DIFFUSION AND SEMI-CONVEXITY OF FUZZY SET MULTIFUNCTIONS

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Abstract: We present some properties regarding diffusion and semi-convexity for fuzzy set multifunctions defined on a ring of subsets of an abstract nonvoid space and taking values in the family of all nonvoid closed subsets of a real normed space.

Key–Words: diffused, semi-convex, fuzzy set multifunction, non-atomicity, Darboux property, regular, multimeasure, null-additive.

1 Introduction

In the last years, because of its utility in a broad spectrum of areas, such as statistics, probability, economy, theory of games, artificial intelligence, computer and system sciences, biology, medicine, physics, human decision making (see, for instance, [3], [8], [27], [30], [32, 33]), a theory of fuzziness became to develop.

In recent years, we were interested in the study of different problems of the classical measure theory (regularity, fuzziness, atoms, pseudoatoms, non-atomicity, finitely purely atomicity, Darboux property, integrability, continuity properties), which we treated in the set valued case (see, for instance, [1, 2], [4-7], [13-22], [28], [31]).

Another important properties, with many applications, are diffusion and semi-convexity. Different problems concerning this property of diffusion have been studied for real valued measures for instance by [11, 12], [25, 26].

Semi-convexity has been considered by Halmos [23] in his study about the range of a measure with values in a finite-dimensional vector space.

In this paper we present different aspects of the properties of diffusion and semi-convexity for fuzzy set multifunctions defined on a ring of subsets of an abstract nonvoid space and taking values in the family of all nonvoid closed subsets of a real normed space. Some results in the context of various notions (such as non-atomicity, Darboux property, regularity, decreasing convergence) are established.

2 Preliminaries

Let X be a real normed space, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of X, $\mathcal{P}_f(X)$ the family of all nonvoid closed subsets of X, $\mathcal{P}_{bf}(X)$ the family of all nonvoid closed subsets of X, $\mathcal{P}_{bfc}(X)$ the family of all nonvoid closed bounded convex subsets of X and h the Hausdorff-Pompeiu pseudometric on $\mathcal{P}_f(X)$. h becomes a metric on $\mathcal{P}_{bf}(X)$ [24].

It is known that $h(M,N) = \max\{e(M,N), e(N,M)\}$, where $e(M,N) = \sup\limits_{x\in M} d(x,N)$, for every $M, N \in \mathcal{P}_f(X)$ is the *excess* of M over N and $d(x, N)$ is the distance from x to N with respect to the distance induced by the norm of X.

On $\mathcal{P}_0(X)$ we consider the Minkowski addition " $\overset{\bullet}{+}$ " [24], defined by:

$$M \overset{\bullet}{+} N = \overline{M + N}, \text{ for every } M, N \in \mathcal{P}_0(X),$$

where $\overline{M + N}$ is the closure of $M + N$ with respect to the topology induced by the norm of X.

We denote $|M| = h(M, \{0\})$, for every $M \in \mathcal{P}_f(X)$, where 0 is the origin of X.
We also denote $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$, where $\mathbb{N}$ is the set of all naturals and $\mathbb{R}_+ = [0, +\infty)$.

We now recall some classical notions of measure theory. Suppose T is an abstract nonvoid space and $\mathcal{C}$ is a ring of subsets of T.

Definition 1 *Let $\nu : \mathcal{C} \to [0, +\infty]$ be a set function. ν is said to be:*

I) monotone if $\nu(A) \leq \nu(B)$, for every $A, B \in \mathcal{C}$ so that $A \subseteq B$;

II) a submeasure if $\nu(\emptyset) = 0$, ν is monotone and $\nu(A \cup B) \leq \nu(A) + \nu(B)$, for every disjoint sets $A, B \in \mathcal{C}$;

III) finitely additive if $\nu(\emptyset) = 0$ and $\nu(A \cup B) = \nu(A) + \nu(B)$, for every disjoint sets $A, B \in \mathcal{C}$;

IV) increasing convergent if

$$\lim_{n\to\infty} \nu(A_n) = \nu(A),$$

for every sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ so that $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}^*$ and $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{C}$;

V) order continuous (shortly, o-continuous) if

$$\lim_{n\to\infty} \nu(A_n) = 0,$$

for every sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, so that $A_n \supseteq A_{n+1}$, for every $n \in \mathbb{N}^*$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$;

VI) super-additive if $\nu(A \cup B) \geq \nu(A) + \nu(B)$, for every disjoint sets $A, B \in \mathcal{C}$.

The following notions generalize the well-known corresponding notions from the non-additive fuzzy measure theory ([9], [10], [29]).

Definition 2 ([4, 5], [7], [13-23], [28], [31]) *Consider $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ a set multifunction, with $\mu(\emptyset) = \{0\}$. μ is said to be:*

I) fuzzy (or monotone) if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$;

II) null-additive if $\mu(A \cup B) = \mu(A)$, for every $A, B \in \mathcal{C}$, with $\mu(B) = \{0\}$;

III) null–null-additive if $\mu(A \cup B) = \{0\}$, for every $A, B \in \mathcal{C}$, with $\mu(A) = \mu(B) = \{0\}$;

IV) decreasing convergent with respect to h if

$$\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0,$$

for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $\bigcap_{n=1}^{\infty} A_n = A \in \mathcal{C}$;

V) a multimeasure if

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

VI) a multisubmeasure if μ is monotone and

$$\mu(A \cup B) \subseteq \mu(A) + \mu(B),$$

for every disjoint sets $A, B \in \mathcal{C}$.

Remark 3 I. If μ is $\mathcal{P}_f(X)$-valued, then in Definition 2-V and VI it usually appears "$\overset{\bullet}{+}$" instead of "$+$", because the sum of two closed sets is not always closed.

II. If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is null-additive, then it is null-null-additive.

The converse is not valid. Indeed, let $T = \{a, b\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(T) = [0, 2]$, $\mu(\{b\}) = [0, 1]$ and $\mu(\emptyset) = \mu(\{a\}) = \{0\}$. Then μ is null-null-additive but it is not null-additive.

III. Any multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is null-additive.

Indeed, let A, B be arbitrary sets of $\mathcal{C}$, so that $\mu(B) = \{0\}$. Since μ is monotone, we have

$$(1) \qquad \mu(A) \subseteq \mu(A \cup B).$$

Since μ is a multisubmeasure, we obtain

(2) $\mu(A \cup B) \subseteq \mu(A) + \mu(B) = \mu(A)$.

From (1) and (2) it results $\mu(A \cup B) = \mu(A)$, which proves that μ is null-additive.

The converse in not true. Indeed, let $T = \{a, b\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(T) = [0, 2], \mu(\{a\}) = \mu(\{b\}) = [0, \frac{1}{2}]$ and $\mu(\emptyset) = \{0\}$. Then μ is null-additive but it is not a multimeasure.

Suppose $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is a set multifunction, with $\mu(\emptyset) = \{0\}$. By $|\mu|$ we denote the extended real valued set function defined by

$$|\mu|(A) = |\mu(A)|, \;\; \forall\, A \in \mathcal{C}.$$

Definition 4 *For a set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(X)$, with $\mu(\emptyset) = \{0\}$, we consider the variation $\overline{\mu}$ of μ defined by*

$$\overline{\mu}(A) = \sup\{\sum_{i=1}^{n} |\mu(A_i)|\},$$

for every $A \in \mathcal{C}$, where the supremum is extended over all finite partitions $\{A_i\}_{i=\overline{1,n}}$ of A.
μ is said to be of finite variation on $\mathcal{C}$ if $\overline{\mu}(A) < \infty$, for every $A \in \mathcal{C}$.

Remark 5 I. $\overline{\mu}$ is fuzzy and super-additive on $\mathcal{P}(T)$, the family of all subsets of T.

II. If μ is fuzzy, then $|\mu|$ is also fuzzy.

The converse is not true. Indeed, let $T = \{a, b\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(T) = \{1\}, \mu(\{a\}) = \mu(\{b\}) = [0, 1]$ and $\mu(\emptyset) = \{0\}$. We have $|\mu(A)| = 1$ if $A \neq \emptyset$ and $|\mu(\emptyset)| = 0$. Then $|\mu|$ is fuzzy, but μ is not fuzzy.

III. If μ is null-additive, then $|\mu|$ is null-additive.

The converse is not valid. Indeed, let $T = \{a, b\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(T) = [0, 1], \mu(\{a\}) = 1$ and $\mu(\{b\}) = \mu(\emptyset) = \{0\}$. We have $|\mu(A)| = 1$ if $A = T$ or $A = \{a\}$ and $|\mu(A)| = 0$ if $A = \{b\}$ or $A = \emptyset$. Then $|\mu|$ is null-additive, but μ is not null-additive.

Definition 6 ([4, 5], [7], [13], [16-18], [21], [28]) *If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is a set multifunction, with $\mu(\emptyset) = \{0\}$, then:*

I) A set $A \in \mathcal{C}$ is said to be an atom of μ if $\mu(A) \supsetneq \{0\}$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \{0\}$ or $\mu(A \backslash B) = \{0\}$;

II) μ is said to be non-atomic if it has no atoms.

Remark 7 I. If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is fuzzy, then μ is non-atomic if and only if for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$, there exists $B \in \mathcal{C}$, with $B \subseteq A$, $\mu(B) \supsetneq \{0\}$ and $\mu(A \backslash B) \supsetneq \{0\}$.

II. Suppose $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is fuzzy. If $A \in \mathcal{C}$ is an atom of μ and $B \in \mathcal{C}$, $B \subseteq A$ has $\mu(B) \supsetneq \{0\}$, then B is an atom of μ and $\mu(A \backslash B) = \{0\}$.

Example 8 Let $T = \{2n | n \in \mathbb{N}\}$, $\mathcal{C} = \mathcal{P}(T)$ and the multisubmeasure defined, for every $A \in \mathcal{C}$, by:

$$\mu(A) = \begin{cases} \{0\}, & \text{if } A = \emptyset \\ \dfrac{1}{2}A \cup \{0\}, & \text{if } A \neq \emptyset, \end{cases}$$

where $\frac{1}{2}A = \{\frac{x}{2} | x \in A\}$.

If A is a set of $\mathcal{C}$ so that card $A = 1$ and $A \neq \{0\}$ or $A = \{0, 2n\}$, $n \in \mathbb{N} \backslash \{0\}$, then A is an atom of μ (by card A we mean the cardinal of A).

If A is a set of $\mathcal{C}$ so that card $A \geq 2$ and there exist $a, b \in A$ such that $a \neq b$ and $ab \neq 0$, then A is not an atom of μ.

Definition 9 *Suppose $\nu : \mathcal{C} \to \mathbb{R}_+$ is an arbitrary set function, with $\nu(\emptyset) = 0$. We say that a set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R}_+)$ is induced by ν if $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$.*

Remark 10 Let $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R}_+)$ be the set multifunction induced by a set function $\nu : \mathcal{C} \to \mathbb{R}_+$ so that $\nu(\emptyset) = 0$. Then:

i) μ is fuzzy if and only if ν is fuzzy.

Indeed, we have $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$. We observe that $\mu(\emptyset) = \{0\}$. Let $A, B \in \mathcal{C}$ so that $A \subseteq B$. Then

$$\mu(A) \subseteq \mu(A) \Leftrightarrow \nu(A) \leq \nu(B).$$

So μ is diffused if and only if ν is diffused.

ii) μ is null-additive (null-null-additive, respectively) if and only if the same is ν.

Indeed, suppose μ is null-additive and let $A, B \in \mathcal{C}$ so that $\nu(B) = 0$. Then $\mu(B) = \{0\}$ and since μ is null-additive, it results $\mu(A \cup B) = \mu(A)$. That is $[0, \nu(A \cup B)] = [0, \nu(A)]$, which implies $\nu(A \cup B) = \nu(A)$ and shows that ν is null-additive. For the converse, we analogously act.

Concerning null-null-additivity, the proof of the equivalence goes on in the same way.

iii) μ is a multimeasure if and only if ν is finitely additive.

Indeed, let $A, B \in \mathcal{C}$ so that $A \cap B = \emptyset$. Then

$$\mu(A) \overset{\bullet}{+} \mu(B) = [0, \nu(A)] \overset{\bullet}{+} [0, \nu(B)] =$$
$$= [0, \nu(A) + \nu(B)] = [0, \nu(A \cup B)] =$$
$$= \mu(A \cup B).$$

iv) μ is non-atomic if and only if ν is non-atomic.

Definition 11 *Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction so that $\mu(\emptyset) = \{0\}$. We say that μ has the Darboux property if for every $A \in \mathcal{C}$ so that $\mu(A) \supsetneq \{0\}$ and every $p \in (0, 1)$, there exists a set $B \in \mathcal{C}$ such that $B \subseteq A$ and $\mu(B) = p\mu(A)$.*

Example 12 Suppose $\mathcal{C}$ is the Borel σ-algebra of $T = [0, +\infty)$ and $\nu : \mathcal{C} \to \mathbb{R}_+$ is an increasing convergent finitely additive set function so that $\nu([t, s]) \leq s - t$, for every $t, s \in T$, with $t \leq s$. Then the set multifunction $\mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{R})$ defined by $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$ has the Darboux property.

Indeed, let $A \in \mathcal{C}$ be a set so that $\mu(A) \supsetneq \{0\}$ and let $p \in (0, 1)$. Consider $f : T \to \mathbb{R}$ the real function defined by $f(t) = \nu(A \cap [0, t])$. Since ν is finitely additive, we have:

$$|f(t) - f(s)| \leq |t - s|, \ \ \forall t, s \in T,$$

which implies that f is continuous on T. We have $f(0) = 0$ and by the increasing convergence of ν it follows $\lim\limits_{t\to\infty} f(t) = \nu(A)$. Consequently, there is $t_0 \in T$ such that $f(t_0) = p\nu(A)$. Denoting $B = A \cap [0, t_0]$, it results $B \in \mathcal{C}$, $B \subseteq A$ and $\nu(B) = p\nu(A)$. But this implies $\mu(B) = p\mu(A)$, which shows that μ has Darboux property.

In Gavriluţ and Croitoru [17] we obtained the following:

Theorem 13 *Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a set multifunction, so that $\mu(\emptyset) = \{0\}$ and $|\mu|(A) < +\infty$, for every $A \in \mathcal{C}$.*

I. Suppose μ is a multisubmeasure. If μ has the Darboux property, then μ is non-atomic.

II. If $\mathcal{C}$ is an algebra, $\overline{\mu}$ is bounded, finitely additive and has the Darboux property, then μ is non-atomic.

III. Suppose $\mathcal{C}$ is a σ-algebra and μ is the multisubmeasure induced by an o-continuous submeasure $\nu : \mathcal{C} \to [0, +\infty)$. Then μ has the Darboux property if and only if μ is non-atomic.

Remark 14 [17] The converse of Theorem 13-I is not true. Indeed, let T be a compact space, $\mathcal{B}$ the Borel δ-ring generated by the compact subsets of T and $\mu : \mathcal{B} \to \mathcal{P}_f(\mathbb{R})$ the multisubmeasure defined by:

$$\mu(A) = \begin{cases} [-\nu(A), \nu(A)], & \text{if } \nu(A) \leq 1 \\ [-\nu(A), 1], & \text{if } \nu(A) > 1, \end{cases}$$

for every $A \in \mathcal{B}$, where $\nu : \mathcal{B} \to [0, +\infty)$ is a bounded finitely additive set function having the Darboux property. Then μ is non-atomic, but μ does not have the Darboux property.

3 Diffusion of a set multifunction

We present in the sequel some properties regarding diffusion for set multifunctions.

Let T be a locally compact Hausdorff space, $\mathcal{C}$ a ring of subsets of T, $\mathcal{B}_0$ the Baire δ-ring generated by the G_δ-compact subsets of T (that is, compact sets which are countable intersections of open sets) and $\mathcal{B}$ the Borel δ-ring generated by the compact subsets of T. It is known that $\mathcal{B}_0 \subset \mathcal{B}$.

Definition 15 ([28]) *Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$. We say that μ is diffused if for every $t \in T$, with $\{t\} \in \mathcal{C}$, $\mu(\{t\}) = \{0\}$.*

We recall now the concept of regularity that we have defined and studied, for instance, in [14, 22] for different types of set multifunctions with respect to the Hausdorff topology induced by the Hausdorff pseudo-metric.

Definition 16 *Let $A \in \mathcal{C}$ be an arbitrary set and $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ a set multifunction, with $\mu(\emptyset) = \{0\}$.*

I) A is said to be regular with respect to μ if for every $\varepsilon > 0$, there exists a compact set $K \subseteq A$, $K \in \mathcal{C}$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, $B \subseteq A \backslash K$.

II) μ is said to be regular if every $A \in \mathcal{C}$ is a regular set with respect to μ.

Remark 17 I. Let $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R}_+)$ be the set multifunction induced by a set function $\nu : \mathcal{C} \to \mathbb{R}_+$. Then:

i) μ is diffused if and only if ν is diffused.

ii) μ is regular if and only if ν is regular.

II. If T is a locally compact Hausdorff space, then $\{t\}$ is a compact set, hence $\{t\} \in \mathcal{B}$.

One can easily observe the following:

Remark 18 I. The Lebesgue measure μ is diffused.

Also, the set functions $m : \mathcal{C} \to \mathbb{R}_+$ defined for every $A \in \mathcal{C}$ by:

i) $m(A) = \sqrt{\mu(A)}$,

ii) $m(A) = \frac{\mu(A)}{1+\mu(A)}$

are diffused. The same are the set multifunctions induced by them.

II. If $\mu, \nu : \mathcal{C} \to \mathcal{P}_f(X)$ are diffused set multifunctions, then the same are:

i) the set multifunction $\mu \overset{\bullet}{+} \nu : \mathcal{C} \to \mathcal{P}_f(X)$ defined by $(\mu \overset{\bullet}{+} \nu)(A) = \mu(A) \overset{\bullet}{+} \nu(A)$, for every $A \in \mathcal{C}$;

ii) the set multifunction $\lambda\mu$ defined by $(\lambda\mu)(A) = \lambda\mu(A)$, for every $A \in \mathcal{C}$.

III. Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$.

The following statements are equivalent:

i) μ is diffused;

ii) $|\mu|$ is diffused;

iii) $\overline{\mu}$ is diffused on $\mathcal{C}$.

In the sequel, we establish that, under several assumptions, non-atomicity and diffusion are equivalent:

Theorem 19 *Let $A \in \mathcal{B}$ with $\mu(A) \supsetneq \{0\}$ and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ be a regular fuzzy null-additive set multifunction, with $\mu(\emptyset) = \{0\}$.*

I. If A is an atom of μ, then there is a compact set $K_0 \in \mathcal{B}$ so that $K_0 \subseteq A$ and $\mu(A \backslash K_0) = \{0\}$.

II. A is an atom of μ if and only if the following property holds:

$$(P) \qquad \begin{array}{l} \text{there is an unique } a \in A \\ \text{such that } \mu(A \backslash \{a\}) = \{0\}. \end{array}$$

III. μ is non-atomic if and only if μ is diffused.

Proof. For every $E \in \mathcal{B}$, let $\mathcal{K}_E = \{K \subseteq E; K$ is a compact set and $\mu(E \backslash K) = \{0\}\} \subset \mathcal{B}$.

I. Let $A \in \mathcal{B}$ be an atom of μ.

One can easily check that $\emptyset \notin \mathcal{K}_A$. Indeed, if on the contrary, $\emptyset \in \mathcal{K}_A$, then it results $\mu(A \backslash \emptyset) = \mu(A) = \{0\}$. But this is false, since A is an atom of μ. First, we prove that $\mathcal{K}_A$ is nonvoid.

Suppose by the contrary that $\mathcal{K}_A = \emptyset$. Then for every compact set $K \subseteq A$, we have $\mu(A \backslash K) \supsetneq \{0\}$.

Since A is an atom of μ, then $\mu(K) = \{0\}$.

Thus, by the regularity of μ, we immediately have that $\mu(A) = \{0\}$ and this is a contradiction.

Let us prove now that every set $K \in \mathcal{K}_A$ is an atom of μ. Indeed, if $K \in \mathcal{K}_A$, then $\mu(A \backslash K) = \{0\}$.

Also, for every $B \in \mathcal{B}$, with $B \subseteq K$, since $K \subseteq A$ and A is an atom of μ, we get $\mu(B) = \{0\}$ or $\mu(A \backslash B) = \{0\}$.

If $\mu(A \backslash B) = \{0\}$, then $\{0\} \subseteq \mu(K \backslash B) \subseteq \mu(A \backslash B) = \{0\}$, so $\mu(K \backslash B) = \{0\}$.

Consequently, $K \in \mathcal{K}_A$ is an atom of μ.

We prove now that $K_1 \cap K_2 \in \mathcal{K}_A$, for every $K_1, K_2 \in \mathcal{K}_A$.

Indeed, if $K_1, K_2 \in \mathcal{K}_A$, then $K_1 \cap K_2$ is a compact set of T and $\mu(A \backslash (K_1 \cap K_2)) = \{0\}$.

It recurrently results:

$$(3) \quad \bigcap_{i=1}^{n} K_i \in \mathcal{K}_A, \forall n \in \mathbb{N}^*, K_1, \ldots, K_n \in \mathcal{K}_A.$$

We prove that $K_0 = \underset{K \in \mathcal{K}_A}{\cap} K$ is a nonvoid set.

Suppose, on the contrary, that $K_0 = \emptyset$.

There are $K_1, K_2, ..., K_{n_0} \in \mathcal{K}_A$ so that $\bigcap_{i=1}^{n_0} K_i = \emptyset$. But from (3) it results $\bigcap_{i=1}^{n_0} K_i \in \mathcal{K}_A$, which is a contradiction.

Now, we prove that $K_0 \in \mathcal{K}_A$. Obviously, K_0 is a compact set.

Let be $K \in \mathcal{K}_A$. Then $\mu(A \backslash K) = \{0\}$.

If $K_0 = K$, then $K_0 \in \mathcal{K}_A$.

If $K_0 \neq K$, then $K_0 \subsetneq K$.

Because $\mu(A \backslash K_0) = \mu(K \backslash K_0)$, it remains to demonstrate that $\mu(K \backslash K_0) = \{0\}$.

Suppose that, on the contrary, $\mu(K \backslash K_0) \supsetneq \{0\}$.

Since K is an atom of μ, by Remark 7-II, it follows that $K \backslash K_0$ is an atom of μ.

Because A is an atom of μ and $\mu(K \backslash K_0) \supsetneq \{0\}$, then $\mu(A \backslash (K \backslash K_0)) = \{0\}$.

Let us consider $C \in \mathcal{K}_{K \backslash K_0}$.

Then $\mu((K \backslash K_0) \backslash C) = \{0\}$ and, since $\mu(A \backslash (K \backslash K_0)) = \{0\}$, by the null-additivity of μ we get that $\mu(A \backslash C) = \{0\}$. This implies $C \in \mathcal{K}_A$.

Therefore, $K_0 \subseteq C$, but $C \subsetneq K \backslash K_0$, which is a contradiction. Therefore, $\mu(K \backslash K_0) = \{0\}$.

So, indeed, if $A \in \mathcal{B}$ is an atom of μ, we find a compact set $K_0 \in \mathcal{B}$ so that $K_0 \subseteq A$ and $\mu(A \backslash K_0) = \{0\}$.

II. Let $A \in \mathcal{B}$ be an atom of μ.

a) The *existence part*:

We show that the set K_0 from the proof of i) is a singleton $\{a\}$.

Suppose that, on the contrary, there exist $a, b \in A$, with $a \neq b$ and $K_0 \supseteq \{a, b\}$.

Since T is a locally compact Hausdorff space, there exists an open neighbourhood V of a so that $b \notin \overline{V}$.

Obviously, $K_0 = (K_0 \backslash V) \cup (K_0 \cap \overline{V})$ and $K_0 \backslash V$, $K_0 \cap \overline{V}$ are nonvoid compact subsets of A.

We prove that $K_0 \backslash V \in \mathcal{K}_A$ or $K_0 \cap \overline{V} \in \mathcal{K}_A$.

Indeed, if $K_0 \backslash V \notin \mathcal{K}_A$ and $K_0 \cap \overline{V} \notin \mathcal{K}_A$, then $\mu(A \backslash (K_0 \backslash V)) \supsetneq \{0\}$ and $\mu(A \backslash (K_0 \cap \overline{V})) \supsetneq \{0\}$.

Since A is an atom of μ, then $\mu(K_0 \backslash V) = \{0\}$ and $\mu(K_0 \cap \overline{V}) = \{0\}$.

Consequently, according to the null-additivity of μ, we get $\mu(K_0) = \{0\}$.

This implies $\{0\} \subsetneq \mu(A) = \{0\}$, which is a contradiction.

Therefore, $K_0 \backslash V \in \mathcal{K}_A$ or $K_0 \cap \overline{V} \in \mathcal{K}_A$.

Because $K_0 \subseteq K$, for every $K \in \mathcal{K}_A$, we get that $K_0 \subseteq K_0 \backslash V$ or $K_0 \subseteq K_0 \cap \overline{V}$, which is impossible.

So, $\exists a \in A$ so that $\mu(A \backslash \{a\}) = \{0\}$.

b) The *uniqueness part*:

Suppose that, on the contrary, there are $a, b \in A$, with $a \neq b$, $\mu(A \backslash \{a\}) = \{0\}$ and $\mu(A \backslash \{b\}) = \{0\}$.

Because $\{b\} \subseteq A \backslash \{a\}$ and $\mu(A \backslash \{a\}) = \{0\}$, by the fuzziness of μ we get that $\mu(\{b\}) = \{0\}$.

Then, according to the null-additivity of μ, by $\mu(A \backslash \{b\}) = \{0\}$ and $\mu(\{b\}) = \{0\}$, we have $\mu(A) = \{0\}$, which is a contradiction.

Now, consider $A \in \mathcal{B}$, with $\mu(A) \supsetneq \{0\}$ having property (P) and let $B \in \mathcal{B}$, with $B \subseteq A$.

If $a \notin B$, then $B \subseteq A \backslash \{a\}$. Because $\mu(A \backslash \{a\}) = \{0\}$ and μ is monotone, we have $\mu(B) = \{0\}$. If $a \in B$, then $A \backslash B \subseteq A \backslash \{a\}$, so $\mu(A \backslash B) = \{0\}$. Consequently, A is an atom of μ.

III. Let μ be diffused and suppose, on the contrary, there is an atom $A_0 \in \mathcal{C}$ of μ.

By ii), there is an unique $a \in A_0$ so that $\mu(A_0 \backslash \{a\}) = \{0\}$.

On the other hand, by the diffusion of μ, $\mu(\{a\}) = \{0\}$.

The null-additivity of μ implies that $\mu(A_0) = \{0\}$, which is a contradiction since $A_0 \in \mathcal{C}$ is an atom of μ. Consequently, μ is non-atomic.

Now, let μ be non-atomic and suppose, on the contrary, that μ is not diffused, so there is $t_0 \in T$ so that $\mu(\{t_0\}) \supsetneq \{0\}$.

Because μ is non-atomic, there is a set $B \in \mathcal{B}$ such that $B \subsetneq \{t_0\}$, $\mu(B) \supsetneq \{0\}$ and $\mu(\{t_0\} \backslash B) \supsetneq \{0\}$. Consequently, $B = \emptyset$ or $B = \{t_0\}$, which is false. The proof is thus finished. □

4 Semi-convexity

In this section we shall present some relationships among semi-convexity, Darboux property, diffusion and non-atomicity of fuzzy set multifunctions.

Definition 20 ([28]) *Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$. We say that μ is semi-convex if for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$, there exists $B \in \mathcal{C}$ so that*

$$B \subseteq A \text{ and } \mu(B) = \frac{1}{2}\mu(A).$$

Example 21 Let $\nu : \mathcal{C} \to [0, +\infty)$ be a set function, so that $\nu(\emptyset) = 0$ and μ the set multifunction defined by $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$. Then μ is semi-convex if and only if ν is semi-convex.

Theorem 22 *I. If the set multifunction $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ has the Darboux property, then it is semi-convex.*

II. If $\mathcal{C}$ is a σ-ring and $\mu : \mathcal{C} \to \mathcal{P}_{bfc}(X)$ is a semi-convex increasing convergent fuzzy multimeasure, then μ has the Darboux property.

Proof. I. It immediately follows.

II. Every $p \in (0, 1)$ has an expansion $p = \sum\limits_{n=1}^{\infty} \frac{a_n}{2^n}$, where $a_n \in \{0, 1\}$, for every $n \in \mathbb{N}^*$.

By the semi-convexity of μ, for every $A \in \mathcal{C}$, there is $B_1 \in \mathcal{C}$ so that $B_1 \subseteq A$ and $\mu(B_1) = \frac{a_1}{2}\mu(A)$.

Analogously, there is $B_2 \in \mathcal{C}$ so that $B_2 \subseteq A \backslash B_1$ and $\mu(B_2) = \frac{a_2}{2^2}\mu(A)$ and so on.

Consider

$$B = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} (\bigcup_{k=1}^{n} B_k).$$

Then $B \in \mathcal{C}$ and

$$\mu(B) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{2^k}\mu(A)$$

(with respect to h).

From [24] it is known that if $M_n \overset{h}{\to} M$, M is bounded and $\alpha_n \to \alpha$ (where $\alpha_n, \alpha \in \mathbb{R}$), then $\alpha_n M_n \overset{h}{\to} \alpha M$. Consequently, $\mu(B) = p\mu(A)$, as claimed. □

Theorem 23 *If the fuzzy set multifunction $\mu : \mathcal{C} \to \mathcal{P}_b(X)$ is null-additive and semi-convex, then μ is non-atomic.*

Proof. Suppose, on the contrary, there is an atom $A_0 \in \mathcal{C}$ of μ. Since μ is semi-convex, there exists $B_0 \in \mathcal{C}$ so that $B_0 \subsetneq A_0$ and $\mu(B_0) = \frac{1}{2}\mu(A_0)$.

Because $A_0 \in \mathcal{C}$ is an atom of μ and $B_0 \in \mathcal{C}$, $B_0 \subseteq A_0$, then:

(i) $\mu(B_0) = \{0\}$. In this case, we get $\mu(A_0) = \{0\}$, which is a contradiction.

(ii) $\mu(A_0 \backslash B_0) = \{0\}$. Then, by the null-additivity of μ, we have $\mu(A_0) = \mu(B_0)$, so $\mu(A_0) = \frac{1}{2}\mu(A_0)$ and this implies $|\mu(A_0)| = \frac{1}{2}|\mu(A_0)|$.

Because $\mu : \mathcal{C} \to \mathcal{P}_{bf}(X)$, then $\mu(A_0)$ is a bounded set, so

$$|\mu(A_0)| = \sup_{x\in\mu(A_0)} \|x\| < +\infty.$$

Consequently, $|\mu(A_0)| = 0$, so $\mu(A_0) = \{0\}$, which is again a contradiction since A_0 is an atom of μ. Then μ is non-atomic. □

Theorem 24 *Let $\mathcal{C} = \mathcal{B}_0$ (or $\mathcal{B}$). If $\mu : \mathcal{C} \to \mathcal{P}_{bfc}(X)$ is a decreasing convergent semi-convex fuzzy multimeasure, then for every $t \in T$, there exists $A_t \in \mathcal{C}$ so that $t \in A_t$ and $\mu(A_t) = \{0\}$.*

Proof. Let $t \in T$ be arbitrarily, but fixed. We first prove that there exists $A_t \in \mathcal{C}$ so that $t \in A_t$.

a) If $\mathcal{C} = \mathcal{B}$, then $A_t = \{t\}$.

b) If $\mathcal{C} = \mathcal{B}_0$, since T is a Hausdorff locally compact space, then for t there exists a compact neighghbourhood V_t, for which there is a relatively compact open set D_t so that $V_t \subseteq D_t$. By [13] we know that there exists $A_t \in \mathcal{B}_0$ so that $V_t \subseteq A_t \subseteq D_t$. Evidently, $t \in A_t$.

Now, let $A = A_t \in \mathcal{C}$ be so that $t \in A$.

If $\mu(A) = \{0\}$, the proof is finished.

If $\mu(A) \supsetneq \{0\}$, since μ is semi-convex, there exists a set $A_1 \in \mathcal{C}$ so that $A_1 \subseteq A$ and $\mu(A_1) = \frac{1}{2}\mu(A)$. Then

$$\mu(A) = \frac{1}{2}\mu(A) \overset{\bullet}{+} \mu(A \backslash A_1) = \frac{1}{2}\mu(A) \overset{\bullet}{+} \frac{1}{2}\mu(A).$$

Using the cancelation law in $\mathcal{P}_{bfc}(X)$, we get that $\mu(A \backslash A_1) = \frac{1}{2}\mu(A)$.

Let $B_1 = A_1$ or $B_1 = A \backslash A_1$ be so that $t \in B_1$. Obviously, $\mu(B_1) = \frac{1}{2}\mu(A)$.

We construct by induction a decreasing sequence of sets $(B_n)_n \subset \mathcal{C}$ such that $\mu(B_n) = \frac{1}{2^n}\mu(A)$, $B_n \subseteq A$ and $t \in B_n$, for every $n \in \mathbb{N}^*$.

We suppose that we have already obtained $B_1, B_2, ..., B_n$. Let us obtain B_{n+1} :

By the Darboux property, there exists $A_{n+1} \in \mathcal{C}$ so that $A_{n+1} \subseteq B_n$ and $\mu(A_{n+1}) = \frac{1}{2}\mu(B_n) = ... = \frac{1}{2^{n+1}}\mu(A)$. Then $\mu(B_n \backslash A_{n+1}) = \frac{1}{2}\mu(B_n) = \frac{1}{2^{n+1}}\mu(A)$.

Let $B_{n+1} = A_{n+1}$ or $B_{n+1} = B_n \backslash A_{n+1}$ be so that $t \in B_{n+1}$. Then $\mu(B_{n+1}) = \frac{1}{2^{n+1}}\mu(A)$.

We consider $B = \bigcap\limits_{n=1}^{\infty} B_n$. Obviously, $t \in B$, $B \in \mathcal{C}$ and, for every $n \in \mathbb{N}^*$, we have

$$|\mu(B)| \le h(\mu(B_n), \mu(B)) + |\mu(B_n)| =$$

$$= h(\mu(B_n), \mu(B)) + \frac{1}{2^n}|\mu(A)|.$$

Because μ is $\mathcal{P}_{bfc}(X)$-valued, then $\mu(A)$ is a bounded set, that is, there exists $M > 0$ so that $|\mu(A)| \leq M$. Also, because μ is decreasing convergent, $\lim_{n\to\infty} h(\mu(B_n), \mu(B)) = 0$. Therefore, $|\mu(B)| = 0$, hence $\mu(B) = \{0\}$.

So, for every $t \in T$, there exists $B \in \mathcal{C}$ so that $t \in B$ and $\mu(B) = \{0\}$, as claimed. □

Theorem 25 *Let $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ be a diffused regular fuzzy multimeasure. Then $\mu_{/\mathcal{B}_0}$ satisfies the condition:*

for every $t \in T$, there exists $A_t \in \mathcal{B}_0$ so that $t \in A_t$ and $\mu(A_t) = \{0\}$.

Proof. Let $t \in T$. By the regularity of μ, for every $n \in \mathbb{N}^*$, there is an open set $\widetilde{D}_n \supseteq \{t\}$, $\widetilde{D}_n \in \mathcal{B}$ such that $|\mu(\widetilde{D}_n \backslash \{t\})| < \frac{1}{n}$. Since for every $n \in \mathbb{N}^*$, $\{t\} \subseteq \widetilde{D}_n$, according to Gavriluţ [13], there exists an open set $D'_n \in \mathcal{B}_0$ so that $\{t\} \subseteq D'_n \subseteq \widetilde{D}_n$. Since $|\mu(\widetilde{D}_n \backslash \{t\})| < \frac{1}{n}$, we get $|\mu(D'_n \backslash \{t\})| < \frac{1}{n}$, for every $n \in \mathbb{N}^*$. So, since μ is diffused, we have:

$$\begin{aligned}
&|\mu(D'_n)| = h(\mu(D'_n), \{0\}) = \\
&= h(\mu(D'_n), \mu(\{t\})) = \\
&= h(\mu(D'_n \backslash \{t\}) \overset{\bullet}{+} \mu(\{t\}), \mu(\{t\})) \leq \\
&\leq |\mu(D'_n \backslash \{t\})| < \frac{1}{n},
\end{aligned}$$

for every $n \in \mathbb{N}^*$.

Denote $D_n = \bigcap_{i=1}^{n} D'_i$, for every $n \in \mathbb{N}^*$ and let $A_t = \bigcap_{n=1}^{\infty} D'_n$. Then $D_n \in \mathcal{B}_0$, for every $n \in \mathbb{N}^*$, $A_t \in \mathcal{B}_0$, $D_n \searrow A_t$ and $|\mu(D_n)| \leq |\mu(D'_n)| < \frac{1}{n}$, for every $n \in \mathbb{N}^*$. Also, $t \in A_t$. It only remains to prove that $\mu(A_t) = \{0\}$. Indeed, since μ is continuous from above, it results

$$\mu(A_t) = \lim_{n\to\infty} \mu(D_n)$$

with respect to h.

On the other hand, for every $n \in \mathbb{N}^*$, we have:

$$\begin{aligned}
|\mu(A_t)| &\leq h(\mu(A_t), \mu(D_n)) + |\mu(D_n)| < \\
&< h(\mu(A_t), \mu(D_n)) + \frac{1}{n}.
\end{aligned}$$

Consequently, $|\mu(A_t)| = 0$, so $\mu(A_t) = \{0\}$, as claimed. □

Corollary 26 *Let $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ be a regular fuzzy multimeasure. Then μ is diffused on $\mathcal{B}$ if and only if for every $t \in T$, there exists $A_t \in \mathcal{B}_0$ so that $t \in A_t$ and $\mu(A_t) = \{0\}$.*

Concluding remarks

In this paper, results concerning diffusion and semi-convexity of fuzzy set multifunctions are discussed in the context of different notions such as non-atomicity, regularity, Darboux property, decreasing convergence.

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