New Generalized Delay Integral Inequalities On Time Scales

Bin Zheng
Shandong University of Technology
School of Science
Zhangzhou Road 12, Zibo, 255049
China
zhengbin2601@126.com

Abstract: In this paper, some new delay integral inequalities with two independent variables on time scales are established, which can be used as a hand tool in the investigation of qualitative properties of solutions of delay dynamic equations on time scales. Some applications for the established inequalities are also presented, and new explicit bounds on unknown functions of delay dynamic equations are obtained. Our results generalize some of the results in [16, 17].

Key–Words: Delay integral inequality; Time scale; Integral equation; Differential equation; Dynamic equation; Bounded

1 Introduction

The development of the theory of time scales was initiated by Hilgewr [1] in 1988, and the purpose of the theory of time scales is to unify continuous and discrete analysis. A time scale is an arbitrary nonempty closed subset of the real numbers. Many integral inequalities on time scales have been established since then, for example [2-11], which have been designed in order to unify continuous and discrete analysis. But to our knowledge, the delay integral inequalities on time scales have been scarcely paid attention to in the literature so far [12,13], and furthermore, nobody has studied the delay integral inequalities with two independent variables on time scales.

Our aim in this paper is to establish some new delay integral inequalities with two independent variables on time scales, and present some applications for them.

For two given sets $G,\ H$, we denote the set of maps from G to H by (G, H), while denote the definition domain and the image of a function f by $Dom(f)$ and $Im(f)$ respectively.

In the rest of the paper, R denotes the set of real numbers and $R_+ = [0, \infty)$. $\mathbb{T}$ denotes an arbitrary time scale and $\mathbb{T}_0 = [x_0, \infty) \bigcap \mathbb{T},\ \widetilde{\mathbb{T}}_0 = [y_0, \infty) \bigcap \mathbb{T}$, where $x_0,\ y_0 \in \mathbb{T}$. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum, otherwise it is $\mathbb{T}$ without the left-scattered maximum. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma(t) \in (\mathbb{T}, \mathbb{T})$ and $\rho(t) \in (\mathbb{T}, \mathbb{T})$ such that $\sigma(t) = inf\{s \in \mathbb{T}, s > t\}$, $\rho(t) = sup\{s \in \mathbb{T}, s < t\}$.

Definition 1: A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t \neq \sup \mathbb{T}$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$.

Definition 2: A function $f \in (\mathbb{T}, R)$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1+\mu(t)f(t) \neq 0$, where $\mu(t) = \sigma(t) - t$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f | f \in \mathfrak{R},\ 1 + \mu(t)f(t) > 0,\ \forall t \in \mathbb{T}\}$.

Definition 3: For some $t \in \mathbb{T}^\kappa$, and a function $f \in (\mathbb{T}, R)$, the *delta derivative* of f is denoted by $f^\Delta(t)$, and satisfies $|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$ for $\forall\varepsilon > 0$, where $s \in \mathfrak{U}$, and $\mathfrak{U}$ is a neighborhood of t. The function f is called *delta differential* on $\mathbb{T}^\kappa$.

Similarly, for some $y \in \mathbb{T}^\kappa$, and a function $f \in (\mathbb{T} \times \mathbb{T}, R)$, the *partial delta derivative* of f with respect to y is denoted by $(f(x, y))^\Delta_y$, and satisfies

$$\begin{aligned} & |f(x,\sigma(y)) - f(x,s) - (f(x,y))^\Delta_y(\sigma(y) - s)| \\ \leq\ & \varepsilon|\sigma(y) - s| \end{aligned}$$

for $\forall\varepsilon > 0$, where $s \in \mathfrak{U}$, and $\mathfrak{U}$ is a neighborhood of y.

Definition 4: For $a,\ b \in \mathbb{T}$ and a function $f : \mathbb{T} \to R$, the Cauchy integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a),$$

where $F^{\Delta}(t) = f(t),\ t \in \mathbb{T}^{\kappa}$.

Similarly, for $a,\ b \in \mathbb{T}$ and a function $f : \mathbb{T} \times \mathbb{T} \to R$, the Cauchy partial integral of f with respect to y is defined by

$$\int_a^b f(x,y)\Delta y = F(x,b) - F(x,a),$$

where $(F(x,y))_y^{\Delta} = f(x,y),\ y \in \mathbb{T}^{\kappa}$.

2 Main Results

We will give some lemmas for further use.

Lemma 2.1 ([14], **Gronwall′s inequality**): Suppose $X \in \mathbb{T}_0$ is a fixed number, and $u(X,y),\ b(X,y) \in C_{rd},\ m(X,y) \in \mathfrak{R}_+$ with respect to y, $m(X,y) \geq 0$. Then

$$u(X,y) \leq b(X,y) + \int_{y_0}^{y} m(X,t)u(X,t)\Delta t,\ y \in \widetilde{\mathbb{T}}_0$$

implies

$$u(X,y) \leq b(X,y) + \int_{y_0}^{y} e_m(y,\sigma(t))b(X,t)m(X,t)\Delta t,$$

$$y \in \widetilde{\mathbb{T}}_0,$$

where $e_m(y, y_0)$ is the unique solution of the following equation

$$(z(X,y))_y^{\Delta} = m(X,y)z(X,y),\ z(X,y_0) = 1.$$

Lemma 2.2 [15]: Assume that $a \geq 0, p \geq q \geq 0$, and $p \neq 0$, then for any $K > 0$

$$a^{\frac{q}{p}} \leq \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}}.$$

Theorem 2.1: Suppose $u,\ f,\ g,\ h \in C_{rd}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, R_+)$, $p,\ q,\ r,\ m,\ C$ are constants, and $p \geq q \geq 0,\ p \geq r \geq 0,\ p \geq m \geq 0,\ p \neq 0,\ C > 0,\ \tau_1 \in (\mathbb{T}_0, \mathbb{T}),\ \tau_1(x) \leq x,\ -\infty < \alpha = inf\{\tau_1(x)\}, x \in \mathbb{T}_0\} \leq x_0$. $\tau_2 \in (\widetilde{\mathbb{T}}_0, \mathbb{T}),\ \tau_2(y) \leq y,\ -\infty < \beta = inf\{\tau_2(y)\}, y \in \widetilde{\mathbb{T}}_0\} \leq y_0$. $\phi \in C_{rd}(([\alpha, x_0] \times [\beta, y_0]) \bigcap \mathbb{T}^2, R_+)$. $K > 0$ is an arbitrary constant. If for $(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality

$$u^p(x,y) \leq C + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)u^q(\tau_1(s),\tau_2(t))\Delta s\Delta t$$

$$+ \int_{y_0}^{y}\int_{x_0}^{x} g(s,t)u^r(\tau_1(s),\tau_2(t))\Delta s\Delta t$$

$$+ \int_{y_0}^{y}\int_{x_0}^{x}\int_{y_0}^{t}\int_{x_0}^{s} h(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t \tag{1}$$

with the initial condition:

$$\begin{cases} u(x,y) = \phi(x,y) \text{ if } x \in [\alpha, x_0]\bigcap\mathbb{T} \text{ or } y \in [\beta, y_0]\bigcap\mathbb{T}, \\ \phi(\tau_1(x),\tau_2(y)) \leq C^{\frac{1}{p}},\ \forall(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0 \\ \qquad if\ \tau_1(x) \leq x_0 \text{ or } \tau_2(y) \leq y_0, \end{cases} \tag{2}$$

then

$$u(x,y) \leq [B_1(x,y) + \int_{y_0}^{y} e_{B_2}(y,\sigma(t))B_2(x,t)B_1(x,t)\Delta t]^{\frac{1}{p}},$$

$$(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \tag{3}$$

where

$$B_1(x,y) = C + \int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)\frac{p-q}{p}K^{\frac{q}{p}} + g(s,t)\frac{p-r}{p}K^{\frac{r}{p}} + \int_{y_0}^{t}\int_{x_0}^{s} h(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t,$$

$$B_2(x,y) = \int_{x_0}^{x}[f(s,y)\frac{q}{p}K^{\frac{q-p}{p}} + g(s,y)\frac{r}{p}K^{\frac{r-p}{p}} + \int_{y_0}^{y}\int_{x_0}^{s} h(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\Delta\xi\Delta\eta]\Delta s.$$

Proof: Given a fixed $X \in \mathbb{T}_0$, and $x \in [x_0, X]\bigcap\mathbb{T},\ y \in \widetilde{\mathbb{T}}_0$. Let the right side of (1) be $v(x,y)$. Then

$$u(x,y) \leq v^{\frac{1}{p}}(x,y) \leq v^{\frac{1}{p}}(X,y),$$

$$\forall x \in [x_0, X]\bigcap\mathbb{T},\ y \in \widetilde{\mathbb{T}}_0. \tag{4}$$

If $\tau_1(x) \geq x_0$ and $\tau_2(y) \geq y_0$, then $\tau_1(x) \in [x_0, X]\bigcap\mathbb{T},\ \tau_2(y) \in \widetilde{\mathbb{T}}_0$, and

$$u(\tau_1(x),\tau_2(y)) \leq v^{\frac{1}{p}}(\tau_1(x),\tau_2(y)) \leq v^{\frac{1}{p}}(x,y). \tag{5}$$

If $\tau_1(x) \leq x_0$ or $\tau_2(y) \leq y_0$, then from (2) we have

$$u(\tau_1(x),\tau_2(y)) = \phi(\tau_1(x),\tau_2(y))$$

$$\leq C^{\frac{1}{p}} \leq v^{\frac{1}{p}}(x,y). \tag{6}$$

From (5) and (6) we always have

$$u(\tau_1(x),\tau_2(y)) \leq v^{\frac{1}{p}}(x,y),\ x \in [x_0, X]\bigcap\mathbb{T},\ y \in \widetilde{\mathbb{T}}_0. \tag{7}$$

So

$$v(X,y)=C+\int_{y_0}^{y}\int_{x_0}^{X}f(s,t)u^q(\tau_1(s),\tau_2(t))\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}g(s,t)u^r(\tau_1(s),\tau_2(t))\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t$$
$$\le C+\int_{y_0}^{y}\int_{x_0}^{X}[f(s,t)v^{\frac{q}{p}}(s,t)+g(s,t)v^{\frac{r}{p}}(s,t)]\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h(\xi,\eta)v^{\frac{m}{p}}(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t. \tag{8}$$

From Lemma 2.2, for $\forall K>0$, we have

$$\begin{cases} v^{\frac{q}{p}}(x,y)\le \frac{q}{p}K^{\frac{q-p}{p}}v(x,y)+\frac{p-q}{p}K^{\frac{q}{p}},\\ v^{\frac{r}{p}}(x,y)\le \frac{r}{p}K^{\frac{r-p}{p}}v(x,y)+\frac{p-r}{p}K^{\frac{r}{p}},\\ v^{\frac{m}{p}}(x,y)\le \frac{m}{p}K^{\frac{m-p}{p}}v(x,y)+\frac{p-m}{p}K^{\frac{m}{p}}.\end{cases} \tag{9}$$

Combining (8), (9) we have

$$v(X,y)\le C+\int_{y_0}^{y}\int_{x_0}^{X}[f(s,t)(\frac{q}{p}K^{\frac{q-p}{p}}v(s,t)+\frac{p-q}{p}K^{\frac{q}{p}})$$
$$+g(s,t)(\frac{r}{p}K^{\frac{r-p}{p}}v(s,t)+\frac{p-r}{p}K^{\frac{r}{p}})]\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h(\xi,\eta)(\frac{m}{p}K^{\frac{m-p}{p}}v(\xi,\eta)+\frac{p-m}{p}K^{\frac{m}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$
$$\le C+\int_{y_0}^{y}\int_{x_0}^{X}[f(s,t)\frac{p-q}{p}K^{\frac{q}{p}}+g(s,t)\frac{p-r}{p}K^{\frac{r}{p}}$$
$$+\int_{y_0}^{t}\int_{x_0}^{s}h(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t$$
$$+\int_{y_0}^{y}\{\int_{x_0}^{X}[f(s,t)\frac{q}{p}K^{\frac{q-p}{p}}+g(s,t)\frac{r}{p}K^{\frac{r-p}{p}}$$
$$+\int_{y_0}^{t}\int_{x_0}^{s}h(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\Delta\xi\Delta\eta]\Delta s\}v(X,t)\Delta t$$
$$=B_1(X,y)+\int_{y_0}^{y}B_2(X,t)v(X,t)\Delta t. \tag{10}$$

From Lemma 2.1 we have

$$v(X,y)\le B_1(X,y)$$
$$+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(X,t)B_1(X,t)\Delta t,\ y\in\widetilde{\mathbb{T}}_0. \tag{11}$$

Then combining (4), (11) we obtain

$$u(x,y)\le[B_1(X,y)+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(X,t)B_1(X,t)\Delta t]^{\frac{1}{p}},$$
$$x\in[x_0,X]\bigcap\mathbb{T},\ y\in\widetilde{\mathbb{T}}_0. \tag{12}$$

Taking $x=X$, it follows

$$u(X,y)\le[B_1(X,y)+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(X,t)B_1(X,t)\Delta t]^{\frac{1}{p}}. \tag{13}$$

Considering $X\in\mathbb{T}_0$ is arbitrary, substituting X with x we can obtain the desired inequality (3).

Remark 1: If we take $\mathbb{T}=R$, $p=q=1$, $g(x,y)=h(x,y)\equiv 0$, then Theorem 2.1 reduces to [16, Theorem 2.2], which is one case of integral inequality for continuous function.

Remark 2: If we take $\mathbb{T}=\mathbb{Z}$, $q=p$, $a(x,y)\equiv C$, $b(x,y)\equiv 1$, $h(x,y)\equiv 0$, the Theorem 2.1 reduces to [17, Corollary 2.6], which is one case of discrete inequality.

Based on Theorem 2.1, we will establish a Volterra-Fredholm type integral inequalities on time scales with two independent variables in the following theorem.

Theorem 2.2: Suppose $u,\ f_i,\ g_i,\ h_i\in C_{rd}(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0,R_+)$, $i=1,2$. $p,\ q,\ r,\ m,\ C,\ \phi,\ \tau_1,\ \tau_2,\ \alpha,\ \beta$ are the same as in Theorem 2.1, $M\in\mathbb{T}_0$, $N\in\widetilde{\mathbb{T}}_0$ are two fixed numbers, $\forall K>0$ is a constant. If for $(x,y)\in([x_0,M]\bigcap\mathbb{T})\times([y_0,N]\bigcap\mathbb{T})$, $u(x,y)$ satisfies the following inequality

$$u^p(x,y)\le C+\int_{y_0}^{y}\int_{x_0}^{x}f_1(s,t)u^q(\tau_1(s),\tau_2(t))\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{x}g_1(s,t)u^r(\tau_1(s),\tau_2(t))\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{x}\int_{y_0}^{t}\int_{x_0}^{s}h_1(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}f_2(s,t)u^q(\tau_1(s),\tau_2(t))\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}g_2(s,t)u^r(\tau_1(s),\tau_2(t))\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t, \tag{14}$$

with the initial condition (2), then we have

$$u(x,y)\le \{[\frac{\widetilde{C}+\widetilde{B}_6}{1-\widetilde{B}_5}]\widetilde{B}_3(x,y)+\widetilde{B}_4(x,y)\}^{\frac{1}{p}},$$

$$(x,y)\in([x_0,M]\bigcap\mathbb{T})\times([y_0,N]\bigcap\mathbb{T}), \tag{15}$$

provided that $\widetilde{B}_5<1$, where

$$\widetilde{C}=C+\int_{y_0}^{N}\int_{x_0}^{M}[f_2(s,t)\frac{p-q}{p}K^{\frac{q}{p}}+g_2(s,t)\frac{p-r}{p}K^{\frac{r}{p}}$$
$$+\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t,$$

$$\widetilde{B}_1(x,y)=\int_{y_0}^{y}\int_{x_0}^{x}[f_1(s,t)\frac{p-q}{p}K^{\frac{q}{p}}+g_1(s,t)\frac{p-r}{p}K^{\frac{r}{p}}$$
$$+\int_{y_0}^{t}\int_{x_0}^{s}h_1(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t,$$

$$\widetilde{B}_2(x,y)=\int_{x_0}^{x}[f_1(s,y)\frac{q}{p}K^{\frac{q-p}{p}}+g_1(s,y)\frac{r}{p}K^{\frac{r-p}{p}}$$
$$+\int_{y_0}^{y}\int_{x_0}^{s}h_1(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\Delta\xi\Delta\eta]\Delta s,$$

$$\widetilde{B}_3(x,y)=1+\int_{y_0}^{y}e_{\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(x,t)\Delta t,$$

$$\widetilde{B}_4(x,y)=\widetilde{B}_1(x,y)+\int_{y_0}^{y}e_{\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(x,t)\widetilde{B}_1(x,t)\Delta t,$$

$$\widetilde{B}_5=\int_{y_0}^{N}\int_{x_0}^{M}f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}\widetilde{B}_3(s,t)\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}\widetilde{B}_3(s,t)\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\widetilde{B}_3(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t,$$

$$\widetilde{B}_6=\int_{y_0}^{N}\int_{x_0}^{M}f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}\widetilde{B}_4(s,t)\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}\widetilde{B}_4(s,t)\Delta s\Delta t$$
$$\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\widetilde{B}_4(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t$$

Proof: Let the right side of (14) be $v(x,y)$. Then

$$u(x,y)\le v^{\frac{1}{p}}(x,y),$$

$$(x,y)\in([x_0,M]\bigcap\mathbb{T})\times([y_0,N]\bigcap\mathbb{T}). \tag{16}$$

Similar to the process of (5)-(7) we have

$$u(\tau_1(x),\tau_2(y))\le v^{\frac{1}{p}}(x,y),$$

$$(x,y)\in([x_0,M]\bigcap\mathbb{T})\times([y_0,N]\bigcap\mathbb{T}). \tag{17}$$

Given a fixed $X\in[x_0,M]\bigcap\mathbb{T}$, and $x\in[x_0,X]\bigcap\mathbb{T},\ y\in[y_0,N]\bigcap\mathbb{T}$, then

$$v(x,y)\le v(X,y),\ x\in[x_0,X]\bigcap\mathbb{T},\ y\in[y_0,N]\bigcap\mathbb{T}. \tag{18}$$

Furthermore, considering

$$v(x_0,y_0)=C+\int_{y_0}^{N}\int_{x_0}^{M}f_2(s,t)u^q(\tau_1(s),\tau_2(t))\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}g_2(s,t)u^r(\tau_1(s),\tau_2(t))\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t, \tag{19}$$

we have

$$v(X,y)=C+\int_{y_0}^{y}\int_{x_0}^{X}f_1(s,t)u^q(\tau_1(s),\tau_2(t))\Delta s\Delta t+$$
$$\int_{y_0}^{y}\int_{x_0}^{X}g_1(s,t)u^r(\tau_1(s),\tau_2(t))\Delta s\Delta t+$$
$$\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h_1(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}f_2(s,t)u^q(\tau_1(s),\tau_2(t))\Delta s\Delta t+$$
$$\int_{y_0}^{N}\int_{x_0}^{M}g_2(s,t)u^r(\tau_1(s),\tau_2(t))\Delta s\Delta t+$$
$$\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t$$

$$\le C+\int_{y_0}^{y}\int_{x_0}^{X}[f_1(s,t)v^{\frac{q}{p}}(s,t)+g_1(s,t)v^{\frac{r}{p}}(s,t)]\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h_1(\xi,\eta)v^{\frac{m}{p}}(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}f_2(s,t)u^q(\tau_1(s),\tau_2(t))\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}g_2(s,t)u^r(\tau_1(s),\tau_2(t))\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t$$

$$= v(x_0,y_0)+\int_{y_0}^{y}\int_{x_0}^{X}[f_1(s,t)v^{\frac{q}{p}}(s,t)+g_1(s,t)v^{\frac{r}{p}}(s,t)$$

$$+\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)v^{\frac{m}{p}}(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t. \qquad (20)$$

Then similar to the process of (8)-(11) we can deduce

$$v(X,y)\le v(x_0,y_0)+\widetilde{B}_1(X,y)+$$
$$\int_{y_0}^{y} e_{\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(X,t)(v(x_0,y_0)+\widetilde{B}_1(X,t))\Delta t$$
$$= v(x_0,y_0)[1+\int_{y_0}^{y} e_{\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(X,t)\Delta t]+\widetilde{B}_1(X,y)$$
$$+\int_{y_0}^{y} e_{\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(X,t)\widetilde{B}_1(X,t)\Delta t,\ y\in[y_0,N]\bigcap\mathbb{T}. \qquad (21)$$

Combining (18), (21), it follows

$$v(x,y)\le v(x_0,y_0)[1+\int_{y_0}^{y} e_{\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(X,t)\Delta t]$$
$$+\widetilde{B}_1(X,y)+\int_{y_0}^{y} e_{\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(X,t)\widetilde{B}_1(X,t)\Delta t,$$
$$x\in[x_0,X]\bigcap\mathbb{T},\ y\in[y_0,N]\bigcap\mathbb{T}. \qquad (22)$$

Taking $x=X$ in (22), then considering X is selected from $[x_0,M]\bigcap\mathbb{T}$ arbitrarily, substituting X with x, yields

$$v(x,y)\le v(x_0,y_0)[1+\int_{y_0}^{y} e_{\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(x,t)\Delta t]$$
$$+\widetilde{B}_1(x,y)+\int_{y_0}^{y} e_{\widetilde{B}_2}(y,\sigma(t))\widetilde{B}_2(x,t)\widetilde{B}_1(x,t)\Delta t,$$
$$(x,y)\in([x_0,M]\bigcap\mathbb{T})\times([y_0,N]\bigcap\mathbb{T}), \qquad (23)$$

that is,

$$v(x,y)\le v(x_0,y_0)\widetilde{B}_3(x,y)+\widetilde{B}_4(x,y),$$
$$(x,y)\in([x_0,M]\bigcap\mathbb{T})\times([y_0,N]\bigcap\mathbb{T}). \qquad (24)$$

On the other hand, from (9), (17) and (19) we obtain

$$v(x_0,y_0)\le C+\int_{y_0}^{N}\int_{x_0}^{M} f_2(s,t)v^{\frac{q}{p}}(s,t)\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M} g_2(s,t)v^{\frac{r}{p}}(s,t)\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)v^{\frac{m}{p}}(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t$$
$$\le C+\int_{y_0}^{N}\int_{x_0}^{M} f_2(s,t)(\frac{q}{p}K^{\frac{q-p}{p}}v(s,t)+\frac{p-q}{p}K^{\frac{q}{p}})\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M} g_2(s,t)(\frac{r}{p}K^{\frac{r-p}{p}}v(s,t)+\frac{p-r}{p}K^{\frac{r}{p}})\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)(\frac{m}{p}K^{\frac{m-p}{p}}v(\xi,\eta)+$$
$$\frac{p-m}{p}K^{\frac{m}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$
$$=\widetilde{C}+\int_{y_0}^{N}\int_{x_0}^{M} f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}v(s,t)\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M} g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}v(s,t)\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}v(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t \qquad (25)$$

Then using (24) in (25) yields

$$v(x_0,y_0)\le\widetilde{C}+$$
$$\int_{y_0}^{N}\int_{x_0}^{M}\{f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}[v(x_0,y_0)\widetilde{B}_3(s,t)$$
$$+\widetilde{B}_4(s,t)]\}\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}\{g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}[v(x_0,y_0)\widetilde{B}_3(s,t)$$
$$+\widetilde{B}_4(s,t)]\}\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}[v(x_0,y_0)\widetilde{B}_3(\xi,\eta)$$
$$+\widetilde{B}_4(\xi,\eta)]\Delta\xi\Delta\eta\Delta s\Delta t$$
$$=\widetilde{C}+v(x_0,y_0)\{\int_{y_0}^{N}\int_{x_0}^{M}[f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}\widetilde{B}_3(s,t)$$
$$+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}\widetilde{B}_3(s,t)]\Delta s\Delta t+$$
$$\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\widetilde{B}_3(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t\}$$
$$+\int_{y_0}^{N}\int_{x_0}^{M} f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}\widetilde{B}_4(s,t)\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M} g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}\widetilde{B}_4(s,t)\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\widetilde{B}_4(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t$$

$$=\widetilde{C}+v(x_0,y_0)\widetilde{B}_5+\widetilde{B}_6, \quad (26)$$

which is followed by

$$v(x_0,y_0)\le\frac{\widetilde{C}+\widetilde{B}_6}{1-\widetilde{B}_5}. \quad (27)$$

Combining (16), (24) and (27) we can obtain the desired inequality (15).

Finally, we will establish a more general inequality than that in Theorem 2.2. Consider the following inequality

$$u^p(x,y)\le C+\int_{y_0}^{y}\int_{x_0}^{x} L(s,t,u(\tau_1(s),\tau_2(t)))\Delta s\Delta t$$

$$+\int_{y_0}^{y}\int_{x_0}^{x}\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)u^q(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M} L(s,t,u(\tau_1(s),\tau_2(t)))\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)u^q(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta]\Delta s\Delta t, \quad (28)$$

with the initial condition (2), where $u,\ p,\ q,\ C,\ \phi,\ \alpha,\ \beta,\ \tau_i,\ h_i,\ i=1,2$ are the same as in Theorem 2.1, $M\in\mathbb{T}_0,\ N\in\widetilde{\mathbb{T}}_0$ are two fixed numbers. $L\in C(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0\times R_+,R_+)$, and $0\le L(s,t,x)-L(s,t,y)\le A(s,t,y)(x-y)$ for $x\ge y\ge 0$, where $A\in C(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0\times R_+,R_+)$.

Theorem 2.3: If for $(x,y)\in([x_0,M]\bigcap\mathbb{T})\times([y_0,N]\bigcap\mathbb{T})$, $u(x,y)$ satisfies (28), and $K>0$ is an arbitrary constant, then the following inequality holds

$$u(x,y)\le\{[\frac{\widehat{C}+\widehat{B}_6}{1-\widehat{B}_5}]\widehat{B}_3(x,y)+\widehat{B}_4(x,y)\}^{\frac{1}{p}},$$

$$(x,y)\in([x_0,M]\bigcap\mathbb{T})\times([y_0,N]\bigcap\mathbb{T}), \quad (29)$$

provided that $\widetilde{B}_5<1$, where

$$\widehat{C}=C+\int_{y_0}^{N}\int_{x_0}^{M} L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\Delta s\Delta t+$$

$$\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)\frac{p-q}{p}K^{\frac{q}{p}}\Delta\xi\Delta\eta\Delta s\Delta t,$$

$$\widehat{B}_1(x,y)=\int_{y_0}^{y}\int_{x_0}^{x} L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\Delta s\Delta t$$

$$+\int_{y_0}^{y}\int_{x_0}^{x}\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)\frac{p-q}{p}K^{\frac{q}{p}}\Delta\xi\Delta\eta\Delta s\Delta t,$$

$$\widehat{B}_2(x,y)=\int_{x_0}^{x}[A(s,y,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}\Delta s$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}\int_{x_0}^{s} h_1(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\Delta\xi\Delta\eta]\Delta s,$$

$$\widehat{B}_3(x,y)=1+\int_{y_0}^{y} e_{\widehat{B}_2}(y,\sigma(t))\widehat{B}_2(x,t)\Delta t,$$

$$\widehat{B}_4(x,y)=\widehat{B}_1(x,y)+\int_{y_0}^{y} e_{\widehat{B}_2}(y,\sigma(t))\widehat{B}_2(x,t)\widehat{B}_1(x,t)\Delta t,$$

$$\widehat{B}_5=\int_{y_0}^{N}\int_{x_0}^{M} A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}\widehat{B}_3(s,t)\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\widehat{B}_3(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t,$$

$$\widehat{B}_6=\int_{y_0}^{N}\int_{x_0}^{M} A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}\widehat{B}_4(s,t)\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\widehat{B}_4(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t.$$

Proof: Let the right side of (28) be $v(x,y)$. Then

$$u(x,y)\le v^{\frac{1}{p}}(x,y),$$

$$(x,y)\in([x_0,M]\bigcap\mathbb{T})\times([y_0,N]\bigcap\mathbb{T}). \quad (30)$$

Similar to the process of (5)-(7) we have

$$u(\tau_1(x),\tau_2(y))\le v^{\frac{1}{p}}(x,y),$$

$$(x,y)\in([x_0,M]\bigcap\mathbb{T})\times([y_0,N]\bigcap\mathbb{T}). \quad (31)$$

Given a fixed $X\in[x_0,M]\bigcap\mathbb{T}$, and let $x\in[x_0,X]\bigcap\mathbb{T},\ y\in[y_0,N]\bigcap\mathbb{T}$. Then

$$v(x,y)\le v(X,y),\ x\in[x_0,X]\bigcap\mathbb{T},\ y\in[y_0,N]\bigcap\mathbb{T}. \quad (32)$$

Furthermore, considering

$$v(x_0,y_0)=C+\int_{y_0}^{N}\int_{x_0}^{M}[L(s,t,u(\tau_1(s),\tau_2(t)))\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)u^q(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t, \quad (33)$$

we have

$$v(X,y)=C+\int_{y_0}^{y}\int_{x_0}^{X} L(s,t,u(\tau_1(s),\tau_2(t)))\Delta s\Delta t$$

$$+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)u^q(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M} L(s,t,u(\tau_1(s),\tau_2(t)))\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)u^q(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t$$
$$\le C+\int_{y_0}^{y}\int_{x_0}^{X} L(s,t,v^{\frac{1}{p}}(s,t))\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)v^{\frac{q}{p}}(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M} L(s,t,u(\tau_1(s),\tau_2(t)))\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)u^q(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t$$
$$= v(x_0,y_0)+\int_{y_0}^{y}\int_{x_0}^{X} L(s,t,v^{\frac{1}{p}}(s,t))\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)v^{\frac{q}{p}}(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t. \quad (34)$$

From Lemma 2.2, we have

$$\begin{cases} v^{\frac{q}{p}}(x,y)\le \frac{q}{p}K^{\frac{q-p}{p}}v(x,y)+\frac{p-q}{p}K^{\frac{q}{p}}, \\ v^{\frac{1}{p}}(x,y)\le \frac{1}{p}K^{\frac{1-p}{p}}v(x,y)+\frac{p-1}{p}K^{\frac{1}{p}}. \end{cases} \quad (35)$$

So combining (34), (35) we obtain

$$v(X,y)\le v(x_0,y_0)+$$
$$\int_{y_0}^{y}\int_{x_0}^{X} L(s,t,\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)+\frac{p-1}{p}K^{\frac{1}{p}})\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)(\frac{q}{p}K^{\frac{q-p}{p}}v(\xi,\eta)$$
$$+\frac{p-q}{p}K^{\frac{q}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$
$$= v(x_0,y_0)+\int_{y_0}^{y}\int_{x_0}^{X}[L(s,t,\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)+\frac{p-1}{p}K^{\frac{1}{p}})$$
$$-L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})+L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})]\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)(\frac{q}{p}K^{\frac{q-p}{p}}v(\xi,\eta)$$
$$+\frac{p-q}{p}K^{\frac{q}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$
$$\le v(x_0,y_0)+$$
$$\int_{y_0}^{y}\int_{x_0}^{X} A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X} L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})]\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}[\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\Delta\xi\Delta\eta]v(X,t)\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)\frac{p-q}{p}K^{\frac{q}{p}}\Delta\xi\Delta\eta\Delta s\Delta t$$
$$\le v(x_0,y_0)+$$
$$\int_{y_0}^{y}[\int_{x_0}^{X} A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}\Delta s]v(X,t)\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X} L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\Delta s\Delta t$$
$$+\int_{y_0}^{y}[\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\Delta\xi\Delta\eta\Delta s]v(X,t)\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)\frac{p-q}{p}K^{\frac{q}{p}}\Delta\xi\Delta\eta\Delta s\Delta t$$
$$= v(x_0,y_0)+\widehat{B}_1(X,y)+\int_{y_0}^{y}\widehat{B}_2(X,t)v(X,t)\Delta t. \quad (36)$$

Similar to the derivation of (23) we obtain

$$v(x,y)\le v(x_0,y_0)[1+\int_{y_0}^{y} e_{\widehat{B}_2}(y,\sigma(t))\widehat{B}_2(x,t)\Delta t]$$
$$+\widehat{B}_1(x,y)+\int_{y_0}^{y} e_{\widehat{B}_2}(y,\sigma(t))\widehat{B}_2(x,t)\widehat{B}_1(x,t)\Delta t,$$
$$(x,y)\in([x_0,M]\bigcap\mathbb{T})\times([y_0,N]\bigcap\mathbb{T}), \quad (37)$$

that is,

$$v(x,y)\le v(x_0,y_0)\widehat{B}_3(x,y)+\widehat{B}_4(x,y),$$
$$(x,y)\in([x_0,M]\bigcap\mathbb{T})\times([y_0,N]\bigcap\mathbb{T}). \quad (38)$$

On the other hand, from (31), (33) and (35) we have

$$v(x_0,y_0)\le C+\int_{y_0}^{N}\int_{x_0}^{M} L(s,t,v^{\frac{1}{p}})\Delta s\Delta t$$
$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)v^{\frac{q}{p}}(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t$$
$$\le C+\int_{y_0}^{N}\int_{x_0}^{M} L(s,t,\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)+\frac{p-1}{p}K^{\frac{1}{p}})\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)(\frac{q}{p}K^{\frac{q-p}{p}}v(\xi,\eta) + \frac{p-q}{p}K^{\frac{q}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$

$$= C + \int_{y_0}^{N}\int_{x_0}^{M}[L(s,t,\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)+\frac{p-1}{p}K^{\frac{1}{p}}) - L(s,t,\frac{p-1}{p}K^{\frac{1}{p}}) + L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})]\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)(\frac{q}{p}K^{\frac{q-p}{p}}v(\xi,\eta) + \frac{p-q}{p}K^{\frac{q}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$

$$\leq C + \int_{y_0}^{N}\int_{x_0}^{M} A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)\Delta s\Delta t + \int_{y_0}^{N}\int_{x_0}^{M} L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)(\frac{q}{p}K^{\frac{q-p}{p}}v(\xi,\eta) + \frac{p-q}{p}K^{\frac{q}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$

$$= \widehat{C} + \int_{y_0}^{N}\int_{x_0}^{M}[A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)\Delta s\Delta t + \int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}v(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t. \tag{39}$$

Then using (38) in (39) yields

$$v(x_0,y_0) \leq \widehat{C} + \int_{y_0}^{N}\int_{x_0}^{M} A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}} \times [v(x_0,y_0)\widehat{B}_3(s,t)+\widehat{B}_4(s,t)]\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}} [v(x_0,y_0)\widehat{B}_3(\xi,\eta)+\widehat{B}_4(\xi,\eta)]\Delta\xi\Delta\eta\Delta s\Delta t$$

$$= \widehat{C} + v(x_0,y_0) \{\int_{y_0}^{N}\int_{x_0}^{M} A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}\widehat{B}_3(s,t)\Delta s\Delta t + \int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\widehat{B}_3(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t\}$$

$$+\int_{y_0}^{N}\int_{x_0}^{M} A(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}\widehat{B}_4(s,t)\Delta s\Delta t + \int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)\frac{q}{p}K^{\frac{q-p}{p}}\widehat{B}_4(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t$$

$$= \widehat{C} + v(x_0,y_0)\widehat{B}_5 + \widehat{B}_6, \tag{40}$$

which is followed by

$$v(x_0,y_0) \leq \frac{\widehat{C}+\widehat{B}_6}{1-\widehat{B}_5}. \tag{41}$$

Combining (30), (38) and (41) we can obtain the desired inequality (29).

3 Some Applications

In this section, we will present some applications for the results we have established above, and try to derive explicit bounds for solutions of certain dynamic equations.

Example 1: Consider the following delay dynamic differential equation

$$(u^p(x,y))^{\Delta\Delta}_{yx} = F(s,t,u(\tau_1(s),\tau_2(t))),$$
$$(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \tag{42}$$

with the initial condition

$$\begin{cases} u(x,y) = \phi(x,y) \text{ if } x \in [\alpha,x_0]\bigcap\mathbb{T} \text{ or } y \in [\beta,y_0]\bigcap\mathbb{T}, \\ |\phi(\tau_1(x),\tau_2(y))| \leq |C|^{\frac{1}{p}},\ \forall(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0 \\ \quad if\ \tau_1(x)\leq x_0 \text{ or } \tau_2(y)\leq y_0, \end{cases} \tag{43}$$

where $u \in C_{rd}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, R)$, $p > 0$ is a constant, $C = u^p(x_0,y_0)$, $\phi \in C_{rd}(([\alpha,x_0]\times[\beta,y_0])\bigcap\mathbb{T}^2, R)$. $\alpha,\ \beta,\ \tau_1,\ \tau_2$ are the same as in Theorem 2.1.

Theorem 3.1: Suppose $u(x,y)$ is a solution of (42), and $|F(s,t,x)| \leq f(s,t)|x|^q + g(s,t)|x|^r$, where $f,\ g,\ q,\ r$ are defined the same as in Theorem 2.1, then

$$|u(x,y)| \leq [B_1(x,y) + \int_{y_0}^{y} e_{B_2}(y,\sigma(t))B_2(x,t)B_1(x,t)\Delta t]^{\frac{1}{p}},$$
$$(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \tag{44}$$

where

$$B_1(x,y) = |C| + \int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)\frac{p-q}{p}K^{\frac{q}{p}} + g(s,t)\frac{p-r}{p}K^{\frac{r}{p}}]\Delta s\Delta t,$$

$$B_2(x,y) = \int_{x_0}^{x}[f(s,y)\frac{q}{p}K^{\frac{q-p}{p}} + g(s,y)\frac{r}{p}K^{\frac{r-p}{p}}]\Delta s.$$

Proof: The equivalent integral equation of (42) can be denoted by

$$u^p(x,y) = C + \int_{y_0}^{y}\int_{x_0}^{x} F(s,t,u(\tau_1(s),\tau_2(t)))\Delta s\Delta t. \tag{45}$$

Then

$$|u^p(x,y)| \le |C| + \int_{y_0}^{y}\int_{x_0}^{x} |F(s,t,u(\tau_1(s),\tau_2(t)))|\Delta s\Delta t$$

$$\le |C| + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)|u(\tau_1(s),\tau_2(t))|^q\Delta s\Delta t + \int_{y_0}^{y}\int_{x_0}^{x} g(s,t)|u(\tau_1(s),\tau_2(t))|^r\Delta s\Delta t,$$

and a suitable application of Theorem 2.1 yields the desired inequality (44).

Example 2: Consider the following delay dynamic integral equation

$$u^p(x,y) = C + \int_{y_0}^{y}\int_{x_0}^{x} F_1(s,t,u(\tau_1(s),\tau_2(t)))\Delta s\Delta t + \int_{y_0}^{N}\int_{x_0}^{M} F_2(s,t,u(\tau_1(s),\tau_2(t)))\Delta s\Delta t,$$

$$(x,y) \in ([x_0,M]\bigcap\mathbb{T}) \times ([y_0,N]\bigcap\mathbb{T}), \tag{46}$$

with the initial condition (43), where $u \in C_{rd}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, R)$, $p > 0$ is a constant, $C = u^p(x_0,y_0)$, $M \in \mathbb{T}_0$, $N \in \widetilde{\mathbb{T}}_0$ are two fixed numbers.

Theorem 3.2: Suppose $u(x,y)$ is a solution of (46), and $|F_i(s,t,x)| \le L(s,t,|x|)$, $i = 1,2$, where L are defined the same as in Theorem 2.3, then the following inequality holds

$$|u(x,y)| \le \{[\frac{\widehat{C} + \widehat{B}_6}{1 - \widehat{B}_5}]\widehat{B}_3(x,y) + \widehat{B}_4(x,y)\}^{\frac{1}{p}},$$

$$(x,y) \in ([x_0,M]\bigcap\mathbb{T}) \times ([y_0,N]\bigcap\mathbb{T}), \tag{47}$$

provided that $\widehat{B}_5 < 1$, where $\widehat{B}_3(x,y)$, $\widehat{B}_4(x,y)$, $\widehat{B}_5$, $\widehat{B}_6$ are defined the same as in Theorem 2.3 with $h_1(x,y) = h_2(x,y) \equiv 0$, and

$$\widehat{C} = |C| + \int_{y_0}^{N}\int_{x_0}^{M} L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})\Delta s\Delta t.$$

Proof: From (46) we have

$$|u^p(x,y)| \le |C| + \int_{y_0}^{y}\int_{x_0}^{x} |F_1(s,t,u(\tau_1(s),\tau_2(t)))|\Delta s\Delta t + \int_{y_0}^{N}\int_{x_0}^{M} |F_2(s,t,u(\tau_1(s),\tau_2(t)))|\Delta s\Delta t$$

$$\le |C| + \int_{y_0}^{y}\int_{x_0}^{x} L(s,t,|u(\tau_1(s),\tau_2(t)|)\Delta s\Delta t + \int_{y_0}^{N}\int_{x_0}^{M} L(s,t,|u(\tau_1(s),\tau_2(t)|)\Delta s\Delta t$$

$$\le |C| + \int_{y_0}^{y}\int_{x_0}^{x} L(s,t,|u(\tau_1(s),\tau_2(t)|)\Delta s\Delta t + \int_{y_0}^{N}\int_{x_0}^{M} L(s,t,|u(\tau_1(s),\tau_2(t)|)\Delta s\Delta t.$$

So by use of Theorem 2.3 we can obtain the desired inequality (47).

4 Conclusions

In this paper, we established some new delay integral inequalities on time scales. As one can see, the presented results provide a handy tool for deriving explicit bounds for solutions of certain delay dynamic integral and differential equations on time scales, and Theorem 2.1 generalizes some known continuous and discrete results in the literature.

References:

[1] S. Hilger, Ein Maβkettenkalk$\ddot{u}$l mit Anwendung auf Zentrmsmannigfaltingkeiten, Ph.D. Thesis, Univarsi. W$\ddot{u}$rzburg, 1988.

[2] F. H. Wong, C. C. Yeh, S. L. Yu, C. H. Hong, Young's inequality and related results on time scales, Appl. Math. Lett. 18 (2005), 983-988.

[3] F. H. Wong, C. C. Yeh, W. C. Lian, An extension of Jensen's inequality on time scales, Adv. Dynam. Syst. Appl. 1 (1) (2006), 113-120.

[4] M. Z. Sarikaya, On weighted Iyengar type inequalities on time scales, Appl. Math. Letters 22 (2009), 1340-1344.

[5] W. J. Liu, C. C. Li, Y. M. Hao, Further generalization of some double integral inequalities and applications, Acta. Math. Univ. Comenian. 77 (1) (2008) 147-154.

[6] X. L. Cheng, Improvement of some Ostrowski-Grüss type inequalities, Comput. Math. Appl. 42 (2001), 109-114.

[7] M. Bohner, T. Matthews, The Grüss inequality on time scales, Commun. Math. Anal. 3 (1) (2007), 1-8.

[8] Q. A. Ngô, Some mean value theorems for integrals on time scales, Appl. Math. Comput. 213 (2009), 322-328.

[9] W. J. Liu, Q. A. Ngô, Some Iyengar-type inequalities on time scales for functions whose second derivatives are bounded, Appl. Math. Comput. 216 (2010), 3244-3251.

[10] W. J. Liu, Q. A. Ngô, A generalization of Ostrowski inequality on time scales for k points, Appl. Math. Comput. 203 (2008), 754-760.

[11] R. Agarwal, M. Bohner, A. Peterson, Inequalities on time scales: a survey, Math. Inequal. Appl. 4 (4) (2001), 535-557.

[12] W. N. Li, Some delay integral inequalities on time scales, Comput. Math. Appl. 59 (2010), 1929-1936.

[13] Q. H. Ma, J. Pečarić, The bounds on the solutions of certain two-dimensional delay dynamic systems on time scales, Comput. Math. Appl., 61 (2011), pp. 2158-2163.

[14] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[15] F. C. Jiang, F. W. Meng, Explicit bounds on some new nonlinear integral inequality with delay, J. Comput. Appl. Math. 205 (2007), 479-486.

[16] H. X. Zhang, F. W. Meng, Integral inequalities in two independent variables for retarded Volterra equations, Appl. Math. Comput. 199 (2008), 90-98.

[17] W. S. Cheung, J. L. Ren, Discrete nonlinear inequalities and applications to boundary value problems, J. Math. Anal. Appl. 319 (2006), 708-724.