Bayes estimators of Modified-Weibull distribution parameters using Lindley's approximation

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Abstract: - For Modified-Weibull distribution we have obtained the Bayes Estimators of scale and shape parameters using Lindley's approximation (L-approximation) under various loss functions. The proposed estimators have been compared with the corresponding MLE for their risks based on corresponding simulated samples.

Key-Words: - Bayesian estimation, Lindley's approximation, Maximum likelihood estimates, Modified Weibull distribution, Monte Carlo simulation

1 Introduction

The Weibull distribution is one of the most popular widely used models of failure time in life testing and reliability theory. The Weibull distribution has been shown to be useful for modeling and analysis of life time data in medical, biological and engineering sciences.

Applications of the Weibull distribution in various fields are given in Zaharim et al [28], Gotoh et al [7], Shamilov et al [21], Vicen-Bueno et al [27], Niola et al [15], Green et al.[8].

A great deal of research has been done on estimating the parameters of the Weibull distribution using both classical and Bayesian techniques, and a very good summary of this work can be found in Johnson et al. [12]. Recently, Hossain and Zimmer in [9] have discussed some comparisons of estimation methods for Weibull parameters using complete and censored samples.

The three-parameter Modified-Weibull distribution is defined by the distribution function:

$$f(x) = \alpha x^{\beta-1}(\beta + \lambda x)e^{\lambda x - \alpha x^{\beta} e^{\lambda x}} \quad (1)$$

$$x \geq 0, \alpha, \beta, \lambda > 0$$

and cumulative distribution function

$$F(x) = 1 - e^{-\alpha x^{\beta} e^{\lambda x}}, \; x \geq 0, \alpha, \beta, \lambda > 0 \quad (2)$$

Here α is the scale parameter, and β and λ are the shape parameters.

It is remarkable that most of the Bayesian inference procedures have been developed with the usual squared-error loss function, which is symmetrical and associates equal importance to the losses due to overestimation and underestimation of equal magnitude. However, such a restriction may be impractical in most situations of practical importance. For example, in the estimation of reliability and failure rate functions, an overestimation is usually much more serious than an underestimation. In this case, the use of symmetrical loss function might be inappropriate as also emphasized by Basu and Ebrahimi in [2].

A useful asymmetric loss known as the LINEX loss function (linear-exponential) was introduced by Varian in [26] and has been widely used by several authors, Zellner in [29], Basu and Ebrahimi in [2], Calabria and Pulcini in [4], Soliman in [23], Singh et al. in [22] and Ahmadi et al. in [1]. This function rises approximately exponentially on one side of zero and approximately linearly on the other side. It may also be noted here that the squared-error loss function can be obtained as a particular member of the LINEX loss function for a specific choice of the loss function parameter.

Despite the flexibility and popularity of the Linex loss function for the location parameter estimation, it appears to be unsuitable for the scale parameter and other quantities (c.f. Basu and Ebrahimi in [2], Parsian and Sanjari Farsipour in [17]). Keeping these points in mind, Basu and Ebrahimi [2] defined a modified Linex loss. A suitable alternative to the modified Linex loss is the general entropy loss proposed by Calabria and Pulcini in [4].

However, Bayesian estimation under the LINEX loss function is not frequently discussed, perhaps, because the estimators under asymmetric loss function involve integral expressions, which are not analytically solvable. Therefore, one has to use the numerical quadrature techniques or certain approximation methods for the solutions. Lindley's approximation technique is one of the methods suitable for solving such problems. Thus, our aim in this paper is to propose the Bayes estimators of the parameters of Modified-Weibull distribution under the squared error loss, LINEX loss and general entropy loss functions using Lindley's approximation technique. In Sections 2 and 3, we discuss estimation of parameters. In Section 4 numerical results are presented, and Section 5 contains the conclusion.

2 Maximum likelihood estimation of the parameters

For a random sample $x = (x_1, x_2, ..., x_n)$ of size n form (1) the likelihood function is

$$L(\alpha, \beta, \lambda x) = \alpha^n e^{\lambda \sum_{i=1}^{n} x_i - \alpha \sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i}} \cdot \prod_{i=1}^{n} \left[x_i^{\beta-1} (\beta + \lambda x_i) \right] \quad (3)$$

and taking the logarithm we get

$$l(\alpha, \beta, \lambda x) = n \ln \alpha + \lambda \sum_{i=1}^{n} x_i - \alpha \sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i} + (\beta - 1) \sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \ln(\beta + \lambda x_i) \quad (4)$$

The maximum likelihood estimates of parameters of the Modified-Weibull distribution are given as solutions of equations

$$\begin{cases} \dfrac{n}{\alpha} - \sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i} = 0 \\ - \alpha \sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i} \ln x_i + \sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \dfrac{1}{\beta + \lambda x_i} = 0 \\ - \alpha \sum_{i=1}^{n} x_i^{\beta+1} e^{\lambda x_i} + \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \dfrac{x_i}{\beta + \lambda x_i} = 0 \end{cases} \quad (5)$$

which maybe solve using a iteration scheme. We propose here to use a bisection or Newton-Raphson method for solving the above-mentioned normal equations.

3 Bayesian estimation of the parameters

In Bayesian estimation, we consider three types of loss functions. The first is the squared error loss function (quadratic loss) which is classified as a symmetric function and associates equal importance to the losses for overestimation and underestimation of equal magnitude. The second is the LINEX (linear-exponential) loss function which is asymmetric, was introduced by Varian in [26]. These loss functions were widely used by several authors; among of them Rojo in [20], Basu and Ebrahimi in [2], Pandey in [16], Soliman in [23], [24], Soliman et al in [25] and Nassar and Eissa in [14]. This function rises approximately exponentially on one side of zero and approximately linearly on the other side. The third is the generalization of the Entropy loss used by several authors (Dey et al. in [5] and Dey and Liu in [6]) where the shape parameter c is taken equal to 1. This more general version allows different shapes of the loss function.

The squared error loss (SEL) function is as follows

$$L_{BS}(\phi^*, \phi) \propto (\phi^* - \phi)^2 \quad (6)$$

Under the assumption that the minimal loss occurs at $\phi^* = \phi$, the LINEX loss function (LINEX) for can be expressed as

$$L_{BL}(\Delta) \propto e^{c\Delta} - c\Delta - 1, c \neq 1 \quad (7)$$

where $\Delta = (\phi^* - \phi)$ is an estimate of ϕ. The sign and magnitude of the shape parameter c represents the direction and degree of symmetry, respectively. (If $c > 0$, the overestimation is more serious than underestimation, and vice-versa.) For c close to zero, the LINEX loss is approximately SEL and therefore almost symmetric.

The posterior expectation of the LINEX loss function (7) is

$$E_\phi(L_{BL}(\phi^* - \phi)) \propto e^{c\phi^*} E_\phi[e^{-c\phi}] - \\ - c(\phi^* - E_\phi[\phi]) - 1 \quad (8)$$

where $E_\phi(\cdot)$ denotes the posterior expectation with respect to the posterior density of ϕ. By a result of Zellner in [29], the (unique) Bayes estimator of ϕ, denoted by ϕ^*_{BL} under the LINEX loss function is the value ϕ^* which minimizes (8). It is

$$\phi^*_{BL} = -\frac{1}{c}\ln\left\{E_\phi[e^{-c\phi}]\right\} \quad (9)$$

provided that the expectation $E_\phi[e^{-c\phi}]$ exists and is finite. The problem of choosing the value of the parameter c is discussed in Calabria and Pulcini in [3].

The modified Linex loss i.e the General Entropy loss (GEL) is defined as:

$$L_{BE}(\phi^*, \phi) \propto \left(\frac{\phi^*}{\phi}\right)^c - c\log\left(\frac{\phi^*}{\phi}\right) - 1 \quad (10)$$

where ϕ^* is an estimate of parameter ϕ. It may be noted that when $c > 0$, a positive error causes more serious consequences than a negative error. On the other hand, when $c < 0$, a negative error causes more serious consequences than a positive error.

The Bayes estimate ϕ^*_{BE} relative to the general entropy loss is given as

$$\phi^*_{BE} = [E\{\phi^{-c}\}]^{-\frac{1}{c}} \quad (11)$$

provided that $E\{\phi^{-c}\}$ exists and is finite. It can be shown that, when $c = 1$, the Bayes estimate (11) coincides with the Bayes estimate under the weighted squared-error loss function. Similarly, when $c = -1$ the Bayes estimate (11) coincides with the Bayes estimate under squared error loss function.

For Bayesian estimation, we need prior distribution for the parameters α, β and λ. It may be noted here that when the shape parameter β is equal to one, the Modified-Weibull distribution reduces to Modified-Exponential distribution (see Preda et al. in [19]) and when the shape parameter λ is equal to one, the Modified-Weibull distribution reduces to Weibull distribution.

Hence, the gamma prior may be taken as a prior distribution for the scale parameter of the Modified-Weibull distribution. It is needless to mention that under the above-mentioned situation, a prior is a conjugate prior. On the other hand, if all the parameters are unknown, a joint conjugate prior for the parameters does not exist. In such a situation, there are a number of ways to choose the priors.

For all the parameters we consider the piecewise independent priors, namely a non-informative prior for the shape parameters and a natural conjugate prior for the scale parameter (under the assumption that shape parameter is known). Thus the proposed priors for parameters α, β and λ may be taken as

$$g_1(\alpha) = \frac{b^a \alpha^{a-1} e^{-b\alpha}}{\Gamma(a)}, \alpha > 0, a, b > 0 \quad (12)$$

$$g_2(\beta) = \frac{1}{\beta}, \beta > 0 \quad (13)$$

and

$$g_3(\lambda) = \frac{1}{\lambda}, \lambda > 0 \quad (14)$$

respectively where $\Gamma(\cdot)$ is the gamma function. Thus the joint prior distribution for λ, β and α is

$$g(\alpha, \beta, \lambda) = \frac{b^a \alpha^{a-1} e^{-b\alpha}}{\beta\lambda\Gamma(a)} \quad (15)$$

$$\alpha, \beta, \lambda > 0, a, b > 0$$

Substituting $L(\alpha, \beta, \lambda)$ and $g(\alpha, \beta, \lambda)$ from (3) and (15) respectively we get the correspond joint posterior $P(\alpha, \beta, \lambda)$ as

$$P(\alpha, \beta, \lambda \mid x) = K \frac{\alpha^{n+a-1} e^{\lambda\sum_{i=1}^n x_i - \alpha\left(b + \sum_{i=1}^n x_i^\beta e^{\lambda x_i}\right)}}{\beta\lambda} \cdot \\ \cdot \prod_{i=1}^n \left[x_i^{\beta-1}(\beta + \lambda x_i)\right]$$

where

$$K^{-1} = \int_0^\infty\int_0^\infty\int_0^\infty \frac{\alpha^{n+a-1} e^{\lambda\sum_{i=1}^n x_i - \alpha\left(b + \sum_{i=1}^n x_i^\beta e^{\lambda x_i}\right)}}{\beta\lambda} \cdot \\ \cdot \prod_{i=1}^n \left[x_i^{\beta-1}(\beta + \lambda x_i)\right] d\lambda d\beta d\alpha$$

It may be noted here that the posterior distribution of (α, β, λ) takes a ratio form that involves an integration in the denominator and cannot be reduced to a closed form. Hence, the evaluation of the posterior expectation for obtaining the Bayes estimator of α, β and λ will be tedious. Among the various methods suggested to approximate the ratio of integrals of the above form, perhaps the simplest one is Lindley's in [13] approximation method, which approaches the ratio of the integrals as a whole and produces a single numerical result.

Thus, we propose the use of Lindley's in [13] approximation for obtaining the Bayes estimator of α, β and λ. Many authors have used this approximation for obtaining the Bayes estimators for some lifetime distributions; see among others, Howlader and Hossain in [10] and Jaheen in [11].

In this paper we calculate $E(\theta_i \mid x)$ and $E(\theta_i^2 \mid x)$ in order to find the posterior variance estimates given by

$$Var(\theta_i \mid x) = E(\theta_i^2 \mid x) - (E(\theta_i \mid x))^2$$

$i = 1,2,3$, where $\theta_1 = \alpha$, $\theta_2 = \beta$, $\theta_3 = \lambda$.

If n is sufficiently large, according to Lindley in [13], any ratio of the integral of the form

$$I(x) = E[u(\theta_1, \theta_2, \theta_3)] = \frac{\int_{(\theta_1,\theta_2,\theta_3)} u(\theta_1, \theta_2, \theta_3) e^{L(\theta_1,\theta_2,\theta_3)+G(\theta_1,\theta_2,\theta_3)} d(\theta_1, \theta_2, \theta_3)}{\int_{(\theta_1,\theta_2,\theta_3)} e^{L(\theta_1,\theta_2,\theta_3)+G(\theta_1,\theta_2,\theta_3)} d(\theta_1, \theta_2, \theta_3)}$$

where

$u(\theta) = u(\theta_1, \theta_2, \theta_3)$ is a function of θ_1, θ_2 or θ_3 only

$L(\theta_1, \theta_2, \theta_3)$ is log of likelihood

$G(\theta_1, \theta_2, \theta_3)$ is log joint prior of θ_1, θ_2 and θ_3,

can be evaluated as

$$I(x) = u(\hat\theta_1, \hat\theta_2, \hat\theta_3) + (u_1a_1 + u_2a_2 + u_3a_3 + a_4 + a_5) + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})]$$

where

$\hat\theta_1$, $\hat\theta_2$ and $\hat\theta_3$ are the MLE of θ_1, θ_2 and θ_3 respectively

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3},\ i = 1,2,3$$

$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23}$$

$$a_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33})$$

$$A = \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{23}L_{231} + \sigma_{22}L_{221} + \sigma_{33}L_{331}$$

$$B = \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{23}L_{232} + \sigma_{22}L_{222} + \sigma_{33}L_{332}$$

$$C = \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{22}L_{223} + \sigma_{33}L_{333}$$

and subscripts 1,2,3 on the rigth-hand sides refer to θ_1, θ_2, θ_3 respectively and

$$\rho_i = \frac{\partial\rho}{\partial\theta_i},\ u_i = \frac{\partial u(\theta_1, \theta_2, \theta_3)}{\partial\theta_i}, i = 1,2,3,$$

$$u_{ij} = \frac{\partial^2 u(\theta_1, \theta_2, \theta_3)}{\partial\theta_i\partial\theta_j},\ i, j = 1,2,3$$

$$L_{ij} = \frac{\partial^2 L(\theta_1, \theta_2, \theta_3)}{\partial\theta_i\partial\theta_j},\ i, j = 1,2,3$$

$$L_{ijk} = \frac{\partial^3 L(\theta_1, \theta_2, \theta_3)}{\partial\theta_i\partial\theta_j\partial\theta_k},\ i, j, k = 1,2,3$$

and σ_{ij} is the (i, j)-th element of the inverse of the matrix $\{L_{ij}\}$, all evaluated at the MLE of parameters.

For the prior distribution (14) we have

$$\rho = \ln g(\alpha, \beta, \lambda) = a \ln b + (a-1)\ln\alpha - b\alpha - \ln\beta - \ln\lambda - \ln\Gamma(a)$$

and then we get

$$\rho_1 = \frac{a-1}{\alpha} - b,\ \rho_2 = -\frac{1}{\beta},\ \rho_3 = -\frac{1}{\lambda}$$

Also, the values of L_{ij} can be obtained as follows for $i, j = 1,2,3$

$$L_{11} = -\frac{n}{\alpha^2},\ L_{12} = -\sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i} \ln x_i = L_{21}$$

$$L_{13} = -\sum_{i=1}^{n} x_i^{\beta+1} e^{\lambda x_i} = L_{31}$$

$$L_{22} = -\sum_{i=1}^{n} \frac{1}{(\beta + \lambda x_i)^2} - \alpha\sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i} \ln^2 x_i$$

$$L_{23} = -\sum_{i=1}^{n} \frac{x_i}{(\beta + \lambda x_i)^2} - \alpha\sum_{i=1}^{n} x_i^{\beta+1} e^{\lambda x_i} \ln x_i = L_{32}$$

$$L_{33} = -\sum_{i=1}^{n} \frac{x_i^2}{(\beta + \lambda x_i)^2} - \alpha\sum_{i=1}^{n} x_i^{\beta+2} e^{\lambda x_i}$$

and the values of L_{ijk} for $i, j, k = 1,2,3$

$$L_{111} = \frac{2n}{\alpha^3},$$

$$L_{112} = L_{113} = L_{121} = L_{131} = L_{211} = L_{311} = 0$$

$$L_{122} = -\sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i} \ln^2 x_i = L_{212} = L_{221},$$

$$L_{123} = -\sum_{i=1}^{n} x_i^{\beta+1} e^{\lambda x_i} \ln x_i = L_{132} = L_{213} = L_{231} = L_{312} = L_{321}$$

$$L_{133} = -\sum_{i=1}^{n} x_i^{\beta+2} e^{\lambda x_i} = L_{313} = L_{331}$$

$$L_{222} = 2\sum_{i=1}^{n} \frac{1}{(\beta + \lambda x_i)^3} - \alpha\sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i} \ln^3 x_i$$

$$L_{223} = 2\sum_{i=1}^{n} \frac{x_i}{(\beta + \lambda x_i)^3} - \alpha \sum_{i=1}^{n} x_i^{\beta+1} e^{\lambda x_i} \ln^2 x_i =$$
$$= L_{232} = L_{332}$$
$$L_{233} = 2\sum_{i=1}^{n} \frac{x_i^2}{(\beta + \lambda x_i)^3} - \alpha \sum_{i=1}^{n} x_i^{\beta+2} e^{\lambda x_i} \ln x_i =$$
$$= L_{323} = L_{332}$$
$$L_{333} = 2\sum_{i=1}^{n} \frac{x_i^3}{(\beta + \lambda x_i)^3} - \alpha \sum_{i=1}^{n} x_i^{\beta+3} e^{\lambda x_i}$$

Now we can obtain the values of the Bayes estimates of various parameters.

a) Case of the squared error loss function

i) If $u(\hat\alpha, \hat\beta, \hat\lambda) = \hat\alpha$ then

$$\hat\alpha_{BS} = \hat\alpha + \frac{a-1-b\hat\alpha}{\hat\alpha}\sigma_{11} - \frac{1}{\hat\beta}\sigma_{12} - \frac{1}{\hat\lambda}\sigma_{13} + + \frac{1}{2}(A\sigma_{11} + B\sigma_{21} + C\sigma_{31})$$

ii) If $u(\hat\alpha, \hat\beta, \hat\lambda) = \hat\beta$ then

$$\hat\beta_{BS} = \hat\beta + \frac{a-1-b\hat\alpha}{\hat\alpha}\sigma_{21} - \frac{1}{\hat\beta}\sigma_{22} - \frac{1}{\hat\lambda}\sigma_{23} + + \frac{1}{2}(A\sigma_{12} + B\sigma_{22} + C\sigma_{32})$$

iii) If $u(\hat\alpha, \hat\beta, \hat\lambda) = \hat\lambda$ then

$$\hat\lambda_{BS} = \hat\lambda + \frac{a-1-b\hat\alpha}{\hat\alpha}\sigma_{31} - \frac{1}{\hat\beta}\sigma_{32} - \frac{1}{\hat\lambda}\sigma_{33} + + \frac{1}{2}(A\sigma_{13} + B\sigma_{23} + C\sigma_{33})$$

b) Case of the Linex loss function

i) If $u(\hat\alpha, \hat\beta, \hat\lambda) = e^{-c\hat\alpha}$ then

$$\hat\alpha_{BL} = \hat\alpha + \log\left(1 - c\left(\frac{a-1-b\hat\alpha}{\hat\alpha}\sigma_{11} - \frac{1}{\hat\beta}\sigma_{12} - - \frac{1}{\hat\lambda}\sigma_{13} - \frac{c}{2}\sigma_{11} + \frac{1}{2}(A\sigma_{11} + B\sigma_{21} + C\sigma_{31})\right)\right)$$

ii) If $u(\hat\alpha, \hat\beta, \hat\lambda) = e^{-c\hat\beta}$ then

$$\hat\beta_{BL} = \hat\beta + \log\left(1 - c\left(\frac{a-1-b\hat\alpha}{\hat\alpha}\sigma_{21} - \frac{1}{\hat\beta}\sigma_{22} - - \frac{1}{\hat\lambda}\sigma_{23} - \frac{c}{2}\sigma_{22} + \frac{1}{2}(A\sigma_{12} + B\sigma_{22} + C\sigma_{32})\right)\right)$$

iii) If $u(\hat\alpha, \hat\beta, \hat\lambda) = e^{-c\hat\lambda}$ then

$$\hat\lambda_{BL} = \hat\lambda + \log\left(1 - c\left(\frac{a-1-b\hat\alpha}{\hat\alpha}\sigma_{31} - \frac{1}{\hat\beta}\sigma_{32} - - \frac{1}{\hat\lambda}\sigma_{33} - \frac{c}{2}\sigma_{33} + \frac{1}{2}(A\sigma_{13} + B\sigma_{23} + C\sigma_{33})\right)\right)$$

c) Case of the general entropy loss function

i) If $u(\hat\alpha, \hat\beta, \hat\lambda) = \hat\alpha^{-c}$ then

$$\hat\alpha_{BE} = \left[\hat\alpha^{-c}\left[1 - \frac{c}{\hat\alpha}\left(\frac{a-1-b\hat\alpha}{\hat\alpha}\sigma_{11} - \frac{1}{\hat\beta}\sigma_{12} - \frac{1}{\hat\lambda}\sigma_{13} + - \frac{c+1}{2\hat\alpha}\sigma_{11} + \frac{1}{2}(A\sigma_{11} + B\sigma_{21} + C\sigma_{31})\right)\right]\right]^{-\frac{1}{c}}$$

ii) If $u(\hat\alpha, \hat\beta, \hat\lambda) = \hat\beta^{-c}$ then

$$\hat\beta_{BE} = \left[\hat\beta^{-c}\left[1 - \frac{c}{\hat\beta}\left(\frac{a-1-b\hat\alpha}{\hat\alpha}\sigma_{21} - \frac{1}{\hat\beta}\sigma_{22} - \frac{1}{\hat\lambda}\sigma_{23} + - \frac{c+1}{2\hat\beta}\sigma_{22} + \frac{1}{2}(A\sigma_{12} + B\sigma_{22} + C\sigma_{32})\right)\right]\right]^{-\frac{1}{c}}$$

iii) If $u(\hat\alpha, \hat\beta, \hat\lambda) = \hat\lambda^{-c}$ then

$$\hat\lambda_{BE} = \left[\hat\lambda^{-c}\left[1 - \frac{c}{\hat\lambda}\left(\frac{a-1-b\hat\alpha}{\hat\alpha}\sigma_{31} - \frac{1}{\hat\beta}\sigma_{32} - \frac{1}{\hat\lambda}\sigma_{33} + - \frac{c+1}{2\hat\lambda}\sigma_{33} + \frac{1}{2}(A\sigma_{13} + B\sigma_{23} + C\sigma_{33})\right)\right]\right]^{-\frac{1}{c}}$$

Also, let $u(\hat\alpha, \hat\beta, \hat\lambda) = \hat\alpha^2$. Then

$$E(\hat\alpha^2 \mid x) = \hat\alpha^2 + 2\hat\alpha\left(\frac{a-1-b\hat\alpha}{\hat\alpha}\sigma_{11} - \frac{1}{\hat\beta}\sigma_{12} - - \frac{1}{\hat\lambda}\sigma_{13}\right) + \sigma_{11} + \frac{1}{2}\hat\alpha(A\sigma_{11} + B\sigma_{21} + C\sigma_{31})$$

Hence the posterior variances are given by

$$Var(\hat\alpha \mid x) = E(\hat\alpha^2 \mid x) - (E(\hat\alpha \mid x))^2 =$$
$$= \sigma_{11} - \left(\frac{a-1-b\hat\alpha}{\hat\alpha}\sigma_{11} - \frac{1}{\hat\beta}\sigma_{12} - \frac{1}{\hat\lambda}\sigma_{13} - - \frac{1}{2}(A\sigma_{11} + B\sigma_{21} + C\sigma_{31})\right)^2 < \sigma_{11} = Var(\hat\alpha)$$

Similary

$Var(\hat\beta \mid x) < \sigma_{22} = Var(\hat\beta)$, $Var(\hat\lambda \mid x) < \sigma_{33} = Var(\hat\lambda)$

4 Numerical Findings

The estimators $\hat\alpha$, $\hat\beta$, and $\hat\lambda$ are maximum likelihood estimators of the parameters of the Modified-Weibull distribution; whereas $\hat\alpha_{BS}$, $\hat\alpha_{BL}$, $\hat\alpha_{BE}$, $\hat\beta_{BS}$, $\hat\beta_{BL}$, $\hat\beta_{BE}$, and $\hat\lambda_{BS}$, $\hat\lambda_{BL}$, $\hat\lambda_{BE}$, are Bayes estimators obtained by using the L-approximation for squared error, Linex and general entropy loss function respectively. As mentioned earlier, the maximum likelihood estimators and hence risks of the estimators cannot be put in a convenient closed form.

Therefore, risks of the estimators are empirically evaluated based on a Monte-Carlo simulation study of samples. A number of values of unknown parameters are considered. Sample size is varied to observe the effect of small and large samples on the estimators. Changes in the estimators and their risks have been determined when changing the shape parameter of loss functions while keeping the sample size fixed. Different combinations of prior parameters α, β and λ are considered in studying the change in the estimators and their risks. The results are summarized in Figures 1-21.

It is easy to notice that the risk of the estimators will be a function of sample size, population parameters, parameters of the prior distribution (hyper parameters), and corresponding loss function parameters.

In order to consider a wide variety of values, we have obtained the simulated risks for sample sizes N=20, 40, 60 and 100.

The various values of parameters of the distribution considered are for:

- scale parameter α=0.2 (.3) 1.4,
- shape parameters β=0.6 (.2) 1.2 and λ=0.2 (.3) 1.4,
- loss parameter c=±1.5, ±1.1, ±0.5 and 0.1 with i=1,2,3.

Prior parameters a and b are arbitrarily taken as 1 respectively 2.

After an extensive study of the results thus obtained, conclusions are drawn regarding the behavior of the estimators, which are summarized below. It may be mentioned here that because of space restrictions, all results are not shown in the graphs.

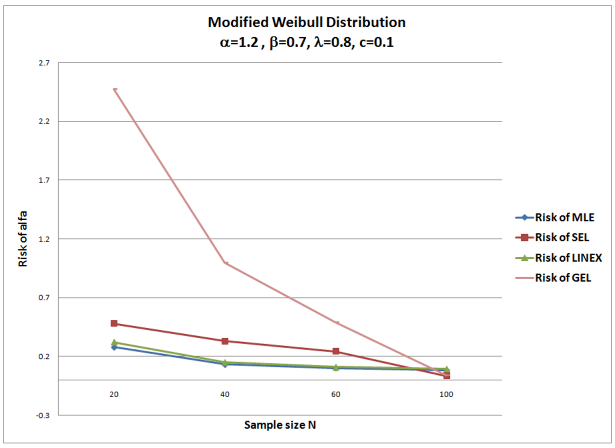

Fig.1 Risk of α for c=0.1

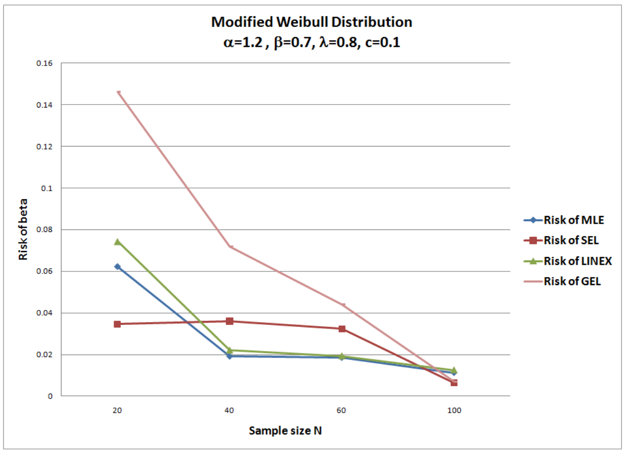

Fig.2 Risk of β for c=0.1

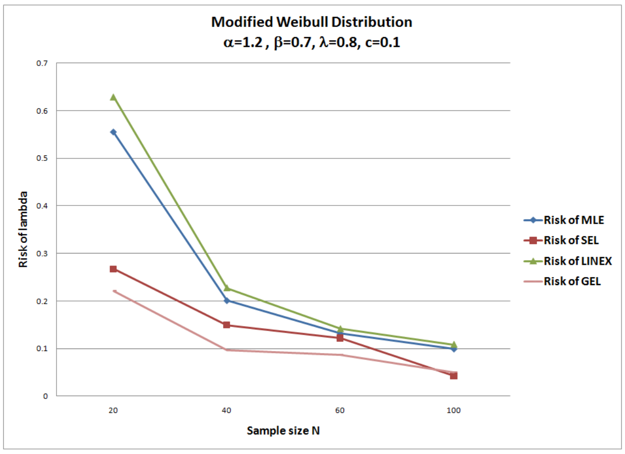

Fig.3 Risk of λ for c=0.1

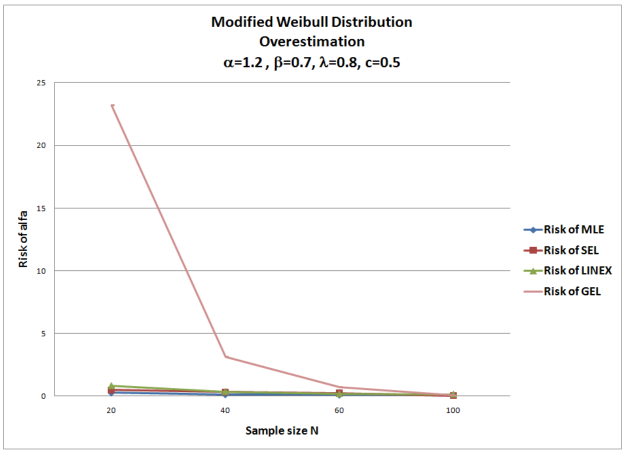

Fig.4 Risk of α for overestimation (c=0.5)

Only a few are presented here to demonstrate the effects found and the conclusion drawn.

However, in most of the cases, the proposed Bayes estimator is better than the Maximum Likelihood Estimator (MLE).

4.1 The Effect of Sample Size

It is noted that as sample size increases, the risk of all the estimators decreases, (see Figures 1-9).

For small samples, the behaviors of risks of estimators depend on the values of c, α, β and λ.

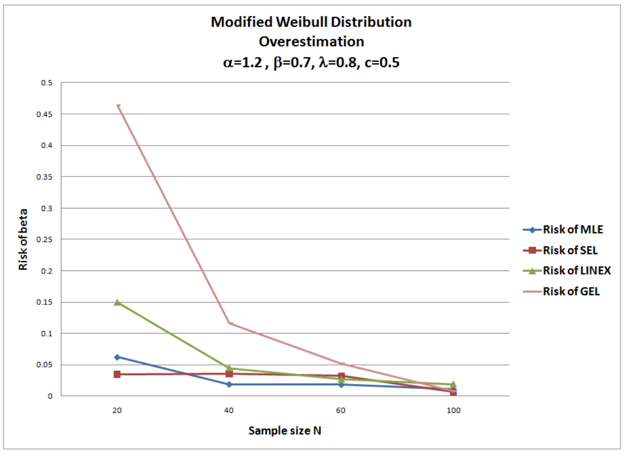

Fig.5 Risk of β for overestimation (c=0.5)

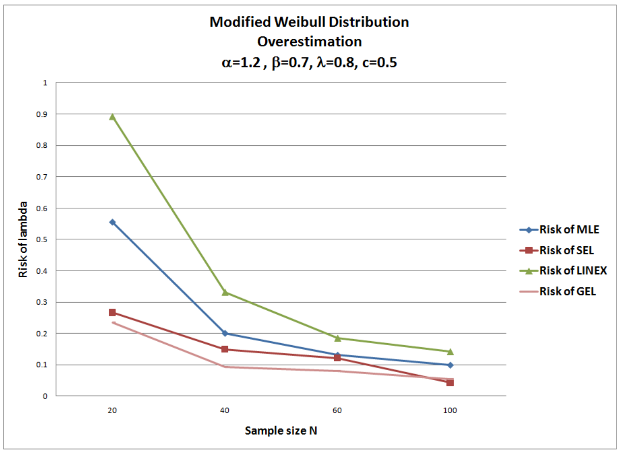

Fig.6 Risk of λ for overestimation (c=0.5)

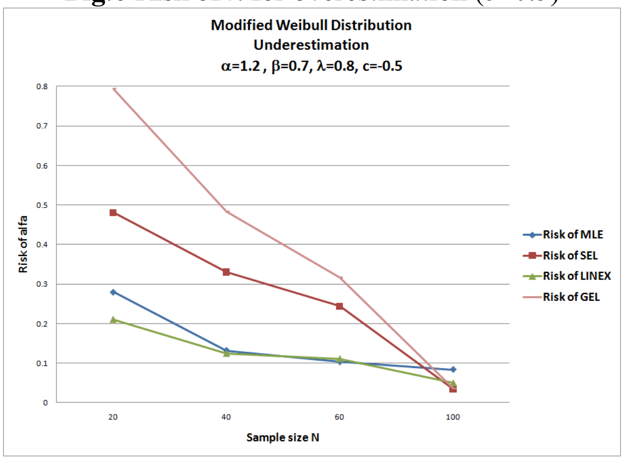

Fig.7 Risk of α for underestimation (c=-0.5)

For any values of c (i.e., when overestimation is more serious then underestimation), the risk of the Bayes estimators GEL of α and β is greater than all other risks (the Bayes estimators SEL and LINEX, the MLE) for a sample size smaller than 75. When sample size increases enough (over 75) the GEL will be the lower risk than all other risks.

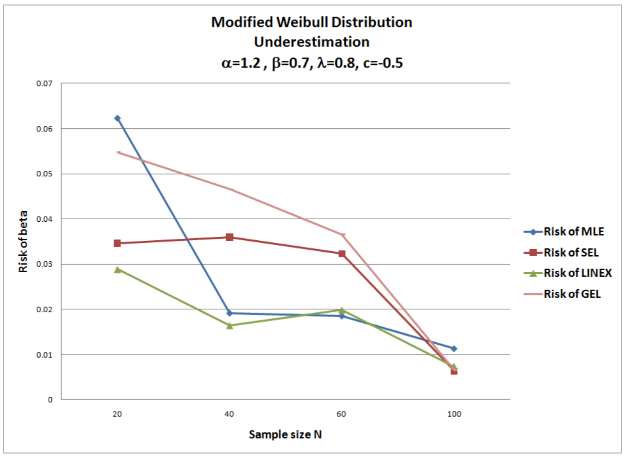

Fig.8 Risk of β for underestimation (c=-0.5)

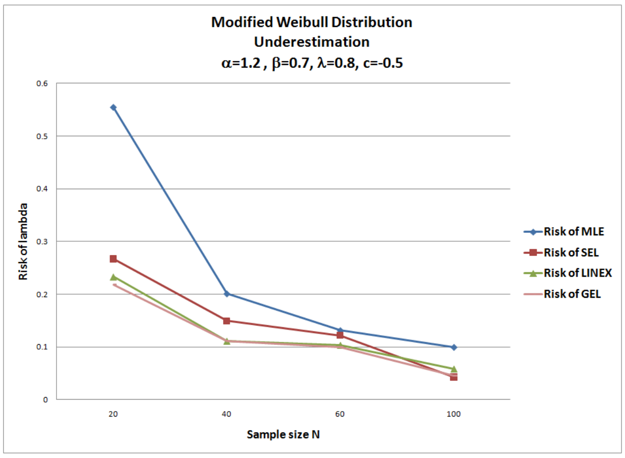

Fig.9 Risk of λ for underestimation (c=-0.5)

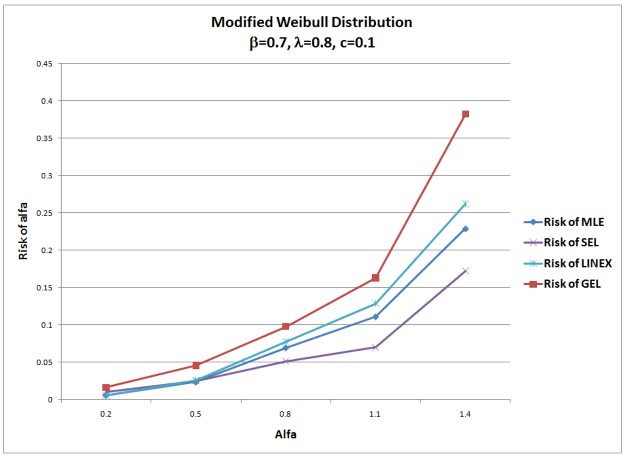

Fig.10 Risk of α as function of α

On the contrary, the risk GEL for λ is lower than all other risks (SEL, LINEX and MLE) for any sample size for any value of c.

The risks LINEX and MLE are higher for $c = 0.1$ and $c = 0.5$ and for $c = -0.5$ only the risk MLE is higher.

4.2 Effect of Population Parameters α, β and λ

The effect of variation of α, β and λ on the risks of the estimator of α, β and λ respectively has also been studied. It has been noticed that for all values of *c*, the risk of the estimators of α and β increases when increase the value of parameter α and β respectively (see Figures 10,11, 13,14, 16 and 17). The same trend is also noticed for λ when $c = 0.1$.

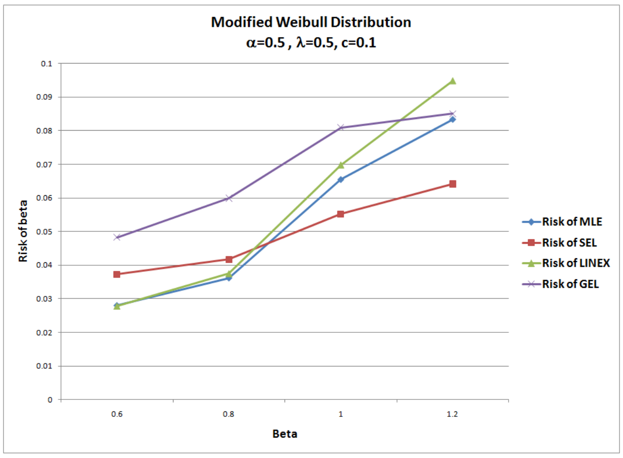

Fig.11 Risk of β as function of β

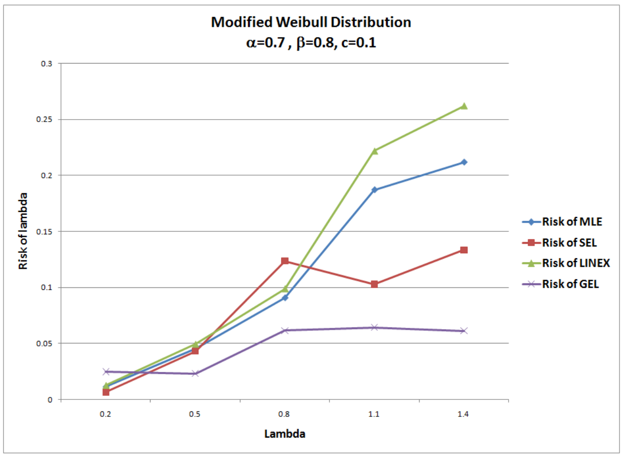

Fig.12 Risk of λ as function of λ

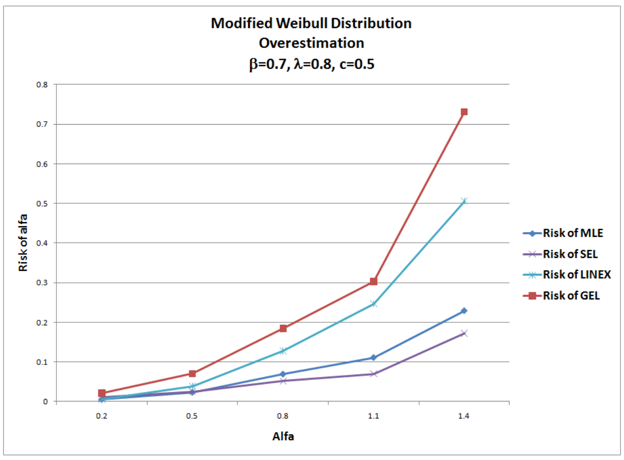

Fig.13 Risk of α for overestimation as function of α

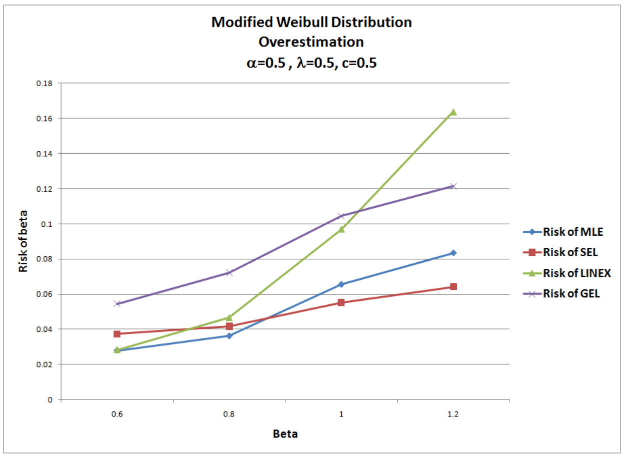

Fig.14 Risk of β for overestimation as function of β

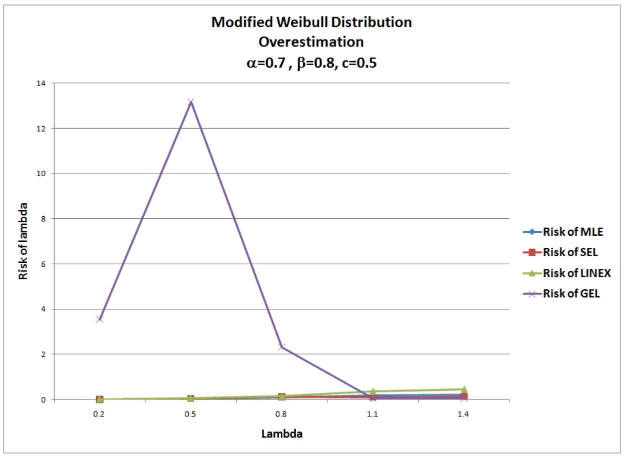

Fig.15 Risk of λ for overestimation as function of λ

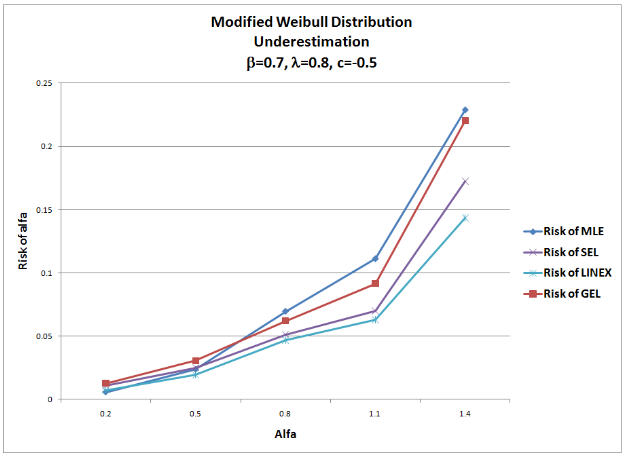

Fig.16 Risk of α for underestimation as function of α

When *c* is positive the risks of the estimators SEL and MLE of α are lower than those for GEL and LINEX of α and when *c* is negative the risks of the estimators SEL and LINEX of α are lower than those for estimators MLE and GEL of α.

The overestimation increase and the underestimation decrease the risks of LINEX and GEL of β in relation to the risk of MLE and SEL of β. In the case of underestimation the risk of estimator GEL is bigger of the risk of estimator MLE if $\beta < 0.85$ and then is less of MLE and the risk of LINEX is smallest of all the risks.

For $c = 0.5$ and $c = -0.5$, it is noted that as the value of λ increases, the risk of the estimator GEL of λ first increases then decreases (see Figures 15 and 18).

The risk of the other all estimators of λ increase when value of λ increase in this cases.

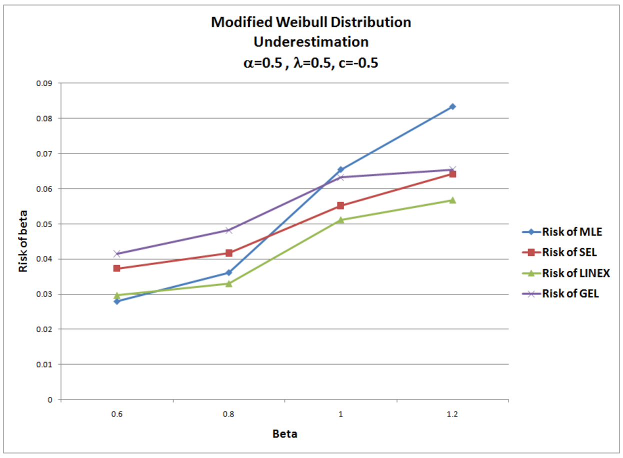

Fig.17 Risk of β for underestimation as function of β

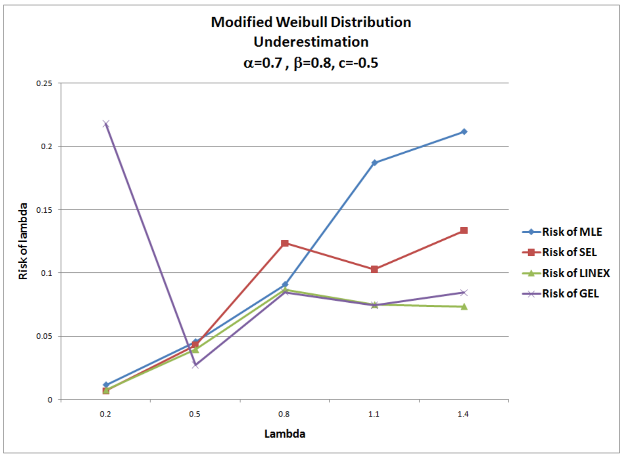

Fig.18 Risk of λ for underestimation as function of λ

4.3 Effect of Loss Parameters

In studying the effect of variation in the values of c, i.e., the LINEX loss parameter and General Entropy loss parameter, on risks of the estimator of α, β and λ, it may be noted that the risks for negative values of c are less than the risks for positive values of c .

As the magnitude of c increases, the risks of the estimators of α, β and λ, increase.

In almost cases, the risks of the Bayes estimator GEL are less than the risks of Bayes estimator LINEX, but this may not be true for $c < -1.1$.

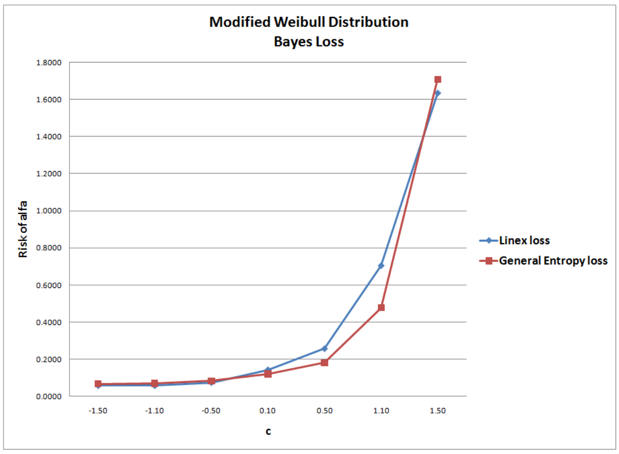

Fig.19 Risk of α as function of c

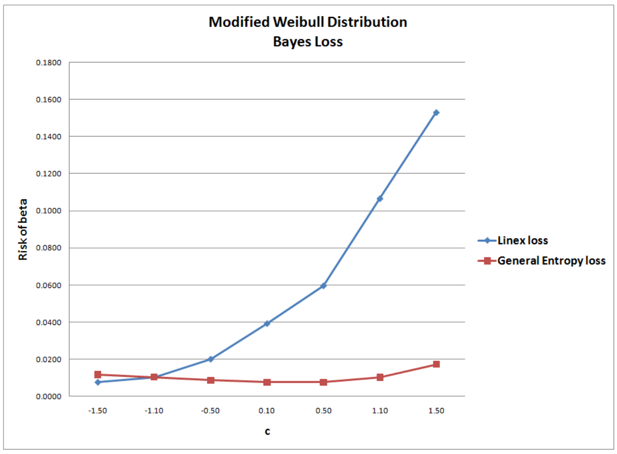

Fig.20 Risk of β as function of c

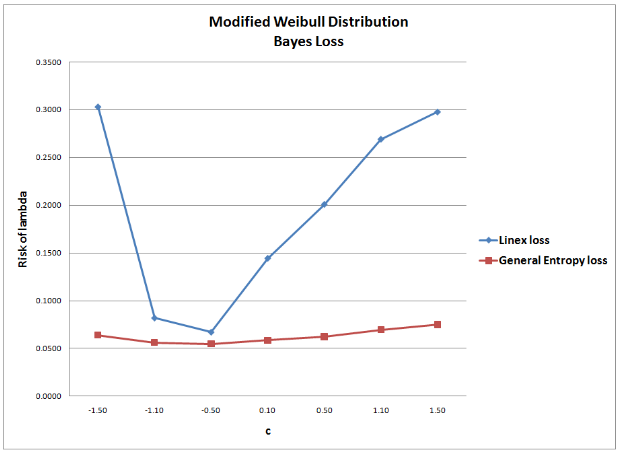

Fig.21 Risk of λ as function of c

5 Conclusion

The performance of the proposed Bayes estimators has been compared to the maximum likelihood estimator. On the basis of these results, we may conclude that for positive c, i.e., overestimation is more serious than underestimation, the Bayes estimators SEL and GEL of λ performs better than the corresponding maximum likelihood estimator and the Bayes estimators LINEX. The maximum likelihood estimators and Bayes estimators SEL of α and β are better for small and moderate sample sizes; whereas risks of the Bayes estimator GEL of all parameters population perform better than any of the estimators for a very large sample sizes.

For negative c, the Bayes estimators LINEX and he maximum likelihood estimators of α and β performs better than the Bayes estimators SEL and GEL. For parameter λ, the Bayes estimators LINEX and GEL are better for small and moderate sample sizes; whereas risks of the Bayes estimators GEL and LINEX of all parameters of population perform better than any of all the estimators for a very large sample sizes.

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