Equivalent Boundary Integral Equations For Plane Poisson's Exterior Boundary Value Problems

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Abstract: - The solution of conventional boundary integral equations (CBIEs) sometimes does not exist or is not unique, which has been demonstrated in a large number of numerical experiments. According to the authors' opinion, there exist two reasons which can lead to this phenomenon. One reason is that the solution of the CBIEs can not describe the behavior of the solution of the corresponding boundary value problem at infinity accurately; the other one is that the form of exterior boundary value problem has deficiency, which is still a problem to be solved but has not attracted adequate attention. Many examples illustrate that CBIEs are not equivalent to the Poisson's exterior boundary value problem. In this paper, the equivalent boundary integral equations (EBIEs) for Poisson's exterior boundary value problems are established.

Key-Words: - Poisson equation, exterior boundary value problem; BEM; EBIEs; Equivalent boundary integral equations; Extremum principle

1 Introduction

Research on numerical methods of differential equations is a hot topic. Many efficient methods for finding numerical solutions of differential equations have been presented so far such as in [1-4].

Many problems of steady field in physics and engineering are directly related to boundary value problem (BVP) about solving Laplace's equation or Poisson's equation, such as in hydrodynamic pressure [5-8], torsion of elastic rod [9, 10], membrane equilibrium [11], stable heat conduction [12-14], steady seepage [15-18], and electromagnetic field [19, 20]. Therefore, the efficient methods of solving Laplace's equation play important roles in the engineer application.

For the mixed interior BVP, it is well known that the existence and uniqueness of the solution to this kind of problem can be determined by the governing differential equations and its corresponding boundary conditions. However, when it comes to the exterior BVP, the existence and uniqueness of the solution depend on not only the governing differential equations and the corresponding boundary conditions but also the behavior of its solution at infinity [21-23]. The conventional Laplace's exterior BVP can be expressed as two forms: one is directly transplanted from the BVP in a finite field, the solution of this BVP is existent but not unique; the other one, which is the most prevalent, can be written as

$$\begin{cases} \Delta u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega_c \\ u(\boldsymbol{x}) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Gamma \\ |u(\boldsymbol{x})| = O(1/|\boldsymbol{x}|), & \boldsymbol{x} \to \infty \end{cases}$$

If the above BVP has a solution, the solution must be unique. However, the question is the solution to this problem is not always existent.

The EBIE refers to the equivalence of the boundary integral equation and the original BVP. The CBIE in infinite domain is directly transplanted from the CBIE in finite domain, and sometimes there is no solution, or sometimes the solution is not unique. In 1977, Jawson and Symm [24] pointed out that, in some situation, the CBIE would be wrong. However, in their work the efficient method to solve such problem was not provided. In this paper, a sufficient and necessary condition with respect to the Dirichlet exterior BVP, which can ensure the existence and uniqueness of the solution, is provided and fully proved. Based on the proposed condition, EBIEs for Poisson's exterior boundary value problems are established.

For Poisson's exterior BVP, the research of its BIE is a very difficult problem to be solved in boundary element analysis. In fact, to the author's best knowledge, there is no similar work can be found in the literature for solving this problem.

According to this, the main work of this paper can be summarized as follows: section 2 of this paper provides a sufficient and necessary condition which can ensure the existence and uniqueness of solution for Dirichlet exterior BVP of the harmonic function. Section 3 establishes the BIE which are equivalent to the BVP on the exterior domain. It then goes on, in the section 4, to present the EBIE for the Poisson's exterior BVP. Section 5 concludes the paper with further discussions.

2 The sufficient and necessary condition to exterior boundary value problem

We assume that Ω is a bounded domain with the boundary Γ in R^2, $\Omega_c = R^2 - (\Omega \cup \Gamma)$ is open complement of Ω.

Lemma 1 For Laplace's equation $\Delta\omega = \frac{\partial^2 \omega}{\partial^2 x_1} + \frac{\partial^2 \omega}{\partial^2 x_2} = 0$, define the following nondegenerate transformation $\begin{cases} x_1 = \varphi(y_1, y_2) \\ x_2 = \psi(y_1, y_2) \end{cases}$,which satisfies the conditions $\frac{\partial \varphi}{\partial y_1} = \frac{\partial \psi}{\partial y_1}$, $\frac{\partial \varphi}{\partial y_2} = -\frac{\partial \psi}{\partial y_2}$,Then we have $\Delta\omega = \frac{\partial^2 \omega}{\partial^2 y_1} + \frac{\partial^2 \omega}{\partial^2 y_2} = 0$.

Lemma 2[22] Assume $H \subset R^2 - \{0\}$, and Let $\widetilde{H}$, $\widetilde{\widetilde{H}}$ be the two sets satisfying

$$\widetilde{H} \subset \{\boldsymbol{y} \,|\, \boldsymbol{y} = \boldsymbol{x}/|\boldsymbol{x}|^2, \boldsymbol{x} \in H\},$$

$$\widetilde{\widetilde{H}} \subset \{\boldsymbol{y} \,|\, \boldsymbol{y} = \boldsymbol{x}/|\boldsymbol{x}|^2, \boldsymbol{x} \in \widetilde{H}\}$$

where $\boldsymbol{x} = (x_1, x_2)$ and $|\boldsymbol{x}|^2 = x_1^2 + x_2^2$, then we have $H = \widetilde{\widetilde{H}}$.

Definition[22] If $H \subset R^2 - \{0\}$ and $u(\boldsymbol{x})$ is a function in H, we can form a new function $\tilde{u}(\boldsymbol{y})$ defined as follows

$$\tilde{u}(\boldsymbol{y}) = u(\boldsymbol{y}/|\boldsymbol{y}|^2), \quad \boldsymbol{y} \in \widetilde{H}$$

where

$$\widetilde{H} = \{(y_1, y_2) \,|\, y_k = x_k / |\boldsymbol{x}|^2, \boldsymbol{x} \in H, k = 1, 2\}.$$

Lemma 3[22] If $u(\boldsymbol{x})$ is the harmonic function in $H \subset R^2 - \{0\}$, then we can obtain $\tilde{u}(\boldsymbol{x})$ is the harmonic function in $\widetilde{H}$.

Lemma 4[22] Assume $u(\boldsymbol{x})$ is the harmonic function in $\Omega - \{\boldsymbol{y}\}$. If $|u(\boldsymbol{x})| = o(\log|\boldsymbol{x} - \boldsymbol{y}|)$ as $\boldsymbol{x} \to \boldsymbol{y}$, then $\boldsymbol{y}$ is removable singular point of u.

Corollary[22] Let $u(\boldsymbol{x})$ is the harmonic function in Ω_c and $0 \notin \overline{\Omega}_c$ (without loss of generality). If $|u(\boldsymbol{x})| = o(\log|\boldsymbol{x}|)$ as $\boldsymbol{x} \to \infty$, then there is constant C such that $\lim_{x \to \infty} u(x) = C$.

Lemma 5[22] There exist the special harmonic function $\hat{u}$ in Ω_c such that

$$\begin{cases} \Delta\hat{u}(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega_c \\ \hat{u}(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Gamma \\ |\hat{u}(\boldsymbol{x})| = O(\ln|\boldsymbol{x}|), & \boldsymbol{x} \to \infty \end{cases}$$

Lemma 6[22] Assume $u(\boldsymbol{x})$, $u_c(\boldsymbol{x})$ are the harmonic functions in Ω and Ω_c respectively, and $u_c(\boldsymbol{x}) = O(1)$ at ∞, then we have

$$\int_\Gamma \frac{\partial u}{\partial \boldsymbol{n}} d\Gamma = 0,$$

$$\int_\Gamma \frac{\partial u}{\partial \boldsymbol{n}_c} d\Gamma = 0$$

Let us consider Dirichlet exterior BVP as follows

$$\Delta u(\boldsymbol{x}) = 0, \ \boldsymbol{x} \in \Omega_c \qquad (1)$$

$$u(\boldsymbol{x}) = u_0(\boldsymbol{x}), \ \boldsymbol{x} \in \Gamma \qquad (2)$$

$$|u(\boldsymbol{x})| \le M, \ \boldsymbol{x} \text{ is large enough} \qquad (3)$$

where M is a constant.

Theorem 7 Let Γ be a piecewise smooth curve, $0 \notin \Omega_c \cup \Gamma$ (without loss of generality) and $u_0(\boldsymbol{x})$ is a continuous function on Γ, there exists a unique solution of problem (1)-(2) if and only if (3) holds.

Proof (Sufficiency). If $\lim_{x\to\infty} u(\boldsymbol{x}) = C$ holds, and then consider the BVP as follows

$$\Delta u = 0, \quad in\ \tilde{\Omega}_c \bigcup \{0\} \qquad (4)$$

$$v(\boldsymbol{y}) = u_0(\boldsymbol{y}/|\boldsymbol{y}|^2), \quad on\ \tilde{\Gamma} \qquad (5)$$

Since $\tilde{\Omega}_c \bigcup \{0\}$ is a finite domain with boundary $\tilde{\Gamma}$, and $v(\boldsymbol{y})$ is a continuous function on Γ. Obviously, there exists a unique solution to the above boundary value problem in mathematical physical equation.

Suppose $\tilde{v}(\boldsymbol{x})$ is a function in $\Omega_c \bigcup \Gamma = \tilde{\tilde{\Omega}}_c \bigcup \tilde{\tilde{\Gamma}}$. According to both Lemma 2 and Lemma 3, $\tilde{v}(\boldsymbol{x})$ is a harmonic functions in $\tilde{\tilde{\Omega}}_c = \Omega_c$, so $\tilde{v}(\boldsymbol{x})$ satisfies (1). On $\tilde{\tilde{\Gamma}} = \Gamma$, we have

$\tilde{v}(\boldsymbol{x}) = v(\boldsymbol{y}) = v(\boldsymbol{x}/|\boldsymbol{x}|^2) = u_0(\boldsymbol{x})$

That is to say, $\tilde{v}(\boldsymbol{x})$ satisfies (2). Thus, $u(\boldsymbol{x}) = \tilde{v}(\boldsymbol{x})$ is the solution of the problem (1)-(2).

(Necessity). Assume (3) is false. When $\boldsymbol{x}$ approaches ∞, either we have $u(\boldsymbol{x}) = O(\ln|\boldsymbol{x}|)$, or we can find a unbounded sequence $\{\boldsymbol{x}_k\}$ satisfying $\lim_{k\to\infty} u(x_k) = b$ for random real number b. Thus, if $u(\boldsymbol{x})$ is the solution of problem (1)-(2), and l is random real number, then $u(\boldsymbol{x}) + l\hat{u}(\boldsymbol{x})$ is also the solution of that according to Lemma 5. But this is a contradiction because there exists a unique solution to the problem (1)-(2).

This completes the proof.

Remark: Let Γ be a piece smooth curve, $u_0(\boldsymbol{x})$ is a continuous function which only contains a finite number of discontinuity points of the first kind, then theorem 7 still is true.

Now we consider inhomogeneous Dirichlet exterior BVP (Poisson's exterior BVP) as follows

$$\Delta u(\boldsymbol{x}) = -f, \quad in\ \Omega_c \qquad (6)$$

$$u|_\Gamma = \bar{u}, \quad on\ \Gamma \qquad (7)$$

where $f(\boldsymbol{x})$ has compact support in Ω_c.

For the above inhomogeneous BVP, solution is the sum of two solutions: $u = u_1 + u_2$, where u_1 is a solution of the associated homogeneous BVP and u_2 is a particular solution of inhomogeneous BVP.

Suppose $w(\boldsymbol{y}) = \int_{\Omega_c} f(\boldsymbol{x}) u^*(\boldsymbol{x}, \boldsymbol{y}) d\Omega_c$, and $\boldsymbol{y} \in \Omega_c$. It is easy to deduce $\Delta w = -f$, so $w(\boldsymbol{y})$ is a particular solution. The problem (6)-(7) can be transformed as follows

$$\Delta v = 0, \quad in\ \Omega_c \qquad (8)$$

$$v|_\Gamma = \bar{u} - w, \quad on\ \Gamma \qquad (9)$$

$$v = O(1), \quad at\ \infty \qquad (10)$$

According to the Theorem 7, a unique solution v exists to the above problem. It is obvious that $u = v + w$ is the solution of the problem (6)-(7).

Theorem 8 Assume $f \in L^1(R^2)$, and $\int_{|\boldsymbol{y}|>1} |f(\boldsymbol{y})| \times \log|\boldsymbol{y}| d\boldsymbol{y}$, then $\boldsymbol{\omega} = f * N$ is local integral, and also is a distribution solution. ($f * N$ denote the convolutions of all the integrable functions f, N in R^2, that is to say, $f*N(\boldsymbol{x}) = \int f(\boldsymbol{y}) N(\boldsymbol{x}-\boldsymbol{y}) d\boldsymbol{y} = \int f(\boldsymbol{x}-\boldsymbol{y}) N(\boldsymbol{y}) d\boldsymbol{y} = N * f(\boldsymbol{x})$, where $N(\boldsymbol{x}) = -\log|\boldsymbol{x}|/2\pi$.)

Proof: Define $N_0(\boldsymbol{x})$ as follows

$$N_0(\boldsymbol{x}) = \begin{cases} N(\boldsymbol{x}) & |\boldsymbol{x}| \le 1 \\ 0 & |\boldsymbol{x}| > 1 \end{cases}$$

Since $N_0 \in L^1$, then $f * N_0$ is defined almost everywhere. Suppose $N_\infty(\boldsymbol{x}) = N(\boldsymbol{x}) - N_0(\boldsymbol{x})$, there exists $\lim_{y\to\infty}(\log|\boldsymbol{x}-\boldsymbol{y}| - \log|\boldsymbol{y}|) = 0$ for any $\boldsymbol{x}$, so $f * N_\infty$ exists everywhere and is local bounded. Thus $\boldsymbol{\omega} = f * N_0 + f * N_\infty$ is defined almost everywhere and local integrable for any case. Now define $f_j(\boldsymbol{x})$ as follows

$$f_j(\boldsymbol{x})=\begin{cases} f(\boldsymbol{x}) & |\boldsymbol{x}|\le j \\ 0 & |\boldsymbol{x}|>j \end{cases}$$

Let $\omega_j = f_j * N$, according the dominated convergence theorem, we have $\omega_j \to \omega$, $\Delta\omega_j \to \Delta\omega$ in the distribution sense. If $\phi \in C_0^\infty$, and $\widetilde{f}_j(\boldsymbol{x}) = f_j(-\boldsymbol{x})$, then

$< \omega_j, \Delta\phi >=< f_j * N, \Delta\phi >=< N, \widetilde{f}_j * \Delta\phi >=< N,$ $\Delta(\widetilde{f}_j * \phi) >= -(\widetilde{f}_j * \phi)(0) =< -f_j, \phi >$

Since $\Delta\omega_j = -f_j$, thus $\Delta\omega = -\lim f_j = -f$.

This completes the proof.

3 Equivalent boundary integral equations for Laplace's exterior problems

Theorem 9 A harmonic function is defined as follow

$$u(\boldsymbol{y})=\int_\Gamma [u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x + C, \quad \boldsymbol{y}\in\Omega_c$$

As $\boldsymbol{y}\to\infty$, $u(\boldsymbol{y})\to C$ if, and only if, $\int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} d\Gamma_x = 0$.

Proof Since

$$2\ln|\boldsymbol{x}-\boldsymbol{y}| = 2\ln|\boldsymbol{y}| + \ln(1-\frac{2(\boldsymbol{x},\boldsymbol{y})}{|\boldsymbol{y}|^2}+\frac{|\boldsymbol{x}|^2}{|\boldsymbol{y}|^2})$$

$$\frac{\partial}{\partial \boldsymbol{n}_x}\ln|\boldsymbol{x}-\boldsymbol{y}| = \boldsymbol{n}\bullet\nabla\ln|\boldsymbol{x}-\boldsymbol{y}| = \frac{\cos((\boldsymbol{x},\boldsymbol{y}),\boldsymbol{n}_x)}{|\boldsymbol{x}-\boldsymbol{y}|}$$

Then, as $\boldsymbol{y}\to\infty$, we have

$$u(\boldsymbol{y}) = -\frac{1}{2\pi}(\int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} d\Gamma_x)\ln|\boldsymbol{y}| + O(\frac{1}{|\boldsymbol{y}|}) + C$$

Therefore $u(\boldsymbol{y})\to C$ if, and only if,

$\int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} d\Gamma = 0$ as $\boldsymbol{y}\to\infty$.

3.1 Equivalent expression of the harmonic function

Theorem 10 Let Γ be a piece smooth curve in plane, and $u(\boldsymbol{x})$ that with the first-order continuous partial derivatives in $\overline{\Omega_c}$ is a harmonic function in Ω_c. Assume $|u(\boldsymbol{x})| = O(1)$ as $\boldsymbol{x}\to\infty$. Then $u(\boldsymbol{x})$ can be expressed as follows

$$\int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} d\Gamma_x = 0 \tag{11}$$

$$u(\boldsymbol{y})=\int_\Gamma [u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x + C, \quad \boldsymbol{y}\in\Omega_c \tag{12}$$

Proof Let $B_R(0)$ including Ω denote a sufficiently large ball with radius R and center at the original point. Setting $\widehat{\Omega} = B_R(0)\cap\Omega_c$ and applying Green theorem on $\widehat{\Omega}$, we can obtain

$$u(\boldsymbol{y})=\int_\Gamma [u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma + \int_{\partial B_R} [u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma, \quad \boldsymbol{y}\in\widehat{\Omega} \tag{13}$$

Since $u(\boldsymbol{x})$ is bounded at infinite, we have $\lim_{x\to\infty} u(\boldsymbol{x}) = C$ according to the corollary of lemma 4, thus

$$u(\boldsymbol{x})=C+O(\frac{1}{R}) \quad and \quad \frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}} = -\frac{1}{2\pi R}+O(\frac{1}{R}) \quad \boldsymbol{x}\in\partial B_R$$

Therefore

$$\lim_{R\to\infty}\int_{\partial B_R} [u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x = C \tag{14}$$

Substituting (14) into (13) and setting $R\to\infty$, we can deduce

$$u(\boldsymbol{y})=\int_{\Gamma}[u^{*}(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^{*}(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_{x}+C, \quad \boldsymbol{y}\in\Omega_{c}$$

This proves (12).
Applying Green theorem to harmonic function $u(\boldsymbol{x})$ and $v(\boldsymbol{x})\equiv 1$ in Ω_c, then we have

$$\int_{\Gamma}\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}d\Gamma_{x}+\int_{\partial B_R}\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}d\Gamma_{x}=0$$

The equation (11) holds as $R\to\infty$
This proves theorem.

3.2 Equivalent boundary integral equations

Letting $\boldsymbol{y}\to\Gamma$, according the equations (11) and (12), we can get the following equivalent boundary integral equations

$$\int_{\Gamma}\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}d\Gamma_{x}=0 \qquad (15)$$

$$ku(\boldsymbol{y})=\int_{\Gamma}[u^{*}(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^{*}(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_{x}+C, \quad \boldsymbol{y}\in\Gamma \qquad (16)$$

where k equals $\alpha/2\pi$ for $2D$ and α denotes the interior angle at point $\boldsymbol{y}$ on the boundary [25].

4 Equivalent boundary integral equations for Poisson's exterior problems

In this section, we shall deduce Equivalent boundary integral equations for problem (6)-(7) (Poisson's exterior BVP).

4.1 Equivalent direct boundary integral equations

Lemma 11 Suppose V is a compact set, $S\subset R^n$ is an open set, and $V\subset S$. Then there is $g\in C_0^\infty(S)$ such that $g\equiv 1$ in v and $0\le g\le 1$ in S.

Proof: Define

$$\delta=\inf\{|\boldsymbol{x}-\boldsymbol{y}|:\boldsymbol{x}\in V,\ \boldsymbol{y}\notin S\}$$
$$and\ U=\{\boldsymbol{x}: exist\ y\in V\ such\ that\ |\boldsymbol{x}-\boldsymbol{y}|<\delta/2\}$$

then we have $\delta>0$, $V\subset U$ and $\overline{U}\subset S$. Suppose T_U is characteristic function of U. Let ϕ is a nonnegative function belonging to $C_0^\infty(B_{\delta/2}(0))$ and $\int\phi=1$. It can be verified easily that $T_U*\phi$ is the g satisfying the conditions of theorem.

Lemma 12 Let $f(\boldsymbol{x})$ have compact support, then the particular solution $w(y)$ has the following properties

$$\int_{\Gamma}[u^{*}(\boldsymbol{x},\boldsymbol{y})\frac{\partial w(\boldsymbol{x})}{\partial \boldsymbol{n}}-w(\boldsymbol{x})\frac{\partial u^{*}(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_{x}=0, \quad \boldsymbol{y}\in\Omega_{c} \qquad (17)$$

$$\int_{\Gamma}\frac{\partial w(\boldsymbol{x})}{\partial \boldsymbol{n}}d\Gamma+\int_{\Omega_c}\frac{\partial w(\boldsymbol{x})}{\partial \boldsymbol{n}}d\Omega_{c}=0 \qquad (18)$$

Let $B_R(0)$ denote a sufficiently large ball with radius R and center at the original point, and $B_R(0)\supset \operatorname{supp} f$ (support of f). According to Lemma 10, there is $g\in C_0^\infty(B_R(0))$ satisfying $g\equiv 1$ in $B_R(0)$. Applying Green theorem to $w(\boldsymbol{y})$ and $gu^*(\boldsymbol{x},\boldsymbol{y})$ in $B_R(0)$, then we have the equation (17); It is similar that we can get the equation(18) by applying green theorem to $w(\boldsymbol{y})$ and g.

It is evident from Theorem 9 the solution $v(\boldsymbol{y})$ of boundary value problem (8)-(9) can be expressed as follows

$$\int_{\Gamma}\frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{n}}d\Gamma_{x}=0 \qquad (19)$$

$$v(\boldsymbol{y})=\int_{\Gamma}[u^{*}(\boldsymbol{x},\boldsymbol{y})\frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{n}}-v(\boldsymbol{x})\frac{\partial u^{*}(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_{x}+C, \quad \boldsymbol{y}\in\Omega_{c} \qquad (20)$$

Since $v(\boldsymbol{y})=u(\boldsymbol{y})-w(\boldsymbol{y})$, so it follows that

$$\int_{\Gamma}\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}d\Gamma_{x}-\int_{\Gamma}\frac{\partial w(\boldsymbol{x})}{\partial \boldsymbol{n}}d\Gamma_{x}=0$$

$$u(\boldsymbol{y})-w(\boldsymbol{y})=\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x+C$$
$$-\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial w(\boldsymbol{x})}{\partial \boldsymbol{n}}-w(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x,\quad \boldsymbol{y}\in\Omega_c$$

According to Lemma 12, we have

$$\int_{\Gamma}\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}d\Gamma_x+\int_{\Omega_c}f(\boldsymbol{x})d\Omega_c=0 \tag{21}$$

$$u(\boldsymbol{y})=\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x$$
$$+\int_{\Omega_c}f(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})]d\Omega_c+C,\quad \boldsymbol{y}\in\Omega_c \tag{22}$$

Letting $\boldsymbol{y}\to\Gamma$, according the equations (21) and (22), we can get the following equivalent boundary integral equations

$$\int_{\Gamma}\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}d\Gamma_x+\int_{\Omega_c}f(\boldsymbol{x})d\Omega_c=0 \tag{23}$$

$$ku(\boldsymbol{y})=\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x$$
$$+\int_{\Omega_c}f(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})]d\Omega_c+C,\quad \boldsymbol{y}\in\Gamma \tag{24}$$

where k equals $\alpha/2\pi$ for $2D$ and α denotes the interior angle at point $\boldsymbol{y}$ on the boundary.

4.2 Equivalent indirect boundary integral equations

Without loss of generality, let us consider Dirichlet exterior BVP as follows

$$\begin{cases}\Delta u_c=-f, & \boldsymbol{x}\in\Omega_c\\ u_c=u_0, & \boldsymbol{x}\in\Gamma\end{cases}$$

According to the theorem 8, $\omega(\boldsymbol{x})$ is a particular solution of the above problem. Assume $u'=u_c-\omega$, we consider the following exterior BVP

$$\begin{cases}\Delta u'=0, & \boldsymbol{x}\in\Omega_c\\ u'|_{\Gamma}=u_c-\omega, & \boldsymbol{x}\in\Gamma\\ u'=O(1), & \boldsymbol{x}\to\infty\end{cases}$$

Let $B_R(0)$ including Ω denote a sufficiently large circle with radius R and center at the original point. Setting $\widehat{\Omega}=B_R(0)\cap\Omega_c$ and applying Green theorem in $\widehat{\Omega}$, we can obtain

$$u'(\boldsymbol{y})=\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u'(\boldsymbol{x})}{\partial \boldsymbol{n}_c}-u'(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}]d\Gamma_x$$
$$+\int_{\partial B_R}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u'(\boldsymbol{x})}{\partial \boldsymbol{n}_c}-u'(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}]d\Gamma_x,\quad \boldsymbol{y}\in\widehat{\Omega} \tag{25}$$

Since $u'(\boldsymbol{x})=O(1)$, and $\lim_{x\to\infty}u'(\boldsymbol{x})=C$ according to lemma 4, thus

$$u'(\boldsymbol{x})=C+O(\frac{1}{R}),\quad \frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}=-\frac{1}{2\pi R}+o(\frac{1}{R}),\quad \boldsymbol{x}\in\partial B_R$$

So

$$\lim_{R\to\infty}\int_{\partial B_R}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u'(\boldsymbol{x})}{\partial \boldsymbol{n}_c}-u'(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}]d\Gamma_x=C$$

then we can deduce

$$u'(\boldsymbol{y})=\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u'(\boldsymbol{x})}{\partial \boldsymbol{n}_c}-u'(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}]d\Gamma_x+C,\quad \boldsymbol{y}\in\Omega_c \tag{26}$$

And since $u'(\boldsymbol{x})=u_c(\boldsymbol{x})-\omega(\boldsymbol{x})$ and

$\frac{\partial u'}{\partial \boldsymbol{n}_c}=\frac{\partial u_c(\boldsymbol{x})}{\partial \boldsymbol{n}_c}-\frac{\partial \omega(\boldsymbol{x})}{\partial \boldsymbol{n}_c}$, Thus

$$u_c(\boldsymbol{y})=\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u_c(\boldsymbol{x})}{\partial \boldsymbol{n}_c}-u_c(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}]d\Gamma_x$$
$$-\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial \omega(\boldsymbol{x})}{\partial \boldsymbol{n}_c}-\omega(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}]d\Gamma_x$$
$$+\int_{\Omega_c}f(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Omega_c+C,\quad \boldsymbol{y}\in\Omega_c \tag{27}$$

According to lemma 12, there exists

$$\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial\omega(\boldsymbol{x})}{\partial\boldsymbol{n}_c}-\omega(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{n}_c}]d\Gamma_x=0$$

then we have

$$u_c(\boldsymbol{y})=\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u_c(\boldsymbol{x})}{\partial\boldsymbol{n}_c}-u_c(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{n}_c}]d\Gamma_x+\int_{\Omega_c}f(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Omega_c+C,\quad \boldsymbol{y}\in\Omega_c \tag{28}$$

Performing non-analytic continuation of $u_c(\boldsymbol{x})$ to the finite domain Ω, we have the following harmonic function $u(\boldsymbol{x})$

$$\begin{cases}\Delta u(\boldsymbol{x})=0, & \boldsymbol{x}\in\Omega\\ u(\boldsymbol{x})=u_c(\boldsymbol{x}), & \boldsymbol{x}\in\Gamma\end{cases}$$

Applying Green theorem in Ω, then the following equation exists

$$0=\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial\boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{n}}]d\Gamma_x,\quad \boldsymbol{y}\in\Omega \tag{29}$$

By the addition of equations (28) and (29), we can get

$$\begin{aligned}u_c(\boldsymbol{y})&=\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u_c(\boldsymbol{x})}{\partial\boldsymbol{n}_c}-u_c(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{n}_c}]d\Gamma_x\\&+\int_{\Gamma}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial\boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{n}}]d\Gamma_x\\&+\int_{\Omega_c}f(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Omega_c+C,\quad \boldsymbol{y}\in\Omega_c\end{aligned} \tag{30}$$

According to $\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{n}}=-\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{n}_c}$ and $u(\boldsymbol{x})=u_c(\boldsymbol{x})$, there are the following equation

$$u_c(\boldsymbol{y})=\int_{\Gamma}\varphi(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma_x+\int_{\Omega_c}f(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Omega_c+C \tag{31}$$

where $\varphi(\boldsymbol{x})=\frac{\partial u(\boldsymbol{x})}{\partial\boldsymbol{n}}+\frac{\partial u_c(\boldsymbol{x})}{\partial\boldsymbol{n}_c}$, $\boldsymbol{y}\in\Omega_c$

By the Lemma 6 we have

$$\int_{\Gamma}\frac{\partial u(\boldsymbol{x})}{\partial\boldsymbol{n}}d\Gamma=0,\ \int_{\Gamma}\frac{\partial u'(\boldsymbol{x})}{\partial\boldsymbol{n}_c}d\Gamma=0$$

Since $\frac{\partial u'}{\partial\boldsymbol{n}_c}=\frac{\partial u_c(\boldsymbol{x})}{\partial\boldsymbol{n}_c}-\frac{\partial\omega(\boldsymbol{x})}{\partial\boldsymbol{n}_c}$, according to the lemma 12, we can get

$$\begin{aligned}\int_{\Gamma}\varphi(\boldsymbol{x})d\Gamma_x&=\int_{\Gamma}[\frac{\partial u(\boldsymbol{x})}{\partial\boldsymbol{n}}+\frac{\partial u_c(\boldsymbol{x})}{\partial\boldsymbol{n}_c}]d\Gamma_x\\&=\int_{\Gamma}\frac{\partial\omega(\boldsymbol{x})}{\partial\boldsymbol{n}_c}d\Gamma_x=-\int_{\Omega_c}f(\boldsymbol{x})d\Omega_c\end{aligned}$$

So

$$\int_{\Gamma}\varphi(\boldsymbol{x})d\Gamma_x+\int_{\Omega_c}f(\boldsymbol{x})d\Omega_c=0 \tag{32}$$

Therefore the equivalent expression of the potential function can be expressed as

$$\begin{cases}\int_{\Gamma}\varphi(\boldsymbol{x})d\Gamma_x=0\\ u_c(\boldsymbol{y})=\int_{\Gamma}\varphi(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma_x+\int_{\Omega}f(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Omega_x+C\end{cases}$$

Letting $\boldsymbol{y}\to\Gamma$, according the expression (31), we can get the following equivalent boundary integral equations

$$u_c(\boldsymbol{y})=\int_{\Gamma}\varphi(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma_x+\int_{\Omega_c}f(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Omega_c+C,\quad \boldsymbol{y}\in\Gamma \tag{33}$$

$$\frac{\partial u_c(\boldsymbol{y})}{\partial\boldsymbol{n}_c}=k\varphi(\boldsymbol{y})+\int_{\Gamma}\varphi(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{n}_c}d\Gamma_x+\int_{\Omega_c}f(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{n}_c}d\Omega_c,\quad \boldsymbol{y}\in\Gamma \tag{34}$$

where k equals $\alpha/2\pi$ for $2D$ and α denotes the interior angle at point $\boldsymbol{y}$ on the boundary. $\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{n}_c}$ in the equation (21) has singularity

$O(\frac{1}{r})$. Thus, considering the nonexistence of regular integral on Γ, we can only get Cauchy principal value integrals.

Equivalent indirect boundary integral equations in three typical boundary conditions are given in the table 1($\bar{u}_n = \frac{\partial u}{\partial \boldsymbol{n}}$)

Table 1: Equivalent indirect boundary integral equations

Boundary value problems	Boundary conditions	Boundary integral equation
Dirichlet	$\bar{u}$	(32), (33)
Neumann	$\bar{u}_n$	(32), (34)
Mix	$\bar{u}$, $\bar{u}_n$	(32), (33), (34)

5 Discussion and conclusion

5.1 Laplace's exterior boundary value problem

Example 1 Let Ω_c be the open complement of the unit circle with the boundary Γ, consider the following Dirichlet exterior BVP

$$\begin{cases} \nabla^2 u = 0, & x \in \Omega_c \\ u(\boldsymbol{x}) \equiv 1, & x \in \Gamma \end{cases}$$

In fact, both $u_1(\boldsymbol{x}) \equiv 1$ and $u_2(\boldsymbol{x}) \equiv \ln|\boldsymbol{x}| + 1$, for any $\boldsymbol{x} \in \overline{\Omega}_c$ are all the solutions of the above problem. It shows that the problem may have not unique solution if any conditions ensuring the behavior of the solution at infinite are not given. However, the constraint condition is not arbitrary. If we give the following condition

$|u(\boldsymbol{x})| = O(1/|\boldsymbol{x}|), \ \boldsymbol{x} \to \infty$

then the solution of the above problem does not exist.

5.2 Boundary integral equation of Laplace's exterior boundary value problem

Now we will certify that the CBIE have no extensive suitability by an example.

Example 2 Suppose Ω is a bounded domain with the boundary Γ, and $\Omega_c = R^2 - (\Omega \cup \Gamma)$ is its open complement. We shall consider the following two Dirichlet exterior boundary value problems (BVPs)

$$(\mathrm{I}): \begin{cases} \Delta u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega^c \\ u(\boldsymbol{x}) = u_0(\boldsymbol{x}) + C & \boldsymbol{x} \in \Gamma \\ u(\boldsymbol{x}) = O(1) & \boldsymbol{x} \to \infty \end{cases} \qquad (\mathrm{II}): \begin{cases} \Delta u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega^c \\ u(\boldsymbol{x}) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Gamma \\ u(\boldsymbol{x}) = O(1) & \boldsymbol{x} \to \infty \end{cases}$$

where C is a constant.

The CBIE base on the harmonic function in Ω_c can be expressed as

$$u(\boldsymbol{x}) = \int_\Gamma [u^*(\boldsymbol{x},\boldsymbol{y}) \frac{\partial u(\boldsymbol{y})}{\partial \boldsymbol{n}} - u(\boldsymbol{y}) \frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}] d\Gamma_y, \quad \boldsymbol{x} \in \Omega_c \tag{35}$$

Supposing the solutions u_1 and u_2 to the above two problem can be get by the use of the CBIE, then $u = u_1 - u_2$ is the solution of the following boundary integral equation

$$\begin{cases} \Delta u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega_c \\ u(\boldsymbol{x}) = C & \boldsymbol{x} \in \Gamma \\ u(\boldsymbol{x}) = O(1) & \boldsymbol{x} \to \infty \end{cases}$$

It is obvious that $u(\boldsymbol{x}) \equiv C$ ($\boldsymbol{x} \in \Omega_c$) is the solution of the above problem. According to the theorem 9, if $\int_\Gamma \frac{\partial u(\boldsymbol{y})}{\partial \boldsymbol{n}} d\Gamma_y = 0$, the equation (35) approaches 0 as $\boldsymbol{x} \to \infty$; if $\int_\Gamma \frac{\partial u(\boldsymbol{y})}{\partial \boldsymbol{n}} d\Gamma_y \neq 0$, the equation (35) approaches ∞ as $\boldsymbol{x} \to \infty$. Thus, at least one solution of problem (Ⅰ) and (Ⅱ) can't obtained by means of the CBIE.

5.3 Conclusions

For any two boundary value problems that the difference of two boundary functions is a constant, at least one problem can not be solved by CBIE. What's more, on account of complexity of boundary conditions in projects, it is difficult to judge that the problem can be solved under whatever conditions. In this situation, the EBIE of the Poisson's exterior problems are developed. The above conclusions also apply to Neumann and mixed BVPs.

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