Optimal conditions for the control problem associated to an economic growth process exposed to a risk factor

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Abstract: The aim of this paper is to extend the study done by [6]. Here two different research lines have been joined together: the one studying an endogenous economical growth model in the case of which the occurrence of a disaster is possible, and the one analyzing the same model but with logistic population growth and the Cobb-Douglas production function. First economical growth model leads to an optimal control problem with an infinite horizon. For this problem we will prove that the solution of the optimal control problem verifies Euler-Lagrange's equation. Using these necessary conditions we will determine the consumption evolution equation. Also, we will prove a necessary optimality condition in terms of value function. In the second model we will analyze an economic growth process with logistic population growth and the Cobb-Douglas production function. The model is a version of the Ramsey model in the case of which the occurrence of a disaster is possible. Mathematical modeling of this economical growth process leads to an optimal control problem with an infinite horizon. The necessary conditions for optimality are given.

Key–Words: mathematical models applied in economies, endogenous growth, natural disaster,logistic population.

1 Introduction

Literatures which deal with catastrophic losses and economic growth are basically based on neoclassical frame work [Tatano, et.al, [16], Kobayashi, et.al., [12]]. These catastrophic events, economically speaking produce a downward jump in production. These models showed us that a catastrophic event brings about a permanent shift in the growth path of the economy. Such effect is called a "level effect".

In the standard Ramsey model [1], population is assumed to grow at a positive constant given rate, yielding an exponential behavior of population size over time. In this paper, based on the Ramsey growth model, we consider two versions of this model taking into account the risk of a natural catastrophe. As in the standard Ramsey model, in the first model the growth rate of population is constant. This economical growth model leads to an optimal control problem. We prove that a necessary condition for the control function to solve our optimal control problem is that it is a solution of Euler equation, as in [8], [15]. Also, we give a necessary condition for optimality in the terms of the value function.

An exponential behavior of population over time is unrealistic. A more realistic approach would be to consider a logistic law for the population growth as in [2] or a growth rate of population which depends on the current level of per capita income as in [9]. In [2], Brida and Accinelli analyzed how the Ramsey model is affected by the choice of a logistic growth of population, considering that the society's welfare is measured by a utility function of per capita consumption. As in [2], Guerrini in [11] analyzed how the Ramsey model is affected by the choice of a logistic growth of population, assuming that the society's welfare over time is measured by weighting the utility index of per capita consumption by numbers, i.e multiplying the utility function of the representative men by the total population.

Further, we consider another model with logistic population growth, the Cobb-Douglas production function, taking into account the risk of a natural catastrophe. This economical growth model leads to an optimal control problem. Using the Pontryagin's principle [14] we give a necessary condition for optimality.

The outline of this paper is as follows. In Section 2, we will present the first model in which we will suppose that the economic growth happens in conditions in which the occurrence of a disaster is possible, and the population growth is constant. In Section 3, we will formulate the mathematical problem associated to the economic model from section 2. Also, we give the necessary conditions for the optimal solution to the economical growth problem when there exists the risk factor of a disaster occurring. In Section 4 we introduce the economical growth model with the endogenous population and the Cobb-Douglas production function. In Section 5 we formulate the mathematical problem associated to the economic model from section 4. Moreover, we obtain the necessary conditions for optimality for the economical growth problem. Finally, in Section 6 some conclusions and remarks are given.

2 The economical growth model

In this paper, based on [1], [3], [4] and [5], we consider an economical growth model in which the risk of a natural catastrophe occurring exists. The economy consists of a fixed number of identical infinitely lived households that, for simplicity, is normalized to one. The representative household is populated by identical and infinitely lived agents. Population (identified with labor force) at moment t is denoted by $L(t)$, which grows at the constant, exogenous rate, n. Time is taken to be continuous. Assume that an arrival of disaster is defined by a Poisson arrival with an arrival rate of λ. Also, we assume the economy closed (i.e. all of the stock capital must be owned by someone in economy and the net foreign debt is zero.)

The representative household has access to a technology described by a neoclassical production function

$$F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+, \;\; Y(t) = F(K(t), L(t)) \quad (1)$$

where $Y(t)$ and $K(t)$ denote aggregate output and aggregate capital stock spent producing goods, and $F(\cdot)$ is a C^2, strictly increasing, strictly concave, linearly homogeneous function , satisfying

$$F(0, L) = F(K, 0) = 0,$$

and the Inada conditions

$$\lim_{K\to 0}\frac{\partial F}{\partial K} = \lim_{L\to 0}\frac{\partial F}{\partial L} = \infty,$$
$$\lim_{K\to \infty}\frac{\partial F}{\partial K} = \lim_{L\to \infty}\frac{\partial F}{\partial L} = 0.$$

The output can be used either for consumption or investment. Consumption and investment levels of the capital per one household at the moment t are represented by $C(t)$ and $I(t)$. Therefore, the household's budget constraint is

$$Y(t) = I(t) + C(t) \quad (2)$$

where $C(t)$ is the aggregate consumption, $I(t)$ is the gross investment. Capital is accumulated in the economy by the investment. Economic growth and the recovering process are formulated as capital accumulation process. The capital accumulation equation is given by

$$\dot{K}(t) = F(K(t), L(t)) - C(t) - \delta K(t), \quad (3)$$

where $\delta \in (0, 1)$ is the depreciation rate of capital stock.

In what follows, expressing all variables of model in per capita units, we obtain new variables of the model:

$y(t) = \dfrac{Y(t)}{L(t)}$ – the output per labor unit ,

$c(t) = \dfrac{C(t)}{L(t)}$ – the consumption per labor unit,

$k(t) = \dfrac{K(t)}{L(t)}$ – the capital stock per labor unit,

$k_0 = \dfrac{K_0}{L(0)}$ – the initial capital stock per labor unit.

We divide both sides of the production function (17) by the labor and using the homogeneity condition of the function F to obtain

$$y(t) = \frac{Y(t)}{L(t)} = \frac{F(K(t), L(t))}{L(t)} = F\left(\frac{K(t)}{L(t)}, 1\right) = f(k(t)), \quad (4)$$

where $y(t)$ and $k(t)$ denote output and capital spent producing goods, all in per capita terms, respectively.

From the assumptions made on F follows that: the function f

$$f : \mathbb{R}_+ \to \mathbb{R}_+, \;\; y(t) = f(k(t)), \quad (5)$$

is a function of class C^2 having the following properties: strictly increasing, strictly concave, linearly homogeneous, satisfying $f(0) = 0$, and the Inada conditions

$$\lim_{k\to 0} f'(k) = +\infty, \quad \lim_{k\to\infty} f'(k) = 0.$$

Therefore, the capital accumulation equation (3) in per capita terms is given by

$$\dot{k}(t) = f(k(t)) - c(t) - (n + \delta)k(t), \quad (6)$$

and the initial capital stock is $k(0) = k_0 > 0$.

If a disaster occurs then capital stock decreases discontinuously and does not follow the above equation. Therefore, assuming that $k(t)$ represents a capital level just before a disaster occurs, the level of the capital just after the disaster occurs jumped to $\beta k(t)$, where $0 \leq \beta \leq 1$.

When a disaster does not occur, $k(t)$ changes according to the differential equation (6).

In this economy, the objective of a social planner is to choose at each moment in time the level of consumption $c(t)$ so as to maximize the household's global expected utility taking into account the budget constraint for the household, relation (6), and the initial stock of capital k_0.

The household's global expected utility is defined as

$$U = E\left[\int_0^\infty e^{-(\rho-n)t} u(c(t))dt\right] \tag{7}$$

where $E[\cdot]$ represents the expectation operator with respect to the stochastic arrival time of disaster, $u(\cdot)$ is the instantaneous utility function,

$$u : \mathbb{R}_+ \to \mathbb{R}_+$$

is of class C^2 and satisfies

$$u(0) = 0, u'(c) > 0, u''(c) < 0, \forall c \geq 0,$$

$$\lim_{c\to 0} u'(c) = \infty, \quad \lim_{c\to\infty} u'(c) = 0$$

and the parameter $\rho > 0$ is the time preference rate.

Initial stock of capital that is available for household is K_0. Thus, the initial stock of capital per labor unit is k_0.

We will assume that a disaster will occur according to the Poisson Process. Therefore, the above problem becomes a problem with the finite interval from a time to the time when a first disaster occurs. Because a disaster occurs according to the Poisson Process, the terminal time of this problem is finite, but uncertain. The global expected utility that a household gets after a disaster depends on the recovering process. We will assume it $V(k)$. This corresponds to the terminal utility of the problem. When a disaster occurs at moment t, the interval from initial time 0 till time t is obeyed with an exponential process. The probability that a disaster occurs just when time t first is λ. Therefore we can rewrite the problem as the following optimal growth model with uncertain terminal time

$$W(k) = \max_{c(t)} E_t\left[\int_0^t e^{-(\rho-n)t}\left(u(c(t)) + \lambda V(k(t))\right)dt\right] \tag{8}$$

$$\dot{k}(t) = f(k(t)) - c(t) - (n+\delta)k(t). \tag{9}$$

Calculating the expectation operator using the property of a Poison process, the global expected utility function is transformed into the global utility function

$$\int_0^\infty e^{-(\rho+\lambda-n)t}\left(u(c(t)) + \lambda V(k(t))\right)dt \tag{10}$$

Taking disaster risk into account, discount rate is now given by the summation of the time preference rate, ρ, and the arrival rate, λ.

Assuming a Poison arrival of disaster, the stochastic arrival of the terminal time of the period is now absorbed into the discount factor and the problem can be treated as a deterministic optimal control problem with an infinite time horizon.

Therefore, we can formulate the optimization problem such as

$$W(k) = \max_{c(t)} \int_0^\infty e^{-(\rho+\lambda-n)t}\left(u(c(t)) + \lambda V(k(t))\right)dt \tag{11}$$

subject to

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - c(t) - (n+\delta)k(t) \quad (12)\\ k(0) &= k_0. \quad (13) \end{aligned}$$

3 Determination of optimality conditions

The economical problem is to choose in every moment t, the size of consumption so as to maximize the global utility taking into account the budget constraint for household and the initial stock of capital k_0, leads us to the following mathematical optimization problem (P) :

The problem P. To determine (k^*, c^*) which maximizes the following functional

$$\int_0^\infty e^{-(\rho+\lambda-n)t}\left(u(c(t)) + \lambda V(k(t))\right)dt \tag{14}$$

in the class of functions $k \in AC([0,\infty), \mathbb{R}_+)$, and $c \in \mathcal{X}$, where
$\mathcal{X} = \{c : [0,\infty) \to [0, A], c\text{–measurable}, A < \infty\}$ which verifies:

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - c(t) - (n+\delta)k(t) \quad (15)\\ k(0) &= k_0. \quad (16) \end{aligned}$$

We assume that the terminal utility function, $V(k(t))$, is known. In our problem **P,** k is a state variable and c is a control variable.

Definition 1 *A trajectory $(k(t), c(t))$ is called an admissible trajectory, with initial capital k_0, for the problem (P) if it verifies the relation (15)-(16).*

Definition 2 *An admissible trajectory, $(k^*(t), c^*(t))$, is called optimal trajectory if:*

$$\int_0^\infty e^{-(\rho+\lambda-n)t}\left(u(c(t))+\lambda V(k(t))\right)dt$$
$$\leq \int_0^\infty e^{-(\rho+\lambda-n)t}\left(u(c^*(t))+\lambda V(k^*(t))\right)dt.$$

for every admissible trajectory $(k(t), c(t))$ of the problem (P).

In the following theorem we prove that a solution of our optimal control problem must satisfy a certain differential equation called the Euler equation.
For this, we denote

$$\phi(k(t)) = f(k(t)) - (n+\delta)k(t). \tag{17}$$

Theorem 1 *If $(c(t), k(t))$ is an optimal trajectory of the problem (P), then it verifies the Euler-Lagrange equation:*

$$-\frac{d}{dt}\left[u'\left(\phi(k(t)) - \dot{k}(t)\right)e^{-(\lambda+\rho-n)t}\right]$$
$$= [u'(\phi(k(t)) - \dot{k}(t))\phi'(k(t)) + \lambda V'(k(t))]\cdot$$
$$\cdot e^{-(\lambda+\rho-n)t}. \tag{18}$$

Proof. Let $(c(t), k(t))$ be an optimal trajectory of the problem (P).
From (15) and (17), we have

$$c(t) = \phi(k(t)) - \dot{k}(t).$$

Choose $[T, T']$ such that $\phi'(k(t))$ is continuous on $[T, T']$. Let h by any C^2-function on $[T, T']$ which satisfies

$$h(T) = h(T') = 0.$$

For each real number $\alpha \in \mathbb{R}$, we define a new function $k_1(t)$ by

$$k_1(t) = k(t) + \alpha h(t).$$

Note that, if α is small, the function $k_1(t)$ is "near" the function $k(t)$.
We define

$$J_h(\alpha) = \int_T^{T'}\{u[\phi(k(t)+\alpha h(t)) - (\dot{k}(t)+\alpha\dot{h}(t))] + \lambda V(k(t)+\alpha h(t))\}e^{-(\lambda+\rho-n)t}dt \tag{19}$$

Because $k(t)$ is an optimal trajectory, we have

$$\int_T^{T'}\left[u\left(\phi(k(t)) - \dot{k}(t)\right) + \lambda V(k(t))\right]e^{-(\lambda+\rho-n)t}dt$$
$$\geq \int_T^{T'}\{u[\phi(k(t)+\alpha h(t)) - (\dot{k}(t)+\alpha\dot{h}(t))] + \lambda V(k(t)+\alpha h(t))\}e^{-(\lambda+\rho-n)t}dt$$

for all $\alpha \in \mathbb{R}$.
Thus, $J_h(\alpha) \leq J_h(0)$ for all $\alpha \in \mathbb{R}$. Hence the function $J_h(\alpha)$ has a maximum at $\alpha = 0$, so that

$$J'_h(0) = 0. \tag{20}$$

Now, looking at (19), we see that, in order to calculate $J'_h(0)$ we must differentiate the integral with respect to a parameter appearing in the integrand.
Conversely, (20) implies

$$\int_T^{T'}\{u'[\phi(k(t)) - \dot{k}(t)][\phi'(k(t))h(t) - \dot{h}(t)] + \lambda V'(k(t))h(t)\}e^{-(\lambda+\rho-n)t}dt = 0 \tag{21}$$

We consider

$$\psi(t) = \int_T^t\{u'[\phi(k(s)) - \dot{k}(s)]\phi'(k(s)) + \lambda V'(k(s))\}e^{-(\lambda+\rho-n)s}ds,$$

$t \in [T, T']$ and

$$g(t) = \psi(t) + G,$$

where G is given by

$$\int_T^{T'}[u'(\phi(k(t)) - \dot{k}(t))e^{-(\lambda+\rho-n)t} + \psi(t)]dt + G(T'-T) = 0$$

Since

$$h(T) = h(T') = 0.$$

we have

$$\int_T^{T'}\{u'[\phi(k(t)) - \dot{k}(t)]\phi'(k(t)) + \lambda V'(k(t))\}\cdot h(t)e^{-(\lambda+\rho-n)t}dt =$$
$$= \int_T^T \dot{g}(t)h(t)dt = -\int_T^{T'} g(t)\dot{h}(t)dt$$

Thus, (21) it becomes:

$$-\int_{T}^{T'}(g(t)+u'[\phi(k(t))-\dot{k}(t)]e^{-(\lambda+\rho-n)t})\dot{h}(t)dt=0, \tag{22}$$

for all functions h which are $\mathcal{C}^1$ on $[T,T']$ and which satisfy

$$h(T)=h(T')=0.$$

We consider

$$\begin{aligned} h(t) &= -\int_{T}^{t}\{u'[\phi(k(s))-\dot{k}(s)]e^{-(\lambda+\rho-n)s}+\psi(s)\}ds- \\ &- G(t-T),\ \ t\le T' \end{aligned} \tag{23}$$

From (42), (23) and the definition for g and G, we obtain

$$\int_{T}^{T'}(g(t)+u'[\phi(k(t))-\dot{k}(t)]e^{-(\lambda+\rho-n)t})^2dt=0.$$

Thus, we have

$$g(t)=-u'(\phi(k(t))-\dot{k}(t))e^{-(\lambda+\rho-n)t}$$

for all $t\in[T,T']$.

Since g is a continuous function on $[T,T']$, we see that

$$c(t)=\phi(k(t))-\dot{k}(t)$$

is a continuous function on $[T,T']$ and $\dot{g}$ is a continuous function on $[T,T']$.
Hence, we conclude

$$\begin{aligned} \dot{g}(t) &= \{u'[\phi(k(t))-\dot{k}(t)]\phi'(k(t))+\lambda V'(k(t))\}\cdot \\ &\cdot e^{-(\lambda+\rho-n)t} \\ &= -\frac{d}{dt}(u'(\phi(k(t))-\dot{k}(t))e^{-(\lambda+\rho-n)t}). \end{aligned}$$

■

This is the Euler-Lagrange equation discovered in 1744 by the mathematician Euler.

From the Euler equation

$$-\frac{d}{dt}\left[u'\left(\phi(k(t))-\dot{k}(t)\right)e^{-(\lambda+\rho-n)t}\right]$$

$$=[u'(\phi(k(t))-\dot{k}(t))\phi'(k(t))+\lambda V'(k(t))]\cdot$$
$$\cdot e^{-(\lambda+\rho-n)t}.$$

we obtain

$$\begin{aligned} &[-u''(c(t))\,\dot{c}(t)+(\lambda+\rho-n)u'(c(t))]e^{-(\lambda+\rho-n)t} \\ &=[u'(c(t))\phi'(k(t))+\lambda V'(k(t))]e^{-(\lambda+\rho-n)t} \end{aligned} \tag{24}$$

equivalent with

$$\begin{aligned} &-u''(c(t))\,\dot{c}(t)+(\lambda+\rho-n)u'(c(t)) \\ &=u'(c(t))\,(f'(k)-(n+\delta))+\lambda V'(k(t)) \end{aligned} \tag{25}$$

Finally, we have the evolution equation of consumption

$$\dot{c}(t)=\frac{u'(c(t))}{u''(c(t))}\left(\lambda+\rho+\delta-f'(k)\right)-\lambda\frac{V'(k(t))}{u''(c(t))}.$$

Definition 3 *We will call the value function associated to the problem (P) the function*

$$W(k_0)=\max_{c(t)}\int_{0}^{\infty}e^{-(\rho+\lambda-n)t}\,(u(c(t))+\lambda V(k(t)))\,dt$$

with

$$\begin{cases} c(t)+\dot{k}(t)=\phi(k(t)),\forall t \\ c(t)\ge 0,\ \ k(t)\ge 0,\ \ \forall t \\ k_0>0\ \ \textit{given.} \end{cases}$$

Proposition 4 *Suppose ϕ' is a continuous function on a neighborhood of k_0, and $W(\cdot)$ is a derivable function in k_0. Let $(c(t),k(t))$ be the optimal trajectory from k_0. Then we have*

$$W'(k_0)=u'(c_0).$$

Proof. Let $(c(t),k(t))$ be an optimal trajectory from k_0. For any t, we have

$$0<c(t)=\phi(k(t))-\dot{k}(t).$$

$\phi'(k(t))$ will be continuous for t in any interval [0,T].

Let $h(t)$ be a derivable function and with the derivative continuous on $[0,\infty)$ with

$$h(0)\ne 0\ \text{ and} h(t)=0\ \text{ for any }\ t\ge T.$$

For τ smaller, the trajectory $(c^\tau(t),k^\tau(t))$ is an admissible trajectory from $k_0+\tau h_0$, where $h_0=h(0)$ and

$$\begin{aligned} c^\tau(t) &= \phi(k(t)+\tau h(t))-\dot{k}(t)-\tau\dot{h}(t) \\ k^\tau(t) &= k(t)+\tau h(t),\quad \text{for any } t\le T \\ c^\tau(t) &= c(t),\quad k^\tau(t)=k(t),\quad \text{for any } t\ge T. \end{aligned}$$

We obtain

$$W(k_0+\tau h_0)-W(k_0)\geq\int_0^T(u(c^\tau(t))-u(c(t))+$$
$$+\lambda(V(k+\tau h)-V(k)))e^{-(\rho+\lambda-n)t}dt. \tag{26}$$

If $\tau h_0>0$, we divide both members of the previous inequality by τh_0 and passing to the limit for $\tau\to 0$ we obtain

$$W'(k_0)\ \geq\ \int_0^T\{\frac{u'(c(t))}{h_0}[\phi'(k(t))h(t)-\dot{h}(t)]+$$
$$+\lambda V'(k(t))\frac{h}{h_0}\}\cdot e^{-(\lambda+\rho-n)t}. \tag{27}$$

From the Euler-Lagrange equation we have

$$-\frac{d}{dt}\left[u'\left(\phi(k(t))-\dot{k}(t)\right)e^{-(\lambda+\rho-n)t}\right]$$
$$=[u'(\phi(k(t))-\dot{k}(t))\phi'(k(t))+\lambda V'(k(t))]\cdot$$
$$\cdot e^{-(\lambda+\rho-n)t}.$$

and using the integration by parts formula we obtain

$$\int_0^T u'(c(t))\dot{h}(t)e^{-(\lambda+\rho-n)t}dt \tag{28}$$
$$=\ -u'(c_0)h_0-\int_0^T h(t)\frac{d}{dt}\left[u'(c(t))e^{-(\lambda+\rho-n)t}\right]dt$$

Using (28) in (27) we have

$$W'(k_0)\geq u'(c_0).$$

Analogue, if we consider $\tau h_0<0$, we obtain

$$W'(k_0)\leq u'(c_0),$$

hence in conclusion we have

$$W'(k_0)=u'(c_0).$$

The proof is finished. ■

Proposition 5 *The function $W(\cdot)$ is nondecreasing.*

Proof. By Proposition 4 we have

$$W'(k_0)=u'(c_0).$$

Because $u'(c)>0$, for any $c>0$, we obtain

$$W'(k_0)>0,$$

for any $k_0>0$, i.e. $W(\cdot)$ is nondecreasing. ■

4 The model with logistic population growth and the Cobb-Douglas production function

In this section, we make the mathematical modeling of the economic process described in section 2, in the case a Cobb-Douglas production function and a logistic population growth rate.

Contrary to most subsequent developments, where the growth rate of the labour force, $n_s=\frac{\dot{L}}{L}$, was treated as exogenously determined, in this paper we consider that it is endogenous.

Following Accinelli and Brida [2] , $L(t)$ is assumed to evolve according to the logistic law

$$\dot{L}(t)=aL(t)-bL^2(t),\ with\ a>b>0. \tag{29}$$

In what follows, we consider that the representative household has access to a technology described by a neoclassical production function. Thus, we consider that the output is determined by the following Cobb-Douglas production function

$$Y(t)=K^\alpha(t)L^{1-\alpha}(t) \tag{30}$$

where $Y(t)$, $L(t)$ and $K(t)$ denote aggregate output, the aggregate labor force (identified with population) and aggregate capital stock spent producing goods and $\alpha\in(0,1)$.

The output can be used either for consumption or investment. Consumption and investment levels of the capital per a household at the moment t are represented by $C(t)$ and $I(t)$.

Therefore, the household's budget constraint is

$$Y(t)=I(t)+C(t) \tag{31}$$

where $C(t)$ is the aggregate consumption, $I(t)$ is the gross investment.

The capital accumulation equation is given by

$$\dot{K}(t)=K^\alpha(t)L^{1-\alpha}(t)-C(t)-\delta K(t), \tag{32}$$

where $\delta\in(0,1)$ is the depreciation rate of capital stock.

Expressing all variables of model in per capita units, as in section 2, the capital accumulation equation (32) is given by

$$\dot{k}(t)=k^\alpha(t)-c(t)-(a-bL(t)+\delta)k(t) \tag{33}$$

and the initial capital stock is $k(0)=k_0>0$.

In this economy, the objective of a social planner is to choose at each moment in time the level of consumption $c(t)$ so as to maximize the household's global expected utility taking into account the budget constraint for the household, relation (33), and the evolution of the labor force, relation (29).

The household's global expected utility is defined as

$$U = E\left[\int_0^\infty e^{-\rho t} u(c(t)) L(t) dt\right] \tag{34}$$

where $E[\cdot]$ represents the expectation operator with respect to the stochastic arrival time of disaster, $u(\cdot)$ is the instantaneous utility function,

$$u : \mathbb{R}_+ \to \mathbb{R}_+$$

is of class C^2 and satisfies

$$u(0) = 0, u'(c) > 0, u''(c) < 0, \forall c \geq 0,$$

$$\lim_{c\to 0} u'(c) = \infty, \quad \lim_{c\to\infty} u'(c) = 0$$

and the parameter $\rho > 0$ is the time preference rate.

Initial stock of capital that is available for household is K_0. Thus, the initial stock of capital per worker is k_0.

As in section 2, we will assume that a disaster will occur according to the Poisson Process. We will assume, that the global expected utility that a household gets after a disaster depends on the recovering process $V(k)$. This corresponds to the terminal utility of the problem. When a disaster occurs at moment t, the interval from initial time 0 till time t is obeyed with an exponential process. The probability that a disaster occurs just when time t first is λ. Therefore we can rewrite the problem as the following optimal growth model with uncertain terminal time.

$$W(k) = \max_{c(t)} E_t \left[\int_0^t e^{-\rho t} L(t) \left(u(c(t)) + \lambda V(k(t))\right) dt\right] \tag{35}$$

subject to

$$\dot{k}(t) = k^\alpha(t) - c(t) - (a - bL(t) + \delta)k(t) \tag{36}$$

$$\dot{L}(t) = aL(t) - bL^2(t), \text{ with } a > b > 0. \tag{37}$$

Using the property of a Poison process, the global expected utility function is transformed into the following global utility function,

$$\int_0^\infty e^{-(\rho+\lambda)t} L(t) \left(u(c(t)) + \lambda V(k(t))\right) dt \tag{38}$$

Taking disaster risk into account, discount rate is now given by the summation of the time preference rate, ρ, and the arrival rate, λ.

Assuming a Poison arrival of disaster, the stochastic arrival of the terminal time of the period is now absorbed into the discount factor and the problem can be treated as a deterministic optimal control problem with an infinite time horizon.

5 Optimality conditions

Now, we reformulate the mathematical problem from section 3, associated the economic process presented in previous section.

The problem P1. To determine (k^*, L^*, c^*) which maximizes the following functional

$$\int_0^\infty e^{-(\rho+\lambda)t} L(t) \left(u(c(t)) + \lambda V(k(t))\right) dt \tag{39}$$

in the class of functions $k, L \in AC([0,\infty), \mathbb{R}_+)$, and $c \in \mathcal{X}$, where

$$\mathcal{X} = \{c : [0,\infty) \to [0, A], c - \text{measurable}, A < \infty\}$$

which verifies:

$$\dot{k}(t) = k^\alpha(t) - c(t) - (a - bL(t) + \delta)k(t) \tag{40}$$

$$\dot{L}(t) = aL(t) - bL^2(t) \tag{41}$$

$$k(0) = k_0, \quad L(0) = L_0. \tag{42}$$

We assume that the terminal utility function, $V(k(t))$, is known.

Definition 6 *A trajectory $(k(t), L(t), c(t))$ is called an admissible trajectory, with initial capital k_0, for the problem $(P1)$ if it verifies the relation (40)-(42).*

Definition 7 *An admissible trajectory*

$$(k^*(t), L^*(t), c^*(t)),$$

*is called optimal trajectory for the problem **P1**, if verifies*

$$\int_0^\infty e^{-(\rho+\lambda)t} L(t) \left(u(c(t)) + \lambda V(k(t))\right) dt$$

$$\leq \int_0^\infty e^{-(\rho+\lambda)t} L^*(t) \left(u(c^*(t)) + \lambda V(k^*(t))\right) dt.$$

*for every admissible trajectory $(k(t), L(t), c(t))$ of the problem **P1**.*

This problem can be solved using Pontryagin's maximum principle as in [15], [8]. The state variables in this problem are $k(t)$, $L(t)$ and the control variable is $c(t)$.

We denote by $\mu(t)$ and $\nu(t)$ the co-state variables corresponding to the equations (40) and (41), respectively.

We will continue to determine the necessary conditions for optimality problem **P1**. For this we define the function of Hamilton-Pontryagin given by

$$H(k, c, \mu, \nu, t) = e^{-(\rho+\lambda)t} L(u(c) + \lambda V(k)) + \\ +\mu(k^\alpha - c - (a - bL + \delta)k) + \nu(aL - bL^2)$$

Theorem 2 *Let $(k^*(t), L^*(t), c^*(t))$ be a optimal solution which solves problem **P1**. Then there exist the adjoint absolutely continuous functions $q(t)$ and $p(t)$ such that for all $t \in [0, \infty)$, the relations*

$$q(t) = L^*(t)u'(c^*(t))$$

$$\dot{q}(t) = q(t)\left(\rho + \lambda + \delta + a - bL^*(t) - \alpha k^{*\alpha-1}(t)\right) - \\ -\lambda L^*(t)V'(k^*(t))$$

$$\dot{p}(t) = p(t)(\rho + \lambda + 2bL^*(t) - a) - bq(t)k^*(t) - \\ -u(c^*(t)) - \lambda V(k^*(t)).$$

hold.

Proof. Let $(k^*(t), L^*(t), c^*(t))$ an optimal solution for **P1**. The Hamilton function associated to the problem **P1** is

$$H(k(t), L(t), c(t), \mu(t), \nu(t), t) = \\ = e^{-(\rho+\lambda)t}L(t)(u(c(t)) + \lambda V(k(t))) + \\ +\mu(t)(k^\alpha(t) - c(t) - (a - bL(t) + \delta)k(t)) + \\ +\nu(t)(aL(t) - bL^2(t)) \tag{43}$$

From the Pontryagin's principle there exist the adjoint absolutely continuous functions $\mu(t)$ and $\nu(t)$ such that

$$\dot{\mu}(t) = -\frac{\partial H}{\partial k} = -\mu(t)(\alpha k^{*\alpha-1}(t) - (a - bL^*(t) + \delta)) - \\ -e^{-(\rho+\lambda)t}\lambda L^*(t)V'(k^*(t)) \tag{44}$$

$$\dot{\nu}(t) = -\frac{\partial H}{\partial L} = -\mu(t)\, bk^*(t) - \nu(t)(a - 2bL^*(t)) - \\ -e^{-(\rho+\lambda)t}(u(c^*(t)) + \lambda V(k^*(t))) \tag{45}$$

and $c^*(t)$ is value $c \in [0, \infty)$ which maximizes

$$H(k^*(t), L^*(t), c, \mu(t), \nu(t), t) = \\ = e^{-(\rho+\lambda)t}L^*(t)(u(c) + \lambda V(k^*(t))) + +\mu(t)(k^{*\alpha}(t) - \\ -c - (a - bL^*(t) + \delta)k^*(t)) + \nu(t)(aL^*(t) - bL^{*2}(t)) \tag{46}$$

Using the transformations

$$\mu(t) = e^{-(\rho+\lambda)t}q(t)$$

and

$$\nu(t) = e^{-(\rho+\lambda)t}p(t),$$

the Hamilton function becomes

$$H(k^*(t), L^*(t), c, q(t), p(t), t) = \\ = e^{-(\rho+\lambda)t}[L^*(t)(u(c) + \lambda V(k^*(t))) + q(t)(k^{*\alpha}(t) - \\ c - (a - bL^*(t) + \delta)k^*(t)) + p(t)(aL^*(t) - bL^{*2}(t))]. \tag{47}$$

The first and second derivatives of function H with respect to c are

$$H'_c(k^*(t), L^*(t), c, q(t), p(t), t) = \\ = e^{-(\rho+\lambda)t}(L^*(t)u'(c) - q(t)) \tag{48}$$

$$H''_{cc}(k^*(t), L^*(t), c, q(t), p(t), t) = \\ = e^{-(\rho+\lambda)t}L^*(t)u''(c). \tag{49}$$

From above relation and $u''(c) < 0$, we obtain that H is a concave function of c.

Because $c^*(t)$ is that $c \in [0, \infty)$ which maximizes (47) and H is a concave function of c we have

$$H'_c(k^*(t), L^*(t), c^*(t), q(t), p(t), t) = 0, \tag{50}$$

thus

$$q(t) = L^*(t)u'(c^*(t)). \tag{51}$$

Using again the transformations

$$\mu(t) = e^{-(\rho+\lambda)t}q(t)$$

and

$$\nu(t) = e^{-(\rho+\lambda)t}p(t),$$

(44) and (45) become

$$\dot{q}(t) = q(t)\left(\rho + \lambda + \delta + a - bL^*(t) - \alpha k^{*\alpha-1}(t)\right) - \\ -\lambda L^*(t)V'(k^*(t)) \tag{52}$$

$$\dot{p}(t) = p(t)(\rho + \lambda + 2bL^*(t) - a) - bq(t)k^*(t) - \\ -u(c^*(t)) - \lambda V(k^*(t)). \tag{53}$$

■

Proposition 8 *The evolution equation of the consumption has the following form*

$$\dot{c}(t) = \frac{u'(t)}{u''(t)}\left(\rho + \lambda + \delta - \alpha k^{\alpha-1}(t)\right) - \lambda\frac{V'(k(t))}{u''(t)}.$$

Proof. Differentiate the condition (51) with respect to t we obtain

$$\dot{q}(t) = \dot{L}(t)u'(c(t)) + L(t)u''(c(t))\dot{c}(t). \quad (54)$$

From above equation and the relation (52) we have

$$\dot{c}(t) = \frac{u'(t)}{u''(t)}\left(\rho + \lambda + \delta - \alpha k^{\alpha-1}(t)\right) - \lambda\frac{V'(k(t))}{u''(t)}$$

■

6 Conclusion

In this article we have considered two economic growth models, taking into consideration the possibility of a disaster appearing. Because the natural catastrophe can occur at any moment in the time interval $(0, \infty)$, the modeling of the economic process, taking into consideration the disaster appearance risk at an unknown moment, has lead us to optimal control problems of uncertain terminal time. Further on assuming that the disaster appearance moment is fit for a Poisson process, we have transformed the optimal control problems of uncertain terminal time into optimal control problems of infinite time horizon. For the problem with the exogenous growth rate of population and neoclassical production function we have proved that the solution of problem **P** verifies Euler-Lagrange's equation, therefore providing the necessary optimal conditions for problem **P**. Using these necessary conditions we have determined the consumption evolution equation. Also, for the problem with the endogenous growth rate of population and Cobb-Douglas production function we have obtained the necessary optimal conditions for problem **P1**. As well, analyzing the two models, we have noticed that the capital evolution equation is the same, because population growth has no effect on the evolution of consumption. Whereas in our models the total utility does depend on the size of the population, the evolution of consumption in not affected by population growth rate as in [2]. Moreover, the equation of capital evolution which results from our mathematical models differs from the equation of capital evolution in the standard Ramsey model, by a term that incorporates the probability as a disaster occurred.

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