# Stochastic, fuzzy and hybrid monetary models with delay

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*Abstract:* - This paper addresses an autonomous stochastic system with delay and associated fuzzy and hybrid models, which describe a monetary system involving interest rate, investment demand and price index. We analyze the deterministic model and the stochastic model with perturbation. For the stochastic system with delay we identify the differential equations for the mean values as well as for the mean square value. The last part of the paper includes numerical simulations and conclusions.

*Key-Words:* Monetary system, deterministic model with delay, stochastic delay system, fuzzy, hybrid system

## 1 Introduction

Systems with delay are ubiquitous in nature and life. Uncertainty can be incorporated either as an expression of our lack of precise knowledge or as a true driving force. In the latter case it is useful to model, as the system by a stochastic or noise driven model, as in familiar in the finite dimensional case. In recent literature, it has been demonstrated that information delay makes dynamic economic models unstable. Stochastic aspects of the models are used to capture the uncertainty about the environment in which the system is operating, the structure and the parameters of the models being studied. A stochastic process can be expressed in two ways, depending on the expected results. One way is to perturbed the initial system with stochastic terms, taking into consideration the equilibrium point of the considered system, but in this case, determining the equilibrium point, if it exists, is quite difficult.

In situations where delays are important, models with stochastic perturbation are framed by stochastic differential delay equations. In this paper, we will investigate the effects of random perturbation for a delayed model, a monetary system composed of product, money, bound and labor force and analyze the steady state of the model with stochastic perturbation. The fuzzy and hybrid models are taken into consideration, as well. The similar analysis was made in [12], [13], [14], [15].

Another approach that we considered was using credibility theory, that represents another uncertainty in nature. In this case we are not working on a probability space, like in the stochastic case, but on a credibility one. Credibility theory is based on five axioms, from which a credibility measure is defined, and it was introduced in order to measure a fuzzy event. This was first given by Li and Liu in their work [9]. This is a new theory that deals with fuzzy phenomena. Fuzzy random theory and random fuzzy theory can be seen as an extension of the credibility theory. A fuzzy random variable can be seen as a function from a probability space to the set of fuzzy variables, and a random fuzzy variable is a function from a credibility space to the set of random variables [8].

The main purpose of this paper is to prove that different time lags can generate different stochastic dynamics. To this end we construct a stochastic delay monetary model and analyze the mean value in the steady state, the mean square and the dispersion of the system variables.

In our present work, we use fuzzy differential equations, that were firstly proposed by Liu [7]. This is a type of differential equation, driven by a

Liu process, just like a stochastic process is described by a Brownian motion. These two processes are different. A stochastic process is characterized by a probability density function, so it is characterized by Fokker-Planck equation. The random variables are represented using normal distribution and the phenomenon has a repetitive characteristic. In the case of credible process, we are working with the distribution of a fuzzy variable, but the process does not have this aspect of repetitivity.

In the case when fuzziness and randomness simultaneously appear in a system, we will talk about hybrid process. In this sense, we have the concept of fuzzy random variable that was introduced by Kwakernaak [5, 6]. A fuzzy random variable is a random variable that takes fuzzy variable values. More generally, hybrid variable was proposed by Liu [8] to describe the phenomena with fuzziness and randomness. Based on the hybrid process, we will work with differential equations characterized by Wiener-Liu process. This can be computed using Itô-Liu formula [9]. In some situations, there exist many Brownian motions (Wiener processes) and Liu processes in a system, therefore, we can take into consideration also multi-dimensional Itô-Liu formula.

The rest of the paper develops as follows. In section 2, we describe a dynamic deterministic monetary system. In section 3 we describe the stochastic model with delay. In section 4 we describe the fuzzy and hybrid models. In section 5, we analyze the linearized of the deterministic model, establishing the existence conditions for the delay parameter's values, for which the monetary system has a Hopf bifurcation. In section 6, we analyze the differential equations systems described by the variables' mean and the mean square associated the stochastic systems with delay. In section 7, we present some numerical simulations. Finally, concluding remarks will be given in section 8.

## 2 The mathematical deterministic model with delay

The monetary system composed of product, money, bound and labor force is shaped by the differential equation system [1]:

$$\begin{aligned}
\dot{x}_1(t) &= -ax_1(t) + x_3(t) + x_1(t)x_2(t) \\
\dot{x}_2(t) &= 1 - bx_2(t) - x_1(t)^2 \qquad (1) \\
\dot{x}_3(t) &= -x_1(t) - cx_3(t)
\end{aligned}$$

The variables $x_1$, $x_2$ and $x_3$ denote the interest rate, the investment demand and the price index, respectively. The positive parameters $a$, $b$ and $c$ denote the saving amount, the per-investment cost, and the demand elasticity of commercials, respectively. The factors that generate the change of the interest rate $(x_1)$ mainly come from two aspects: the price index and the savings amount. The changing rate of $(x_2)$ is determined by the benefit rate of investment, the feedback of the investment demand, and the interest rate. The change of the price index $(x_3)$ is controlled by real interest rate and price index.

Chen [2] investigated the bifurcation behavior occurring in the system (1). In [10] it is demonstrated that the system (1) is not smoothly equivalent to the generalized Lorentz canonical form.

If we take into account that the interest rate, the investment demand and the price index in the moment $t-\tau$, $\tau > 0$, then we have $x_1(t-\tau)$, $x_2(t-\tau)$, $x_3(t-\tau)$ and the monetary system with delay is:

$$\begin{aligned}
\dot{x}_1(t) &= -ax_1(t) - a_1x_1(t-\tau) + x_3(t) + x_1(t)x_2(t) \\
\dot{x}_2(t) &= 1 - bx_2(t) - b_1x_2(t-\tau) - x_1(t)^2 \qquad (2) \\
\dot{x}_3(t) &= -x_1(t) - cx_3(t) - c_1x_3(t-\tau)
\end{aligned}$$

with the initial condition:

$x_1(\theta) = \phi_1(\theta)$, $x_2(\theta) = \phi_2(\theta)$, $x_3(\theta) = \phi_3(\theta)$, $\theta \in [-\tau, 0]$.

Model (2) represents a system of differential equations with delay.

***Proposition 1:***

The equilibrium of the system (2) is $(x_{10}, x_{20}, x_{30})$ where:

$$x_{10} = 0,\ x_{20} = \frac{1}{b+b_1},\ x_{30} = 0 \qquad (3)$$

and if $c + c_1 - b - b_1 > (a + a_1)(b + b_1)(c + c_1)$, then we have $(x_{11}, x_{21}, x_{31})$, $(x_{12}, x_{22}, x_{32})$ with:

$$
\begin{aligned}
&x_{11}=-c_1x_{31},\; x_{21}=\frac{1-(c+c_1)x_{31}^2}{b+b_1},\\
&x_{31}=\frac{1}{c+c_1}\sqrt{\frac{c+c_1-b-b_1-(a+a_1)(b+b_1)(c+c_1)}{c+c_1}}\\
&x_{12}=-c_1x_{32},\; x_{22}=\frac{1-(c+c_1)x_{31}^2}{b+b_1},\\
&x_{32}=-\frac{1}{c+c_1}\sqrt{\frac{c+c_1-b-b_1-(a+a_1)(b+b_1)(c+c_1)}{c+c_1}}
\end{aligned}
\tag{4}
$$

## 3 The mathematical stochastic deterministic model with delay

Let the probability space $(\Omega, F, P)$ be given, and $w(t)\in R$ be a scalar Wiener process defined on $\Omega$ having independent stationary Gaussian increments with $w(0)=0$, $E(w(t)-w(s))=0$. The symbol $E$ denotes the mathematical expectation [3]. The sample trajectories of $w(t)$ are continuous, nowhere differentiable and have infinite variation on any finite time interval.

What we are interested to know is the effect of the noise perturbation on the equilibrium $(x_{10}, x_{20}, x_{30})$. The stochastic perturbation of (1) given by the form of a stochastic differential equation with delay is:

$$
\begin{aligned}
dx_1(t)&=(-ax_1(t)-a_1x_1(t-\tau)+x_3(t)+x_1(t)x_2(t))dt+\\
&\quad+\sigma_1(x_1(t)-x_{10})dw(t)\\
dx_2(t)&=(1-bx_2(t)-b_1x_2(t-\tau)-x_1(t)^2)dt+\\
&\quad+\sigma_2(x_2(t)-x_{20})dw(t)\\
dx_3(t)&=(-x_1(t)-cx_3(t)-c_1x_3(t-\tau))dt+\\
&\quad+\sigma_3(x_3(t)-x_{30})dw(t)
\end{aligned}
\tag{5}
$$

## 4 The fuzzy and hybrid monetary models

Fuzzy differential equations are derived to model fuzzy dynamic systems. Fuzzy differential equations are mainly concerning probabilistic uncertainty based on possibility measure. Recently, Liu (2008) introduced a new kind of fuzzy differential equation based on credibility measure. The study is moved from a probabilistic space to a credibilistic one, as it is described by Li and Liu [9].

This is a new theory that deals with fuzzy phenomena. Fuzzy random theory and random fuzzy theory can be seen as an extension of credibility theory. A fuzzy random variable can be seen as a function from the probability space to the set of fuzzy variables, and a random fuzzy variable is a function from the credibility space to the set of random variables [9].

Let (θ, *P,C*) be a credibility space, where θ is a nonempty set, *P* is a subset of θ, the biggest σ-algebra over θ and C is a credibility measure.

If we consider C(t) a Liu process on credibility space, we call a fuzzy differential equation, the integral equation of Volterra type, given by:

$$
x(t)=x(0)+\int_0^t f(s)ds+\int_0^t g(s)dC(s,z) \tag{6}
$$

with $f$ and $h$ real valued functions. The first integral is a Riemann integral, and the second one is a Liu integral [18]. Formally, the equation (6) can be written in a similar way to a stochastic differential equation, like above:

$$
dx=f(x)dt+g(x)dC(t,z)
$$

The fuzzification is done by considering Liu processes [6], $C(t,z_i)$, with $z_i$, $i=1,2,3$ positive numbers that define the membership functions of the fuzzy normal distributions:

$$
\mu(t,z_i)=2(1+e^{\frac{\pi z_i}{t\sqrt{6}}})^{-1}, \quad i=1,2,3
$$

The system of fuzzy differential equations associated to (1) is:

$$
\begin{aligned}
dx_1(t)&=(-ax_1(t)+x_3(t)+x_1(t)x_2(t))dt+\\
&\quad+\beta_1(x_1(t)-x_{10})dC(t,z_1)\\
dx_2(t)&=(1-bx_2(t)-x_1(t)^2)dt+\\
&\quad+\beta_2(x_2(t)-x_{20})dC(t,z_2)\\
dx_3(t)&=(-x_1(t)-cx_3(t))dt+\beta_3(x_3(t)-x_{30})dC(t,z_3)
\end{aligned}
\tag{7}
$$

The mixture between fuzziness and randomness leads to a hybrid process. In this sense, we have the concept of fuzzy random variable that was introduced by Kwakernaak [5, 6]. A fuzzy random variable is a random variable that takes fuzzy variable values. More generally, hybrid variable was proposed by Liu [7] to describe the phenomena with fuzziness and randomness. Based on the hybrid process, we will work with differential equations characterized by Wiener-Liu process. This can be computed using Itô-Liu formula [9].

Let $(\Omega, F, P)$ be a probabilistic space and $(\theta, P, C)$ a credibility space. We consider $w(t)$ a Wiener process on the probability space, and $C(t, z_i)$, i=1,2,3 the Liu process on the credibility space. We call a hybrid differential system associated to (1) the differential equations system given by:

$$\begin{aligned} dx_1(t) &= (-ax_1(t)+x_3(t)+x_1(t)x_2(t))dt+ \\ &+\beta_1(x_1(t)-x_{10})dC(t,z_1)+\sigma_1(x_1(t)-x_{10})dw(t) \\ dx_2(t) &= (1-bx_2(t)+x_1(t)^2)dt+\beta_2(x_2(t)-x_{20})dC(t,z_2)+ \\ &+\sigma_2(x_3(t)-x_{10})dw(t) \\ dx_3(t) &= (-x_1(t)-cx_3(t))dt+\beta_3(x_3(t)-x_{30})dC(t,z_3)+ \\ &+\sigma_3(x_3(t)-x_{10})dw(t) \end{aligned} \tag{8}$$

## 5 The analysis of a linear differential delay equation

Linearizing (5) around the equilibrium yields the linear stochastic differential delay equation:

$$dy(t) = (Ay(t) + By(t-\tau))dt + Cy(t)dw(t) \tag{9}$$

where $y(t) = (y_1(t), y_2(t), y_3(t))^T$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}, \quad C = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}, \tag{10}$$

where,

$$\begin{aligned} &a_{11} = -a, \quad a_{12} = 0, \quad a_{13} = 1, \quad b_{11} = -a_1, \\ &a_{21} = 0, \quad a_{22} = -b, \quad a_{23} = 0, \quad b_{22} = -b_1, \\ &a_{31} = -1, \quad a_{32} = 0, \quad a_{33} = -c, \; b_{33} = -c_1. \end{aligned} \tag{11}$$

The linear differential equation with delay is given by:

$$\dot{y}(t) = Ay(t) + By(t-\tau) \tag{12}$$

From (10) and (12) we obtain:

***Proposition 2:***

**1.** The characteristic function of the system (12) is:

$$\begin{aligned} h(\lambda,\tau) &= (\lambda^2+\lambda(a+c)+ac+1+(\lambda(a_1+c_1)+ \\ &+ac_1+a_1c)e^{-\lambda\tau}+a_1c_1e^{-2\lambda\tau})(\lambda+b+b_1e^{-\lambda\tau}) \end{aligned} \tag{13}$$

**2.** If $a_1 = b_1 = c_1 = 0$ and $\tau = 0$, the characteristic equation $h(\lambda, 0) = 0$ has roots with negative real parts. The solutions of the system (1) are asymptotically stable.

**3.** If $a_1 = c_1 = 0$, $b_1 > b$ and $\tau \in (0, \tau_0)$ the characteristic equation $h(\lambda, \tau) = 0$ has roots with negative real parts, where $\tau_0$ is given by:

$$\tau_0 = \frac{1}{\sqrt{b_1^2 - b^2}} \arccos\left(-\frac{b}{b_1}\right) \tag{14}$$

If $\tau = \tau_0$ system (1) has a limit cycle with a period of oscillation given by:

$$T_0 = \frac{2\pi}{\sqrt{b_1^2 - b^2}} \tag{15}$$

**4.** If $c_1 = 0$, $b_1 = 0$, $a^2 + c^2 > 2$, $a_1 \in \left(\frac{1+ac}{a}, \sqrt{a^2+c^2-2}\right)$ and $\tau \in (0, \tau_1)$ the characteristic equation $h(\lambda, \tau) = 0$ has roots with negative real parts, where $\tau_1$ is given by:

$$\tau_1 = \frac{1}{\omega_0} arctg\left(\frac{\omega(1-\omega^2-c^2)}{a(\omega^2+c)+c}\right) \tag{16}$$

and $\omega_0$ is the positive root of the equation:

$$\omega^4 + (a^2+c^2-a_1^2-2)\omega^2 + (1+ac)^2 - a_1^2c^2 = 0. \tag{17}$$

**5.** If $a_1 = 0$, $b_1 = 0$, $a^2 + c^2 > 2$, $c_1 \in \left(\frac{1+ac}{a}, \sqrt{a^2+c^2-2}\right)$ and $\tau \in (0, \tau_2)$ the characteristic equation $h(\lambda, \tau) = 0$ has roots with negative real parts, where $\tau_2$ is given by:

$$\tau_2 = \frac{1}{\omega_0} arctg\left(\frac{\omega(1-\omega^2-a^2)}{c(\omega^2+a^2)+a}\right) \tag{18}$$

and $\omega_0$ is the positive root of the equation:

$$\omega^4 + (a^2 + c^2 - c_1^2 - 2)\omega^2 + (1 + ac)^2 - a_1^2 c^2 = 0. \quad (19)$$

*Proof:*

1. The characteristic function of (12) is:

$$h(\lambda, \tau) = \det(\lambda I - A - Be^{-\lambda\tau}) \quad (20)$$

From (17) with (10) and (11), we get formula (13).

2. Since $a_1 = c_1 = 0$ and $\tau = 0$, the characteristic equation is:

$$(\lambda^2 + \lambda(a + c) + ac + 1)(\lambda + b) = 0 \quad (21)$$

Since $a, b, c$ are positive numbers, it ensues that the equation (17) has roots with negative real parts.

3. Since $a_1 = c_1 = 0$ and $b_1 > 0$, the characteristic equation is:

$$(\lambda^2 + \lambda(a + c) + ac + 1)(\lambda + b + b_1 e^{-\lambda\tau}) = 0$$

It is well known [4], [11] that the necessary and sufficient condition for $\mathrm{Re}(\lambda) < 0$ for the equation:

$$\lambda + b + b_1 e^{-\lambda\tau} = 0$$

is given by $0 < \tau < \tau_0$, where $\tau_0$ is given by (14).

The period $T_0$ of oscillations is given by (15). For $\tau = \tau_0$ the normal form for the system (2) is obtained using the method from [4], [11].

## 6 The analysis of the linear differential stochastic equation with delay

Let $Y(t)$ be the fundamental solution of the system (12). The solution of (8) is a stochastic process given by:

$$y(t, \phi) = y_\phi(t) + \int_0^t Y(t - s) C y(t - \tau, \phi) dw(s) \quad (22)$$

where $y_\phi(t)$ is the solution given by:

$$y_\phi(t) = Y(t)\phi(0) + \int_{-\tau}^{0} Y(t - \tau - s)\phi(s) ds \quad (23)$$

and $\phi : [-\tau, 0] \to R^3$ is a family of continuous functions.

The existence and uniqueness theorem for the stochastic differential delay equations has been established in [3].

The solution $y(t, \phi)$ is a stochastic process with distribution at any time $t$ determined by the initial function $\phi(\theta)$. From the Chebyshev inequality, the possible range of $y$, at time $t$ is "almost" determined by its mean and variance at time $t$. Thus, the first and second moments of the solutions are important for investigation the solution behavior.

We have used $E$ to denote the mathematical expectation and we denote $y(t, \phi)$ by $y(t)$.

***Proposition 3:***

The mean of the solution of (9) is given by:

$$\frac{dE(y(t))}{dt} = AE(y(t)) + BE(y(t - \tau)). \quad (24)$$

The mean of the solution for (9) behaves precisely like the solution of the unperturbed deterministic equation (12).

The proof results from taking into account the mathematical expectation of both sides of (9) as well as the properties of the Ito calculus.

To examine the stability of the second moment of $y(t)$ for linear stochastic differential delay equation (9) we use Ito's rule to give the stochastic differential of $y(t)y(t)^T$ where $y(t) = (y_1(t), y_2(t), y_3(t))^T$:

$$\begin{aligned} \frac{d}{dt} E(y(t)y^T(t)) &= E\left\{y(t)dy^T(t) + y(t)dy^T(t)\right\} \\ &= E\{Ay(t)y^T(t) + y(t)y^T(t)A^T + By(t-\tau)y^T(t) + \\ &+ y(t)y^T(t-\tau)B^T + Cy^T(t)C\} \end{aligned} \quad (25)$$

Let $R(t, s) = E\left\{y(t)y^T(s)\right\}$ be the covariance matrix of the process y(t) so that $R(t, t)$ satisfies:

$$\begin{aligned}\dot{R}(t,t) &= AR(t,t)+R(t,t)A^T+BR(t-\tau,t)+\\ &+R(t,t-\tau)B^T+CR(t,t)C\end{aligned} \tag{26}$$

***Proposition 4:***

1. The differential system (26) is given by:

$$\begin{aligned}\dot{R}_{11}(t,t) &= (2a_{11}+\sigma_1^2)R_{11}(t,t)+2b_{11}R_{11}(t,t-\tau)+\\ &+2a_{12}R_{12}(t,t)+2a_{13}R_{13}(t,t)\\ \dot{R}_{22}(t,t) &= (2a_{22}+\sigma_2^2)R_{22}(t,t)+2b_{22}R_{22}(t,t-\tau)+\\ &+2a_{21}R_{12}(t,t)+2a_{23}R_{23}(t,t)\\ \dot{R}_{33}(t,t) &= (2a_{33}+\sigma_3^2)R_{33}(t,t)+2b_{33}R_{33}(t,t-\tau)+\\ &+2a_{31}R_{13}(t,t)+2a_{32}R_{23}(t,t)\\ \dot{R}_{12}(t,t) &= a_{21}R_{11}(t,t)+a_{12}R_{22}(t,t)+\\ &+(a_{11}+a_{22}+\sigma_1\sigma_2)R_{12}(t,t)+\\ &+b_{11}R_{12}(t-\tau,t)+b_{22}R_{12}(t,t-\tau)\\ &+a_{23}R_{13}(t,t)+a_{13}R_{32}(t,t)\\ \dot{R}_{13}(t,t) &= a_{31}R_{11}(t,t)+a_{13}R_{33}(t,t)+\\ &+(a_{11}+a_{33}+\sigma_1\sigma_3)R_{13}(t,t)+\\ &+b_{11}R_{13}(t-\tau,t)+b_{33}R_{13}(t,t-\tau)+\\ &+a_{32}R_{12}(t,t)+a_{12}R_{23}(t,t)\\ \dot{R}_{23}(t,t) &= a_{32}R_{22}(t,t)+a_{23}R_{33}(t,t)+\\ &+a_{31}R_{12}(t,t)+a_{21}R_{13}(t,t)+\\ &+(a_{22}+a_{33}+\sigma_2\sigma_3)R_{23}(t,t)+\\ &+b_{22}R_{23}(t-\tau,t)+b_{33}R_{23}(t,t-\tau)\end{aligned} \tag{27}$$

2. The characteristic function of (27) is given by:

$$h(\lambda,\tau)= =\det\begin{pmatrix}2\lambda I-A_{11}-B_{11}e^{-\lambda\tau} & A_{12}\\ A_{21} & 2\lambda I-A_{22}-B_{22}e^{-\lambda\tau}\end{pmatrix} \tag{28}$$

where,

$$A_{11}=\begin{pmatrix}2a_{11}+\sigma_1^2 & 0 & 0\\ 0 & 2a_{22}+\sigma_2^2 & 0\\ 0 & 0 & 2a_{33}+\sigma_3^2\end{pmatrix},$$
$$A_{12}=\begin{pmatrix}2a_{12} & 2a_{13} & 0\\ 2a_{21} & 0 & 2a_{23}\\ 0 & 2a_{31} & 2a_{32}\end{pmatrix},\quad B_{11}=\begin{pmatrix}2b_{11} & 0 & 0\\ 0 & 2b_{22} & 0\\ 0 & 0 & 2b_{33}\end{pmatrix},$$
$$A_{21}=\begin{pmatrix}a_{21} & a_{12} & 0\\ a_{31} & 0 & a_{13}\\ 0 & a_{32} & a_{23}\end{pmatrix}, \tag{29}$$
$$B_{22}=\begin{pmatrix}b_{11}+b_{22} & 0 & 0\\ 0 & b_{11}+b_{33} & 0\\ 0 & 0 & b_{22}+b_{33}\end{pmatrix},$$
$$A_{22}=\begin{pmatrix}a_{11}+a_{22}+\sigma_1\sigma_2 & a_{23} & a_{13}\\ a_{32} & a_{11}+a_{33}+\sigma_1\sigma_3 & a_{12}\\ a_{31} & a_{21} & a_{22}+a_{33}+\sigma_2\sigma_3\end{pmatrix}.$$

*Proof:*

1. The system (27) derives from (26) by taking into account that $R_{ij}(t,s)=R_{ji}(t,s)$, $i,j=1,2.$
2. Let $R_{ij}(t,s)=e^{\lambda(t+s)}K_{ij}$, $i,j=1,2.$ where $K_{ij}$ are constants. Replacing $R_{ij}(t,s)$ in (27) and setting the condition that the system we obtain should accept results different than 0, we get $h_2(\lambda,\tau)=0$.

The characteristic function (28) with the coefficients given by (11) is:

$$h_2(\lambda,\tau)=h_{21}(\lambda,\tau)h_{22}(\lambda,\tau) \tag{30}$$

where

$$\begin{aligned}h_{21}(\lambda,\tau) &= (2\lambda-2a-\sigma_1^2-2a_1e^{-\lambda\tau})(2\lambda-2b-\sigma_2^2-2b_1e^{-\lambda\tau})\\ &\quad(2\lambda-2c-\sigma_3^2-2c_1e^{-\lambda\tau})(2\lambda-a-c-\sigma_1\sigma_3-(a_1+c_1)e^{-\lambda\tau});\\ h_{22}(\lambda,\tau) &= (2\lambda-a-b-\sigma_1\sigma_2)(2\lambda-b-c-\sigma_2\sigma_3)-(2\lambda-a-b-\sigma_1\sigma_2)(b_1+c_1)+\\ &\quad+(2\lambda-b-c-\sigma_2\sigma_3)(a_1+b_1))e^{-\lambda\tau}+(a_1+b_1)(b_1+c_1)e^{-2\lambda\tau}-1\end{aligned} \tag{31}$$

The stability of the second moment is done by analyzing the roots of the characteristic equation $h_2(\lambda,\tau)=0$.

Because the equation $h_{22}(\lambda,\tau)=0$ encompasses the functions $e^{-\lambda\tau}$ and $e^{-2\lambda\tau}$ the study of the roots is done for cases in which $a_1=b_1=0$ and $b_1=c_1=0$.

## 7 Numerical simulations

For the numerical simulation, the following values were taken into consideration: $a=2$, $b=0.15$, $c=1.5$, $a_1=1$, $b_1=2$, $c_1=1$, $\sigma_1=2.5$, $\sigma_2=0.65$, $\sigma_3=2$. The equilibrium point is $x_{10}=0$, $x_{20}=0.465$, $x_{30}=0$. For $\tau=0$ it ensues that $h(\lambda,0)=0$, where $h(\lambda,0)$ is given by (13), and has roots with negative real parts. Thus, the equilibrium is asymptotically stable. The mean values of the interest rate, the investment demand and the price index are asymptotically stable.

The roots of the equation $h(\lambda,0)=0$, where $h(\lambda,0)$ is given by (13), have negative real parts. Thus the mean square values, the interest rate variance, the investment demand and the price index are asymptotically stable.

For $\tau = 3$, Fig.1, Fig.2, Fig.3 present the interest rate variation $(n, x_1(n,\omega))$, the investment demand $(n, x_2(n,\omega))$, and the price index $(n, x_3(n,\omega))$. The figures Fig.4, Fig.5, Fig.6 display $(n, E(x_1(n,\omega))$, $(n, E(x_2(n,\omega))$, $(n, E(x_3(n,\omega))$.

The figures Fig. 7, Fig.8, Fig. 9 show the variances of the variables $y_1(t)$, $y_2(t)$, $y_3(t)$.

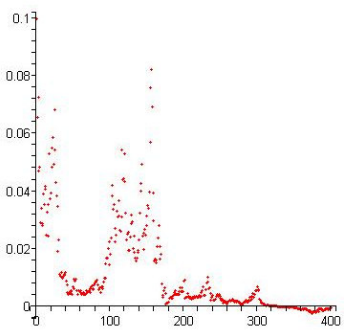


Fig.1 The orbit $(n, x_1(n,\omega))$ of the interest rate

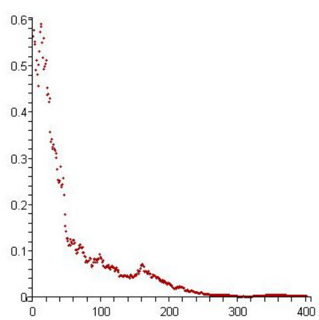


Fig.2 The orbit $(n, x_2(n,\omega))$ of the investment demand

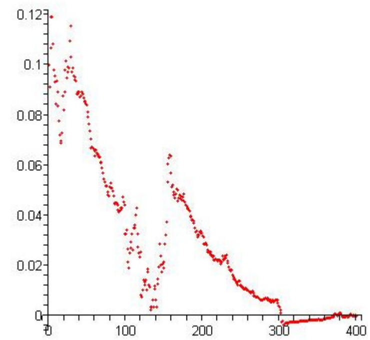


Fig.3 The orbit $(n, x_3(n,\omega))$ of the price index

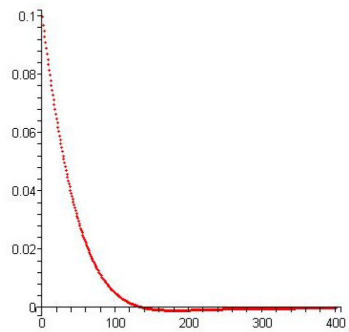


Fig.4 The orbit $(n, E(x_1(n,\omega))$

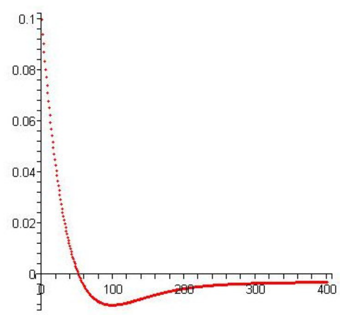


Fig.5 The orbit $(n, E(x_2(n,\omega))$

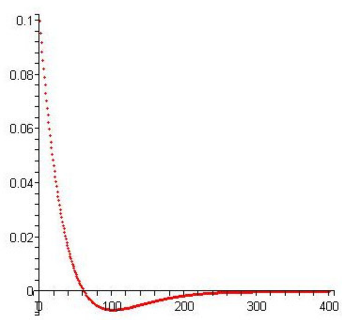


Fig.6 The orbit $(n, E(x_3(n,\omega))$

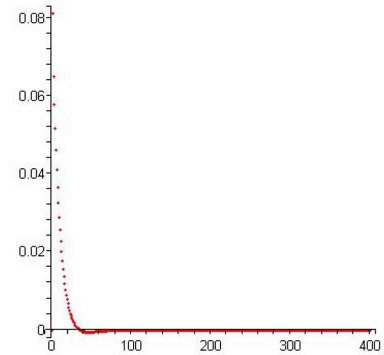

Fig.7 The plot $V(y_1(t,\omega))$

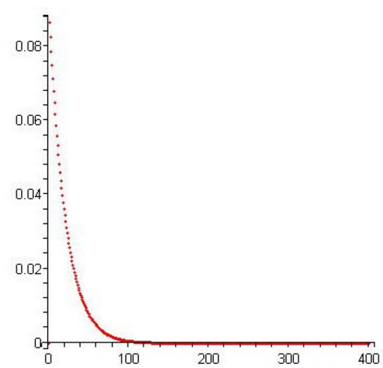

Fig.8 The plot $V(y_2(t,\omega))$

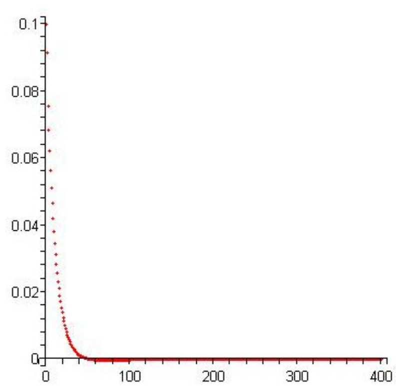

Fig.9 The plot $V(y_3(t,\omega))$

In what follows we consider the numerical simulation of the fuzzy model (6) and hybrid model (7), as well.

The numerical simulation of the terms $\beta_i(x_1(t)-x_{i0})dC(t,z_i)$ is done using the formula $\beta_i(x_i(j)-x_{i0})L(j,z_i)$, where:

$$L(j,z_i)=2(1+e^{\frac{\pi z_i}{\sqrt{6S(i,j)}}})^{-1}$$
$$S(i,j)=\beta_i\sum_{k=1}^{j-1}x_i[k]$$

For $a=2$, $b=0.15$, $c=1.5$, $\sigma_1=2.5$, $\sigma_2=0.65$, $\sigma_3=2$, $\beta_1=1$, $\beta_2=1$, $\beta_3=1$, $z_1=5$, $z_2=6$, $z_3=6$, $\sigma=1$, the hybrid model is obtained and the figures: Fig.10 $(n,x_1(n,\omega,z_1))$, Fig.11 $(n,x_2(n,\omega,z_2))$ and Fig.12 $(n,x_3(n,\omega,z_3))$:

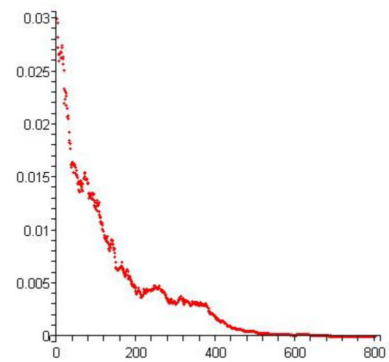

Fig.10 The orbit $(n,x_1(n,\omega,z_1))$

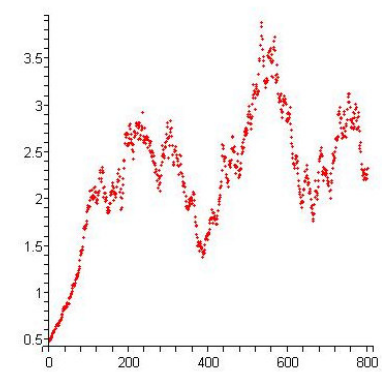

Fig.11 The orbit $(n,x_2(n,\omega,z_2))$

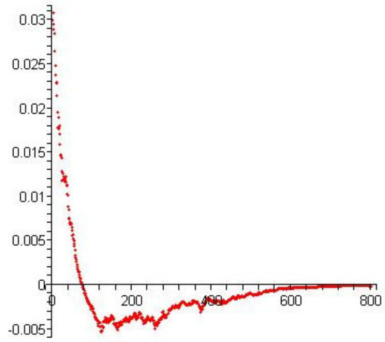

Fig.12 The orbit $(n, x_3(n, \omega, z_3))$

For $a = 2$, $b = 0.15$, $c = 1.5$, $\beta_1 = 1$, $\beta_2 = 1$, $\beta_3 = 1$, $z_1 = 5$, $z_2 = 6$, $z_3 = 6$, $\sigma = 1$, the fuzzy model and corresponding figures are obtained:
Fig. 13 $(n, x_1(n, z_1))$, Fig.14 $(n, x_2(n, z_2))$, Fig.15 $(n, x_3(n, z_3))$:

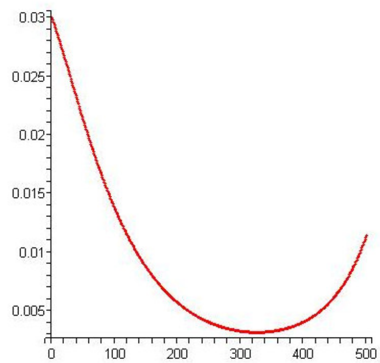

Fig.13 The orbit $(n, x_1(n, z_1))$

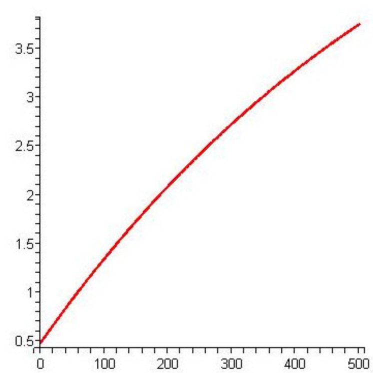

Fig.14 The orbit $(n, x_2(n, z_2))$

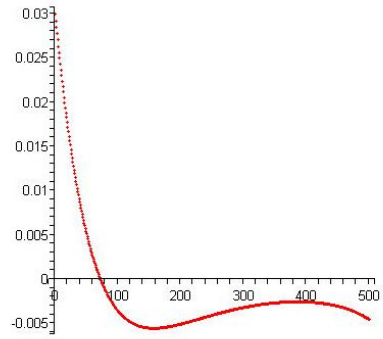

Fig.15 The orbit $(n, x_3(n, z_3))$

# 8 Conclusion

In this paper, we study a nonlinear monetary model by considering stochastic approach (by writing stochastic system of differential equations), fuzzy approach (the fuzzy system of differential equations associated to the deterministic system) and hybrid system of differential equations, as a combination of randomness and fuzziness.

We have analyzed the deterministic model and the stochastic system with perturbations. The model investigation with specific technique of differential equations determinist and hybrid systems is obtained by using a Maple 12 program.

For the numerical values of the saving amount, investment cost and demand elasticity of commercials, respectively, we have established with the help the numerical simulations that the system's solution is asymptotically stable. At the same time, we have established the mean and mean square values. The presented analysis can be employed for other economic models and processes [16], [17].

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