\tau(t)}^{\tau(t)} \tilde{f}(s, t) \}
$$

$$
+ \tilde{g}(s, t) \int_{\tau(t)}^{s} \tilde{h}(s, t) ds, \quad i = 1, 2, ..., n + 1.
$$

**Proof:** From (32) and (33), for all $t \in (t_i, t_{i+1}]$, $i = 1, 2, ..., n + 1$, we have

$$
|u(t)| \leq C + \right.
\int_{t_0}^{t} \left[ M_1[s, t, u(\tau(s))], \int_{t_0}^{s} N_1(\xi, u(\tau(\xi))) d\xi \right] ds
$$

$$
+ \right.
\int_{t_0}^{t} \left[ M_2[s, t, u(\tau(s))], \int_{t_0}^{s} N_2(\xi, u(\tau(\xi))) d\xi \right] ds
$$

$$
+ \sum_{t_0 < t_j < t} \beta_j |u(t_j - 0)|
$$

$$
\leq C + \right.
\int_{t_0}^{t} \left[ f(s, t)|u(\tau(s))|^q + g(s, t)|u(\tau(\xi))|^q d\xi \right] ds
$$

$$
+ \right.
\int_{t_0}^{t} \left[ f(s, t)|u(\tau(s))|^q + g(s, t)|u(\tau(\xi))|^q d\xi \right] ds
$$

$$
+ \sum_{t_0 < t_j < t} \beta_j |u(t_j - 0)|
$$

$$
\leq C + \right.
\int_{t_0}^{t} \left[ f(s, t)|u(\tau(s))|^q + g(s, t)|u(\tau(\xi))|^q d\xi \right] ds
$$

$$
+ \right.
\int_{t_0}^{t} \left[ f(s, t)|u(\tau(s))|^q + g(s, t)|u(\tau(\xi))|^q d\xi \right] ds
$$

$$
+ \sum_{t_0 < t_j < t} \beta_j |u(t_j - 0)|
$$

$$
= C + \right.
\int_{t_0}^{t} \left[ f(s, t)|u(\tau(s))|^q + g(s, t)|u(\tau(\xi))|^q d\xi \right] ds
$$

$$
+ \right.
\int_{t_0}^{t} \left[ f(s, t)|u(\tau(s))|^q + g(s, t)|u(\tau(\xi))|^q d\xi \right] ds
$$

$$
+ \sum_{t_0 < t_j < t} \beta_j |u(t_j - 0)|
$$

$$
= C + \right.
\int_{t_0}^{t} \left[ f(s, t)|u(\tau(s))|^q + g(s, t)|u(\tau(\xi))|^q d\xi \right] ds
$$

$$
+ \right.
\int_{t_0}^{t} \left[ f(s, t)|u(\tau(s))|^q + g(s, t)|u(\tau(\xi))|^q d\xi \right] ds
$$

$$
+ \sum_{t_0 < t_j < t} \beta_j |u(t_j - 0)|
$$

$$
= C + \right.$$
+ \int_{\tau(t_0)}^{\tau(t_1)} [\tilde{f}(\lambda, t)|u(\lambda)|^q + \tilde{g}(\lambda, t) \int_{t_0}^{\lambda} \tilde{h}(\eta)|u(\eta)|^q d\eta] d\lambda
+ \sum_{t_0 < j < t} \beta_j |u(t_j - 0)|
= C + 
\int_{\tau(t_0)}^{\tau(t)} [\tilde{f}(s, t)\omega(|u(s)|) + \tilde{g}(s, t) \int_{\tau(t_0)}^{s} \tilde{h}(\xi)\omega(|u(\xi)|) d\xi] ds
\int_{\tau(t_0)}^{\tau(t_1)} [\tilde{f}(s, t)\omega(|u(s)|) + \tilde{g}(s, t) \int_{t_0}^{s} \tilde{h}(\xi)\omega(|u(\xi)|) d\xi] ds
+ \sum_{t_0 < j < t} \beta_j |u(t_j - 0)|. \tag{35}

From the definition of \( H_i \) we have
\[ H_i'(t) = |G_i(2t - b_i) - G_i(t)'| = \frac{2}{(2t - b_i)^{\frac{2}{p}}} - \frac{1}{t^{\frac{2}{p}}} \]
\[ = \frac{2t^{\frac{2}{p}} - (2t - b_i)^{\frac{2}{p}}}{t^{\frac{2}{p}}(2t - b_i)^{\frac{2}{p}}} \geq 0 \tag{36} \]
for \( t \geq b_i, \ i = 0, 1, \ldots, n \).

So \( H_i(t) \) are nondecreasing on \( t \geq b_i, \ i = 0, 1, \ldots, n \). Then by an suitable application of Corollary 2.1 we can obtain the desired inequality (34).

**Theorem 3.2:** Under the conditions of Theorem 3.1, we have
\[ u(t) \leq \left\{ \frac{p - q}{p} \left\{ \frac{p}{p - q} \left( m_i^{\frac{p}{q}} - b_i^{\frac{p}{q}} \right) + \int_{\tau(t_1)}^{\tau(t)} [\tilde{f}(s, t) + \tilde{g}(s, t) \int_{\tau(t_0)}^{s} \tilde{h}(\xi) d\xi] ds + b_i^{\frac{p}{p - q}} \right\}^{\frac{1}{p - q}} \right\}^{\frac{1}{p - q}}, \tag{37} \]
where \( m_i, \ b_i \) are the same as in Theorem 3.1.

**Proof:** As long as we notice
\[ G_i(v) = \int_{b_i}^{v} \frac{1}{s^{\frac{2}{p}}(t)} ds = \frac{p}{p - q} \left( v^{\frac{p}{p - q}} - b_i^{\frac{p}{p - q}} \right), \ i = 0, 1, \ldots, n, \tag{38} \]
then combining (38) and Theorem 3.1 we can easily deduce the desired result.

### 4 Conclusions

In this paper, we have established a new generalized Volterra-Fredholm type integral inequality, which provide a handy tool in the investigation of making estimates for bounds of solutions of certain integral equations. From the example one can see new explicit bounds are derived by the presented inequality. The process of establishing the inequality in Theorem 2.1 can be applied to the situation with two independent variables, which can be left to the further research.

**References:**


