Stochastic, fuzzy, hybrid delayed dynamics heterogeneous competitions with product differentiation

[1]MIHAELA NEAMŢU, [2]MARILEN PIRTEA, [3]GABRIELA MIRCEA, [4]DUMITRU OPRIŞ

[1,2,3,4]West University of Timişoara
[1,3]Department of Economic Informatics and Statistics
[2]Department of Finance
[4]Department of Mathematics
[1,2,3]Str. Pestalozzi, nr 16 A, Timişoara, Romania
[4]B-dul. V. Pârvan, nr 4, Timişoara, Romania
mihaela.neamtu@feaa.uvt.ro
marilen.pirtea@feaa.uvt.ro
gabriela.mircea@feaa.uvt.ro
opris@math.uvt.ro

Abstract: In this paper, we study the heterogeneous duopoly with product differentiation, in which one firm is quantity setter and the other is price setter. The deterministic and stochastic models with delay are presented and fuzzy and hybrid models, as well. For the stochastic disturbance model with delay, sufficient conditions are done so that the steady state is asymptotically stable in mean and in mean square. Using programs in Maple 13 we make some numerical simulations that verify the theoretical results.

Key–Words: Delay dynamics, stochastic differential equations, heterogeneous duopoly competition, product differentiation, fuzzy model, hybrid model, fixed time delay

1 Introduction

In the recent literature, it has been proved that information delay makes dynamic economic models unstable [11], [16], [3]. Information delay in dynamic economic models has been introduced by Invernizzi and Medio [4], and its application to dynamic oligopolies has been examined by Chiarella and Khomin [1] and Chiarella and Szidarovsky [2]. Also, its application to IS-LM models has been studied by Neamtu et al. [14], [15], [13]. For the Kaldor's business cycle model Takeuchi and Yamamura [20] have been analyzed the effects caused by the fiscal policy with a fixed time lag on the stability of economics. The destabilizing effect of the time delay indicates that delay dynamic models may explain various cyclic behavior of economic variables. In the existing literature few studies are given for price/quantity adjusting oligopolies, for heterogeneous competitors and for markets with nonlinear demand functions.

Regarding the practical situations, where the delay plays an important role, models with stochastic perturbation are framed by stochastic differential delay equation. In this paper, we investigate the effects of random perturbation for delayed dynamics rent seeking game and heterogeneous competition with product differentiation analyzing the steady state of the models with stochastic perturbation.

The main purpose of this paper is to provide a positive answer that different time lags can generate different stochastic dynamics. Stochastic aspects of the models are used to capture the uncertainty about the environment in which the system is operating, the structure and the parameters of the models being studied. A stochastic process can be expressed in two ways, depending on the expected results. One way is to perturbed the initial system with stochastic terms, taking into consideration the equilibrium point of the considered system, but in this case, determining the equilibrium point, if it exists, is quite difficult. We study a stochastic nonlinear duopoly model in a heterogenous competition and we prove sufficient conditions that the steady state to be exponential asymptotically stable in mean, in mean square and to admit bounded variance. Also, we take into consideration the fuzzy and hybrid models.

In our present work, we use fuzzy differential equations, that were firstly proposed by Liu [8]. This is a type of differential equation, driven by a Liu process, just like a stochastic process is described by a Brownian motion. These two processes are different.

A stochastic process is characterized by a probability density function, so it is characterized by Fokker-Planck equation. The random variables are represented using normal distribution and the phenomenon has a repetitive characteristic. In the case of credible process, we are working with the distribution of a fuzzy variable, but the process does not have this aspect of repetitivity.

In the case when fuzziness and randomness simultaneously appear in a system, we will talk about hybrid process. In this sense, we have the concept of fuzzy random variable that was introduced by Kwakernaak [5], [6]. A fuzzy random variable is a random variable that takes fuzzy variable values. More generally, hybrid variable was proposed by Liu [9] to describe the phenomena with fuzziness and randomness. Based on the hybrid process, we will work with differential equations characterized by Wiener-Liu process. This can be computed using It-Liu formula [10]. In some situations, there exist many Brownian motions (Wiener processes) and Liu processes in a system, therefore, we can take into consideration also multidimensional It-Liu formula.

After this introduction, in Section 2 we describe the dynamic deterministic and stochastic models for heterogeneous competition with product differentiation with isoelastic price functions. Also, the perturbated stochastic model for the deterministic model is presented. In Section 3, the analysis of the linear stochastic differential equations with delay is showed. Section 4 studies the linearized models for the Cournot-Bertrand and Cournot-Cournot stochastic disturbance models, where sufficient conditions are done so that the steady state is asymptotically stable in mean and in mean square. In section 5 the fuzzy and hybrid models, corresponding to the Cournot-Bertrand and traditional Cournot models with delay, are presented. Some numerical simulations are performed in Section 6, using programs in Maple 13. Concluding comments are presented in the last section.

2 The dynamic deterministic and stochastic models for heterogeneous competition with product differentiation

Assume that 2 agents compete for a rent, which will earn a unit profit for the firm that actually will win the rent. Let x_i, $i=1,2$, denote the effort of agent i, $i=1,2$, spends in order to win the rent, and let b_i, $i=1,2$, $b_i>0$ be its cost. Consider firm 1 as leader and firm 2 the follower. Let $\tau_1\geq 0$ be the parameter that characterize the delay. The deterministic mathematical model with delay is described by the differential equations with delay given by:

$$\begin{aligned}\dot{x}_1(t)&=k_1\left[\frac{x_2(t)}{(x_1(t)+x_2(t))^2}-b_1\right]\\ \dot{x}_2(t)&=k_2\left[\frac{x_1(t)}{(x_1(t-\tau_1)+x_2(t))^2}-b_2\right],\end{aligned}\tag{1}$$

where $k_i>0$, $i=1,2$ and the initial conditions:

$$x_1(\theta)=\phi_1(\theta),\theta\in[-\tau_1,0],x_2(0)=\phi_2.$$

Now, we consider the Cournot-Bertrand competition in which two firms produce differentiated products. Firm 1 is a quantity setter and firm 2 is a price setter. Let x_1 and p_2 be the output for the first firm and respectively the firm 2's market price. Let $R_1(p_2)$ and $R_2(x_1)$ be the reaction functions of the firms and we assume that each firm has a fixed time delay τ_i, $i=1,2$ on its competitor's variable.

The delayed deterministic dynamic model is given by:

$$\begin{aligned}\dot{x}_1(t)&=k_1[R_1(p_2(t-\tau_2))-a_1x_1(t)]\\ \dot{p}_2(t)&=k_2[R_2(x_1(t-\tau_1))-a_2p_2(t)]\end{aligned}\tag{2}$$

where $k_i>0$, $i=1,2$ and $a_i>0$, $i=1,2$ and the initial conditions:

$$x_1(\theta)=\phi_1(\theta),\theta\in[-\tau_2,0],p_2(\theta)=\phi_2(\theta),\theta\in[-\tau_1,0].$$

The reaction functions can be considered as [17]:

1.

$$\begin{aligned}R_1(p_2)&=\frac{1}{1-\theta_1\theta_2}\left(\sqrt{\frac{\theta_1}{p_2c_1}}-\frac{\theta_1}{p_2}\right),\\ R_2(x_1)&=\sqrt{\frac{c_2}{\theta_2x_1}},\end{aligned}\tag{3}$$

where $0<\theta_i<1$, $i=1,2$ and c_1, c_2 denote the constant marginal costs for the Cournot Bertrand competition.

2.

$$R_1(p_2)=\sqrt{\frac{\theta_1p_2}{c_1}}-\theta_1p_2,R_2(x_1)=\sqrt{\frac{\theta_2x_1}{c_2}}-\theta_2x_1,\tag{4}$$

for the traditional Cournot competition in which both firms are quantity setters.

3.

$$R_1(p_2)=\sqrt{\frac{c_1p_2}{\theta_1}},R_2(x_1)=\sqrt{\frac{c_2x_1}{\theta_2}},\tag{5}$$

in the case of Bertrand-Bertrand competition in which both firms are price setters.

The models (1), (2) are determinist models described by a system of differential equation with delay given by:

$$\begin{aligned}\dot{x}_1(t)&=k_1(f_1(x_1(t),x_2(t),x_2(t-\tau_2))-a_1x_1(t))\\ \dot{x}_2(t)&=k_2(f_2(x_1(t),x_2(t),x_1(t-\tau_1))-a_2x_2(t))\end{aligned} \tag{6}$$

with the initial conditions

$$x_1(\theta)=\phi_1(\theta),\theta\in[-\tau_1,0], x_2(\theta)=\phi_2(\theta),\theta\in[-\tau_2,0].$$

Let the probability space $(\Omega,\mathcal{F},P)$ be given, and $w(t)\in R$ be a scalar Wiener process defined on Ω having independent stationary Gaussian increments with $w(0)=0$, $E(w(t)-w(s))=0$ and $E(w(t)-w(s))=min(t,s)$. The symbol E denotes the mathematical expectation [12]. The sample trajectories of $w(t)$ are continuous, nowhere differentiable and have infinite variation on any finite time interval [7].

Assume that $x_1=x_1^*$, $x_2=x_2^*$ is a steady state of (6), that means

$$\begin{aligned}f_1(x_1^*,x_2^*,x_2^*)-a_1x_1^*=0\\ f_2(x_1^*,x_2^*,x_1^*)-a_2x_2^*=0.\end{aligned}$$

We are interested in finding the effect of the noise perturbation on the steady state. Let the stochastic disturbance model of (6) given by a system of stochastic differential equations with delay:

$$\begin{aligned}&dx_1(t)=k_1((f_1(x_1(t),x_2(t),x_2(t-\tau_2)))-\\ &-k_1a_1x_1(t))dt-k_1\sigma_1(x_1(t)-x_1^*)dw(t),\\ &dx_2(t)=k_2((f_2(x_1(t),x_2(t),x_1(t-\tau_1)))-\\ &-k_2a_2x_2(t))dt-k_2\sigma_2(x_2(t)-x_2^*)dw(t).\end{aligned} \tag{7}$$

From a formal point of view, we can solve (7) and write the stochastic process $x_1(t)=x_1(t,\omega)$, $x_2(t)=x_2(t,\omega)$,

$$\begin{aligned}&x_1(t)=x_1(0)+\int_0^t k_1(f_1(x_1(t),x_2(t),x_2(t-\tau_2))-\\ &-a_1x_1(t))ds-k_1\sigma_1\int_0^t(x_1(t)-x_1^*)dw(s),\\ &x_2(t)=x_2(0)+\int_0^t k_2(f_2(x_1(t),x_2(t),x_1(t-\tau_1))-\\ &-a_2x_2(t))ds-k_2\sigma_2\int_0^t(x_2(t)-x_2^*)dw(s).\end{aligned} \tag{8}$$

Linearizing (7) around the steady state yields the linear stochastic differential delay equations:

$$\begin{aligned}&y_1(t)=k_1[(a_{11}-a_1)y_1(t)+a_{12}y_2(t)+\\ &+b_{12}y_2(t-\tau_2)]dt-k_1\sigma_1y_1(t)dw(t),\\ &y_2(t)=k_2[a_{21}y_1(t)+(a_{22}-a_2)y_2(t)+\\ &+b_{21}y_1(t-\tau_1)]dt-k_2\sigma_2y_2(t)dw(t),\end{aligned} \tag{9}$$

where $a_{11}=\frac{\partial f_1}{\partial x_1}|_{(x_1^*,x_2^*)}$, $a_{12}=\frac{\partial f_1}{\partial x_2}|_{(x_1^*,x_2^*)}$, $b_{12}=\frac{\partial f_1}{\partial x_2(t-\tau_2)}|_{(x_1^*,x_2^*)}$, $a_{21}=\frac{\partial f_2}{\partial x_1}|_{(x_1^*,x_2^*)}$, $a_{22}=\frac{\partial f_2}{\partial x_2}|_{(x_1^*,x_2^*)}$, $b_{21}=\frac{\partial f_2}{\partial x_1(t-\tau_1)}|_{(x_1^*,x_2^*)}$.

3 The analysis of the linear stochastic differential delay equations

Consider the matrices:

$$\begin{aligned}&A=\begin{pmatrix}k_1(a_{11}-a_1) & k_1a_{12}\\ k_2a_{21} & k_2(a_{22}-a_2)\end{pmatrix},\\ &B_1=\begin{pmatrix}0 & 0\\ k_2b_{21} & 0\end{pmatrix}, B_2=\begin{pmatrix}0 & k_1b_{12}\\ 0 & 0\end{pmatrix},\\ &C=\begin{pmatrix}k_1\sigma_1 & 0\\ 0 & k_2\sigma_2\end{pmatrix}, y=\begin{pmatrix}y_1\\ y_2\end{pmatrix}.\end{aligned} \tag{10}$$

From (9) we have:

$$\begin{aligned}&dy=(Ay(t)+B_1y(t-\tau_1)+B_2y(t-\tau_2))dt-\\ &-Cy(t)dw(t).\end{aligned} \tag{11}$$

When $\sigma_1=\sigma_2=0$, the stochastic differential delay equation (11) is given by:

$$\dot{y}(t)=Ay(t)+B_1y(t-\tau_1)+B_2y(t-\tau_2). \tag{12}$$

From (12), if $a_1=a_2=0$, $b_{12}=0$, the characteristic function of (12) is given by:

$$\begin{aligned}&h_1(\lambda,\tau_1)=\lambda^2-(k_1a_{11}+k_2a_{22})\lambda+\\ &+k_1k_2a_{11}a_{22}-k_1k_2a_{12}a_{21}-k_1k_2a_{12}b_{21}e^{-\lambda\tau_1}.\end{aligned} \tag{13}$$

If $a_{11}=a_{12}=a_{21}=a_{22}=0$, the characteristic functions of (12) is given by:

$$\begin{aligned}&h_2(\lambda,\tau_1,\lambda_2)=\lambda^2-(k_1a_1+k_2a_2)\lambda+\\ &+k_1k_2a_1a_2-k_1k_2b_{12}b_{21}e^{-\lambda(\tau_1+\tau_2)}.\end{aligned} \tag{14}$$

If $a_1 = a_2 = 0$, $b_{12} = 0$, let $\mathcal{C}_1([-\tau_1, 0], \mathbb{R}^2)$ be the family of continuous functions Φ^1 from $[-\tau_1, 0]$ to $\mathbb{R}^2$.

If $a_{11} = a_{12} = a_{21} = a_{22} = 0$, let $\mathcal{C}_2([-(\tau_1 + \tau_2), 0], \mathbb{R}^2)$ be the family of continuous function Φ^2 from $[-(\tau_1 + \tau_2), 0]$ to $\mathbb{R}^2$.

Using the fundamental solution Y^1, Y^2 the solution of (12) with the initial condition $y(\theta) = \Phi^1(\theta) \in \mathcal{C}_1([-\tau_1, 0], \mathbb{R}^2)$ respectively $y(\theta) = \Phi^2(\theta) \in \mathcal{C}_2([-(\tau_1 + \tau_2), 0], \mathbb{R}^2)$ is given by:

$$y_{\Phi^1(t)} = Y^1(t)\Phi^1(0) + \int_{-\tau_1}^{0} Y^1(t - \tau_1 - s)\Phi^1(s)ds, \tag{15}$$

respectively

$$y_{\Phi^2(t)} = Y^2(t)\Phi^2(0) + \int_{-(\tau_1+\tau_2)}^{0} Y^2(t - \tau_1 - \tau_2 - s)\Phi^2(s)ds. \tag{16}$$

From (15), (16), the asymptotic behavior of $y_{\Phi^1}(t)$, $y_{\Phi^2}(t)$ are determined by the fundamental solutions $Y^1(t)$, $Y^2(t)$.

We have the following result:

Theorem 1. [3] If $\alpha_1 = max\{Re(\lambda) : h_1(\lambda) = 0\}$, respectively $\alpha_2 = max\{Re(\lambda) : h_2(\lambda) = 0\}$, then for $\alpha > \alpha_1$ respectively $\alpha > \alpha_2$ there is the constants $k_1 = k_1(\alpha)$, $k_2 = k_2(\alpha)$ such that the fundamental solution Y^1, respectively Y^2 satisfies the inequality:

$$||Y^1(t)|| \leq k_1 e^{\alpha_1 t}, respectively ||Y^2(t)|| \leq k_2 e^{\alpha_2 t}, \tag{17}$$

$t \geq 0$.

From Theorem 1, the solutions (15), respectively (16) approach 0 as $t \to \infty$ if and only if $\alpha_1 > 0$ respectively $\alpha_2 > 0$.

When the characteristic equations $h_1(\lambda) = 0$, $h_2(\lambda) = 0$ have pure imaginary roots, then the study of the solutions for (12) leads to the existence of the Hopf bifurcation and will be presented in the next sections.

Consider system (11) with $\sigma_1 \neq 0$, $\sigma_2 \neq 0$. From the fundamental solution $Y^1(t)$, respectively $Y^2(t)$, the solution of (11) is a stochastic process given by:

$$y(t, \Phi^1) = y_{\Phi^1}(t) + \int_0^t Y^1(y - s)Cy(s - \tau_1; \Phi^1)dw(s) \tag{18}$$

respectively

$$y(t, \Phi^2) = y_{\Phi^2}(t) + \int_0^t Y^1(y - s)Cy(s - (\tau_1 + \tau_2); \Phi^2)dw(s), \tag{19}$$

where $y_{\Phi^1(t)}$, respectively $y_{\Phi^2(t)}$ is the solution given by (15), respectively (16). The existence and uniqueness theorem for the stochastic delay equation has been established in [7].

The solutions $y(t, \Phi^1)$, respectively $y(t, \Phi^2)$ are stochastic processes with distribution at any time t determined by the initial function $\Phi^1(\theta)$, respectively $\Phi^2(\theta)$. From the Chebyshev inequality, the possible rang of y at time t is a "almost" determined by its mean and variance at time t. Thus, the first and second moments of the solutions are important for investigating the solutions behavior.

We have used E to denote the mathematical expectation and we denote $y(t, \Phi^1)$ respectively $y(t, \Phi^2)$ by $y(t)$.

Proposition 1 *If $a_1 = a_2 = 0$, $b_{12} = 0$ the first moments of the solution of (12) are given by:*

$$\frac{E(y(t))}{dt} = AE(y(t)) + B_1 E(y(t - \tau_1)), \tag{20}$$

and

$$E(y(t)) = Y^1(t)\Phi^1(0) + \int_{-\tau_1}^{0} Y^1(t - \tau_1 - s)\Phi^1(s)ds.$$

If $\alpha_1 = max\{Re(\lambda) : h_1(\lambda) = 0\}$, then for any $\alpha > \alpha_1$ there is a constant $k_1 = k_1(\alpha)$ so that:

$$||dE(y(t))|| < k_1 ||\Phi^1|| e^{\alpha t}, t \geq 0. \tag{21}$$

If $\alpha_1 < 0$, then (11) is the first moment exponentially stable.

The proof can be obtained by taking the mathematical expectation of both sides of (11) and taking into account the properties of Itô integral.

Proposition 2 *If $a_{11} = a_{12} = a_{21} = a_{22} = 0$, the first moment of (12) is given by:*

$$\frac{E(y(t))}{dt} = B_1 E(y(t - \tau_1)) + B_2 E(y(t - \tau_2)), \tag{22}$$

and

$$E(y(t)) = Y^2(t)\Phi^2(0) + \int_{-(\tau_1+\tau_2)}^{0} Y^2(t - (\tau_1 + \tau_2) - s)\Phi^2(s)ds. \tag{23}$$

If $\alpha_2 = max\{Re(\lambda) : h_2(\lambda) = 0\}$, then for any $\alpha > \alpha_2$ there is a constant $k_2 = k_2(\alpha)$ so that:

$$||E(y(t))|| < k_1 ||\Phi^2|| e^{\alpha t}, t \geq 0. \tag{24}$$

If $\alpha_2 < 0$, then (11) is the first moment exponentially stable.

Thus in the mean, the solution for the linear stochastic equations (12) behaves precisely like the solution of the unperturbed deterministic equations.

To examine the stability of the second moment of $y(t)$ for the linear stochastic differential delay equation (11). We use Itô rule to given the stochastic differential of $y(t)y(t)^T$.

We have:

$$\begin{aligned}&\frac{d}{dt}E(y(t)y^T(t)) = E(dy(t)y(t)^T + y(t)dy^T +\\&+ Cy(t)^T C) =\\&E(Ay(t)y(t)^T + y(t)y(t)^T A^T + B_1 y(t-\tau_1)y(t)^T +\\&+ y(t)y^T(t-\tau_1)B_1^T + B_2 y(t-\tau_2)y^T(t) +\\&+ y(t)y^T(t-\tau_2)B_2^T + Cy^T(t)C).\end{aligned} \tag{25}$$

Let $R(t,s) = E(y(t)y^T(t))$ be the covariances matrix of the process $y(t)$ so that $R(t,t)$ satisfies:

$$\begin{aligned}&\dot{R}(t,t) = AR(t,t) + R(t,t)A^T + B_1 R(t-\tau_1,t) +\\&+ R(t,t-\tau_1)B_1^T + B_2 R(t-\tau_2,t) + R(t,t-\tau_2)B_2^T +\\&+ CR(t,t)C^T.\end{aligned} \tag{26}$$

Proposition 3 *The characteristic function of (26) is given by:*

$$\begin{aligned}&h(\lambda,\tau_1,\tau_2) = (2\lambda - 2k_1(a_{11}-a_1) - k_1^2\sigma_1^2)(2\lambda -\\&- 2k_2(a_{22}-a_2) - k_2^2\sigma_2^2)(2\lambda - k_1(a_{11}-a_1) -\\&- k_2(a_{22}-a_2) - k_1k_2\sigma_1\sigma_2) -\\&- 2k_1k_2(4\lambda - 2k_1(a_{11}-a_1) - 2k_2(a_{22}-a_2) -\\&- k_1^2\sigma_1^2 - K_2^2\sigma_2^2)(a_{12} + b_{12}e^{-\lambda\tau_1})(a_{21} + b_{21}e^{-\lambda\tau_1}).\end{aligned} \tag{27}$$

Proof. From (26) we have:

$$\begin{aligned}&\dot{R}_{11}(t,t) = (2k_1(a_{11}-a_1) +\\&+ k_1^2\sigma_1^2)R_{11}(t,t) + 2k_1a_{12}R_{12}(t,t) +\\&+ k_1b_{12}R_{21}(t,t-\tau_2) + k_1b_{12}R_{12}(t,t-\tau_2)\\&\dot{R}_{12}(t,t) = k_2a_{21}R_{11}(t,t) + (k_1(a_{11}-a_1) +\\&+ k_2(a_{22}-a_2))R_{12}(t,t) + k_2b_{21}R_{11}(t-\tau_1,t) +\\&+ k_2(a_{22}-a_2)R_{22}(t,t) + k_1b_{12}R_{22}(t,t-\tau_2) +\\&+ k_1k_2\sigma_1\sigma_2R_{12}(t,t)\\&\dot{R}_{22}(t,t) = 2k_2a_{21}R_{12}(t,t) + k_2b_{21}R_{12}(t-\tau_1,t) +\\&+ (2k_2(a_{22}-a_2) + k_2^2\sigma_2^2)R_{22}(t,t).\end{aligned} \tag{28}$$

Consider $R_{11}(t,s) = e^{\lambda(t+s)}K_{11}$, $R_{12}(t,s) = e^{\lambda(t+s)}K_{12}$, $R_{22}(t,s) = e^{\lambda(t+s)}K_{22}$, with K_{11}, K_{12}, K_{22} constants. We replace them in (28) and setting the condition that the system we obtain should accept results different than 0, we get (27).

The stability of the second moments is analyzed studying the roots of the characteristic equation $h(\lambda,\tau_1,\tau_2) = 0$, for the cases $a_1 = a_2 = 0$, $b_{12} = 0$ and $a_{11} = a_{12} = a_{21} = a_{22} = 0$. We use the Routh-Hurwitz theorem for determining the necessary and sufficient conditions that this equation admits roots with negative real part.

4 The analysis of the Cournot-Bertrand and traditional Cournot stochastic models with delay

The Cournot-Bertrand model with delay is given by (2) with the reaction functions (3) and has the following properties:

Proposition 4 *1. The steady state is given by the point (x_1^*, p_2^*), where*

$$x_1^* = \frac{c_2}{\theta_2 a_2 \alpha^2}, p_2^* = \alpha,$$

where α is the positive root of the equation:

$$\begin{aligned}&\theta_1\theta_2^2a_2^4x^3 - c_1\theta_2^2a_2^4x^2 - 2c_1c_2a_1a_2^2\theta_2(1-\theta_1\theta_2)x -\\&- c_1c_2a_1^2(1-\theta_1\theta_2)^2 = 0;\end{aligned}$$

2. The characteristic function of the linearized system (9) in (x_1^*, p_2^*) is given by:

$$\begin{aligned}&h(\lambda,\tau) = \lambda^2 + (k_1a_1 + k_2a_2)\lambda + k_1k_2a_1a_2 -\\&- k_1k_2\gamma e^{-\lambda\tau}\end{aligned} \tag{29}$$

where $\tau = \tau_1 + \tau_2$, $\gamma = b_{12}b_{21}$ and

$$b_{12} = \frac{1}{(1-\theta_1)\alpha}\left(\frac{\theta_1}{\alpha} - \frac{1}{2}\sqrt{\frac{\theta_1}{c_1}}\frac{1}{\sqrt{\alpha}}\right),$$

$$b_{21} = \frac{1}{2}\sqrt{\frac{c_2}{\theta_2}}\frac{1}{x_1^*\sqrt{x_1^*}};$$

3. If $\gamma \in [-\sqrt{a_1a_2}, \frac{\sqrt{a_1a_2}}{4})$, no stability switching occurs and the delay system is always locally asymptotically stable;

4. Given $\gamma < -\sqrt{a_1a_2}$, the dynamic system with fixed time delay is locally asymptotically stable when $\tau < \tau_0$ and unstable when $\tau > \tau_0$, *where* $\tau_0 = \dfrac{\theta_0}{\omega_0}$ *with:*

$$\omega_0^2 = \frac{-(k_1^2a_1^2 + k_2^2a_2^2)}{2} \pm$$
$$\pm \frac{\sqrt{(k_1^2a_1^2 + k_2^2a_2^2)^2 - 4k_1^2k_2^2(a_1a_2 - \gamma^2)}}{2},$$
$$\sin\theta_0 = -\frac{(a_1k_1 + a_2k_2)\omega_0}{k_1k_2\gamma},$$
$$\cos\theta_0 = \frac{a_1k_1a_2k_2 - \omega_0^2}{k_1k_2\gamma},$$

and $\theta_0 \in [\frac{\pi}{2}, \pi)$ *if* $k_1k_2 - \omega_0^2 \geq 0$ *and* $\theta_0 \in (0, \frac{\pi}{2})$ *otherwise.*

If $a_1 = a_2 = 1$ Theorem 2 from [17] is obtained.

Now, consider the stochastic perturbated model of (2) given by the following system of stochastic differential equations with delay:

$$\begin{aligned} dx_1(t) &= k_1[R_1(p_2(t-\tau_1)) - a_1x_1(t)]dt- \\ &- k_1\sigma_1(x_1(t) - x_1^*)dw(t), \\ dp_2(t) &= k_2[R_2(x_1(t-\tau_2)) - a_2p_2(t)]dt- \\ &- k_2\sigma_2(p_2(t) - p_2^*)dw(t). \end{aligned} \tag{30}$$

The linearized of the (30) is given by:

$$\begin{aligned} dy_1(t) &= (-k_1a_1y_1(t) + k_1b_{12}y_2(t-\tau_1))dt- \\ &- k_1\sigma_1y_1(t)dw(t), \\ dy_2(t) &= (k_2b_{21}y_1(t-\tau_2) - k_2a_2y_2(t))dt- \\ &- k_2\sigma_2y_2(t)dw(t). \end{aligned} \tag{31}$$

We have considered $y_1(t) = y_1(t,\omega)$ and $y_2(t) = y_2(t,\omega)$. If we use E to denote the mathematical expectation and taking it in both sides of (31) by using properties for Itô integral we obtain the first moments of the solution of (31) that are given by:

$$\begin{aligned} \dot{E}(y_1(t)) &= -k_1a_1E(y_1(t)) + k_1b_{12}E(y_2(t-\tau_2)), \\ \dot{E}(y_2(t)) &= k_2b_{21}E(y_1(t-\tau_1)) - k_2a_2E(y_2(t-\tau)). \end{aligned} \tag{32}$$

To examine the stability of the second moments of $y_1(t)$ and $y_2(t)$, for the linear stochastic differential equations with delay (31), we use Itô's rule given by the stochastic differential of $(y_1(t), y_2(t))^T(y_1(t), y_2(t))$. Denoted by $R_{11}(t,s) = E(y_1(t)y_1(s))$, $R_{12}(t,s) = E(y_1(t)y_2(s))$, $R_{22}(t,s) = E(y_2(t)y_2(s))$, the second moments from (31) are given by:

$$\begin{aligned} &\dot{R}_{11}(t,t) = (-2k_1a_1 + \sigma_1^2k_1^2)R_{11}(t,t)+ \\ &+ k_1b_{12}R_{21}(t-\tau_1,t) + k_1b_{12}R_{12}(t,t-\tau_2), \\ &\dot{R}_{12}(t,t) = (-k_1a_1 - k_2a_2 + k_1k_2\sigma_1\sigma_2)R_{12}(t,t)+ \\ &+ k_2b_{21}R_{11}(t-\tau_1,t) + k_1b_{12}R_{22}(t,t-\tau_2), \\ &\dot{R}_{22}(t,t) = (-2k_2a_2 + \sigma_2^2k_2^2)R_{22}(t,t)+ \\ &+ k_2b_{21}R_{12}(t-\tau_1,t) + k_2b_{21}R_{21}(t-\tau_1,t). \end{aligned} \tag{33}$$

The characteristic function of system (33) is:

$$\begin{aligned} &h_2(\lambda,\tau) = (2\lambda + 2k_1a_1 - k_1^2\sigma_1^2)(2\lambda + 2k_2a_2- \\ &- k_2^2\sigma_2^2)(2\lambda + k_1a_1 + k_2a_2 - k_1k_2\sigma_1\sigma_2)- \\ &- 2k_1k_2(4\lambda + 2(k_1a_1 + k_2a_2) - k_1^2\sigma_1^2 - k_2^2\sigma_2^2)\cdot \\ &\cdot b_{12}b_{21}e^{-\lambda\tau}, \end{aligned} \tag{34}$$

where $\tau = \tau_1 + \tau_2$.

Using the notations $n_1 = k_1a_1$, $n_2 = k_2a_2$, from (34) we deduce:

$$h_3(\lambda,\tau) = Q_3(\lambda) - Q_1(\lambda)e^{-\lambda\tau},$$

where

$$\begin{aligned} &Q_3(\lambda) = (2\lambda + 2n_1 - k_1^2\sigma_1^2)(2\lambda + 2n_2 - k_2^2\sigma_2^2)\cdot \\ &\cdot (2\lambda + n_1 + n_2 - k_1k_2\sigma_1\sigma_2) \\ &Q_1(\lambda) = 2k_1k_2b_{12}b_{21}(4\lambda + 2n_1 + 2n_2 - k_1^2\sigma_1^2- \\ &- k_2^2\sigma_2^2). \end{aligned}$$

If $\tau = 0$, the roots of the equation $h_3(\lambda, 0) = 0$ have negative real part if the Routh-Hurwitz conditions hold.

If we denote by:

$$\begin{aligned} &c_1 = 4(3(n_1+n_2) - k_1^2\sigma_1^2 - k_2^2\sigma_2^2 - k_1k_2\sigma_1\sigma_2), \\ &c_2 = 2(2n_1 - k_1^2\sigma_1^2)(2n_2 - k_2^2\sigma_2^2) + 2(2(n_1+n_2)- \\ &- k_1^2\sigma_1^2 - k_2^2)(n_1 + n_2 - k_1k_2\sigma_1\sigma_2) \\ &c_3 = (2n_1 - k_1^2\sigma_1^2)(2n_2 - k_2^2\sigma_2^2)(n_1 + n_2 - k_1k_2\sigma_1\sigma_2) \\ &c_4 = 8k_1k_2b_{12}b_{21} \\ &c_5 = 2k_1k_2b_{12}b_{21}(2(n_1+n_2) - k_1^2\sigma_1^2 - k_2\sigma_2^2), \end{aligned}$$

then

$$h(\lambda,\tau) = 8\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 - (c_4\lambda + c_5)e^{-\lambda\tau}.$$

If

$$\phi_2(\lambda) = h(\lambda, 0) = 8\lambda^3 + c_1\lambda^2 + (c_2 - c_4)\lambda + c_3 - c_5$$

and $\phi_2(0) = c_3 - c_5 < 0$, $\phi_2'(0) = c_2 - c_4 > 0$ or $\phi_2(0) = c_3 - c_5 > 0$, $\phi_2'(0) = c_2 - c_4 < 0$, then the equation $\phi_2(\lambda) = 0$ admits a positive root.

We obtain:

$$sin\theta = \frac{\omega^3(8c_5 - c_1c_4) - \omega(c_2c_5\omega - c_3c_4)}{c_5^2 - \omega^2 c_4^2}$$
$$cos\theta = \frac{\omega^4 c_4 - \omega^2(c_5c_1 + c_4c_2) + c_3c_5}{c_5^2 - \omega^2 c_4^2}$$

and

$$P(\omega) = 64\omega^6 + (c_1^2 - 16c_2)\omega^4 + (c_2^2 - 2c_1c_3 - c_4^2)\omega^2 + + c_3^2 - c_5^2.$$

If ω_0 is a positive solution of $P(\omega) = 0$ then:

$$\tau_0 = \frac{1}{\omega_0} arctg \frac{\omega^3(8c_5 - c_1c_4) - \omega(c_2c_5\omega - c_3c_4)}{\omega^4 c_4 - \omega^2(c_5c_1 + c_4c_2) + c_3c_5}$$

In what follows the analysis for the traditional Cournot competition with the reaction functions given by (4) is summarized in:

Proposition 5 *1. The steady state is given by* (x_{10}, p_{20})*, where*

$$x_{10} = \frac{\theta_2}{c_2(\theta_2 + a_2 z^*)}, p_{20} = \frac{\theta_2 z^*}{c_2(\theta_2 + a_2 z^*)}$$

with z^* *the positive root of the equation:*

$$m^2 a_2^2 z^3 + (2a_2\theta_2 m^2 - \theta_2^2) z^2 + (m^2\theta_2^2 - 2a_1\theta_1) z - a_1^2 = 0$$

and $m = \frac{\theta_1 c_2}{\theta_2 c_1}$*;*

2. The characteristic equation of the linearized system in (x_{10}, p_{20}) *is (9), where*

$$b_{12} = -\theta_1 + \frac{1}{2\sqrt{p_{20}}}\sqrt{\frac{\theta_1}{c_1}}, b_{21} = -\theta_2 + \frac{1}{2\sqrt{x_{10}}}\sqrt{\frac{\theta_2}{c_2}}; \quad (35)$$

3. The first moments of the solution are (32) with b_{12} *and* b_{21} *from (35);*

4. The second moments of the solution are (33) with b_{12} *and* b_{21} *from (35);*

5. The characteristic function of (33) is given by (34) with b_{12} *and* b_{21} *from (35).*

The analysis of the behavior for the mean and mean square values can be done using the characteristic equations $h_1(\lambda, \tau) = 0$ and $h_2(\lambda, \tau) = 0$.

5 The fuzzy and hybrid Cournot-Bertrand and traditional Cournot models with delay

Fuzzy differential equations are derived to model fuzzy dynamic systems. Fuzzy differential equations are mainly concerning probabilistic uncertainty based on possibility measure. Recently, Liu (2008) introduced a new kind of fuzzy differential equation based on credibility measure. The study is moved from a probabilistic space to a credibilistic one, as it is described by Li and Liu [10]. This is a new theory that deals with fuzzy phenomena. Fuzzy random theory and random fuzzy theory can be seen as an extension of credibility theory. A fuzzy random variable can be seen as a function from the probability space to the set of fuzzy variables, and a random fuzzy variable is a function from the credibility space to the set of random variables [10]. Let (θ, P, C) be a credibility space, where θ is a nonempty set, P is a subset of θ, the biggest σ-algebra over θ and C is a credibility measure.

If we consider C(t) a Liu process on credibility space, we call a fuzzy differential equation, the integral equation of Volterra type, given by:

$$x(t) = x(0) + \int_0^t f(s)ds + \int_0^t g(s)dC(s, z) \quad (36)$$

with f and g real valued functions. The first integral is a Riemann integral, and the second one is a Liu integral [21]. Formally, the equation (36) can be written in a similar way to a stochastic differential equation, like above:

$$dx = f(x)dt + g(x)dC(t, z).$$

The fuzzification is done by considering Liu processes [6], $C(t, z_i)$, with z_i, $i = 1, 2, 3$ positive numbers that define the membership functions of the fuzzy normal distributions

$$\mu_i(t, z_i) = 2(1 + \exp(\pi z_i / t\sqrt{6}))^{(-1)}, i = 1, 2, 3.$$

The system of fuzzy differential equations associated to (2) has the form:

$$\begin{aligned}\dot{x}_1(t) &= k_1[R_1(p_2(t - \tau_2)) - a_1 x_1(t)]dt - \\ &- k_1\beta_1(x_1 - x_1^*)dC(t, z_1) \\ \dot{p}_2(t) &= k_2[R_2(x_1(t - \tau_1)) - a_2 p_2(t)]dt - \\ &- k_2\beta_2(x_2 - x_2^*)dC(t, z_2)\end{aligned} \quad (37)$$

The mixture between fuzziness and randomness leads to a hybrid process. In this sense, we have the concept of fuzzy random variable that was introduced by Kwakernaak [5], [6]. A fuzzy random variable is a random variable that takes fuzzy variable values. More generally, hybrid variable was proposed by Liu [8] to describe the phenomena with fuzziness and randomness. Based on the hybrid process, we will work with differential equations characterized by Wiener-Liu process. This can be computed using It-Liu formula [10], [18].

Let (Ω, F, P) be a probabilistic space and (θ, P, C) a credibility space. If we consider w(t) a Wiener process on the probability space, and $C(t, z_i)$, $i = 1, 2, 3$ Liu process on credibility space, we call a hybrid differential system associated to (2) the differential equations system given by:

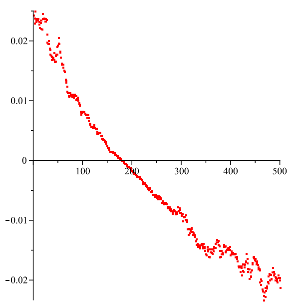
$$\begin{aligned} dx_1 &= k_1[R_1(p_2(t-\tau_2)) - a_1 x_1(t)]dt- \\ &- k_1\beta_1(x_1 - x_1^*)dC(t, z_1)- \\ &- k_1\sigma_1(x_1 - x_1^*)dw(t) \\ dp_2 &= k_2[R_2(x_1(t-\tau_1)) - a_2 p_2(t)]dt- \\ &- k_2\beta_2(x_2 - x_2^*)dC(t, z_2)- \\ &- k_2\sigma_2(x_2 - x_2^*)dw(t) \end{aligned} \quad (38)$$

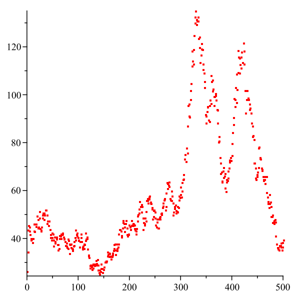
6 Numerical simulations

The numerical simulation was made using a program in Maple 13.

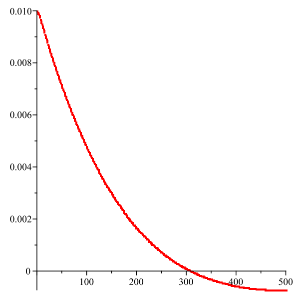
For the Cournot-Bertrand model we consider the following parameters: $k_1 = 0.2$, $k_2 = 0.5$, $\sigma_1 = 2$, $\sigma_2 = 1$, $a_1 = 2$, $a_2 = 1$, $c_1 = 4$, $c_2 = 3$, $\theta_1 = 0.4$, $\theta_2 = 0.2$. We obtained the steady state $x_1^* = 0.01593$, $p_2^* = 30.68$. The real parts of roots for the equations $h_1(\lambda, 0) = 0$ and $h_2(\lambda, 0) = 0$ are negative. The mean values $E(y_i(t))$, $i = 1, 2$, the mean square values $E(y_i(t)^2)$, $i = 1, 2$ and the variances $D(y_i(t)) = E(y_i(t)^2) - (E(y_i(t)))^2$, $i = 1, 2$ are asymptotically stable. In what follows we consider $\tau = 3$.

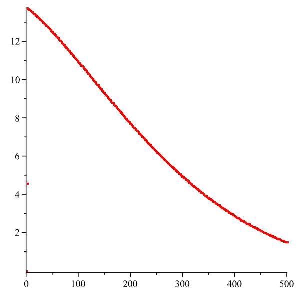
The orbits $(n, y_1(n, \omega))$, $(n, y_2(n, \omega))$ are presented in Fig1 and in Fig2 respectively:



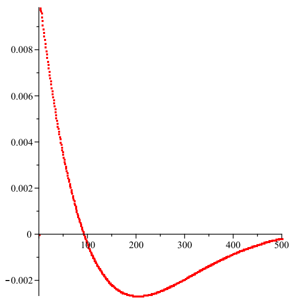


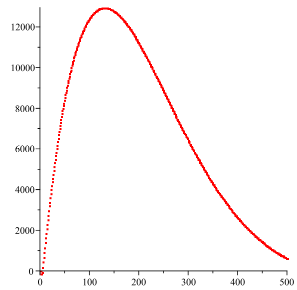
In figures Fig3 and Fig4 the orbits $(n, E(y_1(n, \omega)))$, $(n, E(y_2(n, \omega)))$ are displayed:





The variances $(n, D(y_1(n, \omega)))$, $(n, D(y_2(n, \omega)))$ are showed in Fig5 and Fig6:

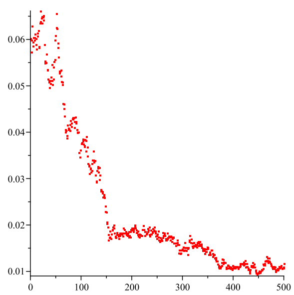


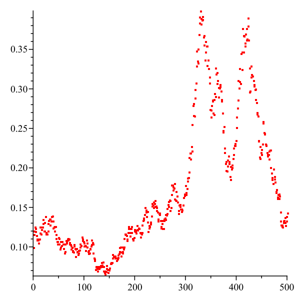


From the above simulations we can notice that the mean values and the variances of the state variables are asymptotically stable.

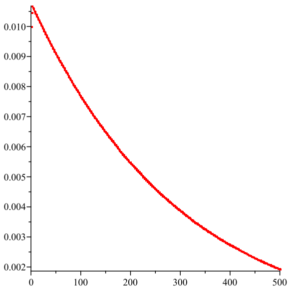
For the traditional Cournot model the parameters for the model are: $k_1 = 0.2$, $k_2 = 0.5$, $\sigma_1 = 2$, $\sigma_2 = 1$, $a_1 = 2$, $a_2 = 1$, $c_1 = 4$, $c_2 = 3$, $\theta_1 = 0.4$, $\theta_2 = 0.2$. We obtained the steady state $x_{10} = 0.046$, $p_{20} = 0.057$. The real parts of roots for the equation $h_2(\lambda, 0) = 0$ are negative and the roots of equation $h_1(\lambda, 0) = 0$ are given by: $\lambda_1 = 0.011$ and $\lambda_2 = -0.911$. Therefore, the mean values $E(y_i(t))$, $i = 1, 2$ are unstable and the mean square values $E(y_i(t)^2)$, $i = 1, 2$ are asymptotically stable. In what follows we consider $\tau = 3$.

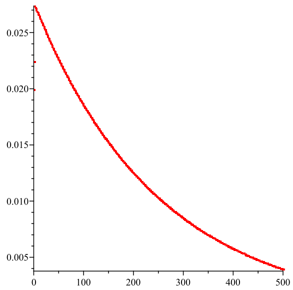
The orbits $(n, y_1(n, \omega))$, $(n, y_2(n, \omega))$ are presented in Fig7 and in Fig8 respectively:



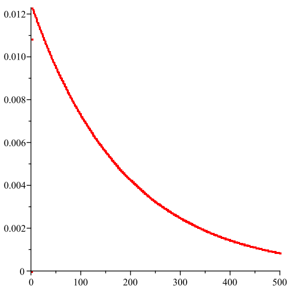


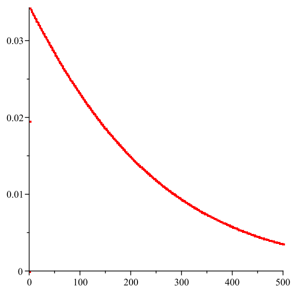
Figures Fig9 and Fig10 display the orbits $(n, E(y_1(n, \omega)))$, $(n, E(y_2(n, \omega)))$:





The variances $(n, D(y_1(n, \omega)))$, $(n, D(y_2(n, \omega)))$ are showed in Fig11 and Fig12:





From the above graphics we notice that for the traditional Cournot model the mean values and the variances of the state variables are asymptotically stable.

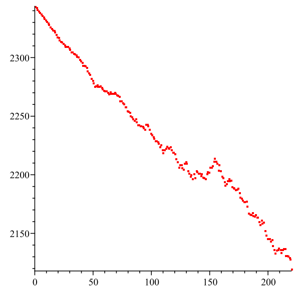
All these graphics justify the behaviors of the models solutions as obtained in the theoretical section.

In what follows we consider the numerical simulation of the fuzzy model (37) and hybrid model (38), as well. The numerical simulation of the terms $\beta_i(x_1(t) - x_i^*)dC(t, z_i)$ is done using the formula $\beta_i(x_i[j] - x_i^*)L(j, z_i)$, where:

$$L(j, z_i) = 2(1 + \exp(\pi z_i/(hS(i,j)\sqrt{6})))^{(-1)},$$
$$S(i,j) = \beta_i \sum_{k=0}^{j-1} x_i[k].$$

We use $\sigma_1 = 3$, $\sigma_2 = 5$, $\beta_1 = 0$, $\beta_2 = 0$, $z_1 = 0.2$, $z_2 = 0.3$, $a_1 = 2$, $a_2 = 1$, $c_1 = 4$ $c_2 = 3$, $\theta_1 = 0.4$, $\theta_2 = 0.2$, $k_1 = 0.2$, $k_2 = 0.5$ and obtain the graphics:

Fig 13 $(n, x_1(n, \omega, z_1))$, Fig 14 $(n, x_2(n, \omega, z_2))$:



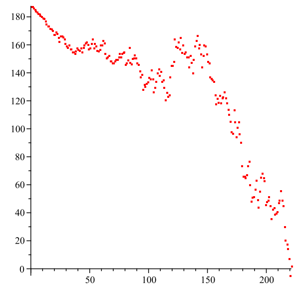
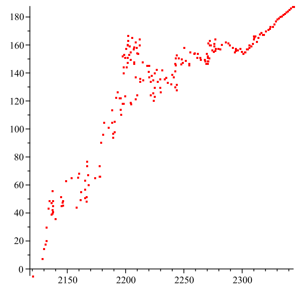
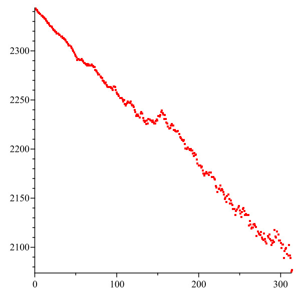


Fig 15 $(x_1(n, \omega, z_1), x_2(n, \omega, z_2))$:



If $\beta_1 = \beta_2 = 0$ we get the stochastic model. In this case we have:

Fig 16 $(n, x_1(n, \omega))$, Fig 17 $(n, x_2(n, \omega))$:



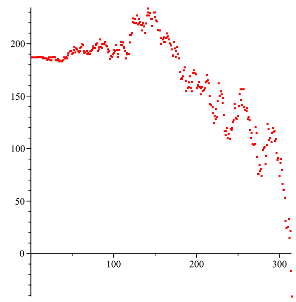
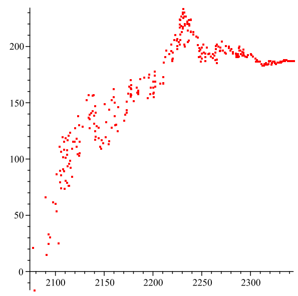


Fig 18 $(x_1(n, \omega), x_2(n, \omega))$:



A similar simulation can be done for Cournot-Cournot and Bertrand-Bertrand models.

7 Conclusions

In this paper we have studied the Cournot-Bertrand and Cournot-Cournot models described by the stochastic differential equations with delay. The study is done in the neighborhood of the steady state. Conditions are found for the asymptotically stability of the mean values and the variance of the state variables. These conditions are given using the characteristic equations associated to the differential systems that describe the mean values and the mean square values. A similar analysis can be done for the Bertrand-Bertrand competition with the reaction functions given by (4).

We have presented Cournot-Bertrand and Cournot-Cournot models by considering stochastic approach (by writing stochastic system of differential equations), fuzzy approach (the fuzzy system of differential equations associated to the deterministic system) and hybrid system of differential equations, as a combination of randomness and fuzziness.

The models from this paper can be extended considering the fractional integral [19].

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