A Note For Plane Laplace's Exterior Boundary Value Problems

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Abstract: - The solution of conventional boundary integral equations (CBIEs) sometimes does not exist or is not unique, which has been demonstrated in a large number of numerical experiments. According to the authors' opinion, there exist two reasons which can lead to this phenomenon. One reason is that the solution of the CBIEs can not describe the behavior of the solution of the corresponding boundary value problem at infinity accurately; the other one is that the form of exterior boundary value problem has deficiency, which is still a problem to be solved but has not attracted adequate attention. In this paper, a sufficient and necessary condition with respect to the Dirichlet exterior boundary value problem, which can ensure the existence and uniqueness of the solution, is provided and fully proved. Based on the proposed condition, equivalent boundary integral equations (EBIEs) for exterior problems are established. In addition, an extremum principle on the exterior domain is introduced in this paper.

Key-Words: - Laplace equation, exterior boundary value problem; BEM; Equivalent boundary integral equations; Extremum principle ; Numerical method

1 Introduction

Research on numerical methods of differential equations is a hot topic. Many efficient methods for finding numerical solutions of differential equations have been presented so far such as in [1-4].

Many problems of steady field in physics and engineering are directly related to boundary value problem (BVP) about solving Laplace's equation, such as in hydrodynamic pressure [5-8], torsion of elastic rod [9,10], membrane equilibrium [11], stable heat conduction [12-14], steady seepage [15-18], and electromagnetic field [19-20]. Therefore, the efficient methods of solving Laplace's equation play important roles in the engineer application.

For the mixed interior BVP, it is well known that the existence and uniqueness of the solution to this kind of problem can be determined by the governing differential equations and its corresponding boundary conditions. However, when it comes to the exterior BVP, the existence and uniqueness of the solution depend on not only the governing differential equations and the corresponding boundary conditions but also the behavior of its solution at infinity [21-23]. The conventional Laplace's exterior BVP can be expressed as two forms: one form is directly transplanted from the BVP in a finite field, the solution of this BVP is existent but not unique; the other one, which is the most prevalent, can be written as

$$\begin{cases} \Delta u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega_c \\ u(\boldsymbol{x}) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Gamma \\ |u(\boldsymbol{x})| = O(1/|\boldsymbol{x}|), & \boldsymbol{x} \to \infty \end{cases}$$

If the above BVP is solvable, the solution must be unique. However, the solution of the problem is not always existent.

The EBIE refers to the equivalence of the boundary integral equation and the original BVP. The CBIE in infinite domain is directly transplanted from the CBIE in finite domain, and sometimes there is no solution, or sometimes the solution is not unique. In 1977, Jawson and Symm [24] pointed out that, in some situation, the CBIE would be wrong. However, in their work the efficient method to solve such problem was not provided. The objective of this paper is to develop an EBIE in infinite domain, and the solution of the constructed EBIE is existent and unique.

For the first kind Fredholm integral equation, there is no systematic theory until now and thus it is still a problem to be solved. Even though the integral kernel behaves degenerate properties, the solution of this kind of integral equation sometimes does not exist or is not unique. The indirect CBIE of Laplace problems belongs to this kind of integral equation, and therefore, its solution is not existence or not unique in some situation. For this, some modifications for the first kind of Fredholm integral equation are constructed in this paper, and proved the modified integral equation has a unique solution.

According to this, the main work of this paper can be summarized as follows: (1) section 2 of this paper provides a sufficient and necessary condition which can ensure the existence and uniqueness of solution for Dirichlet exterior BVP of the harmonic function. All arguments presented in this paper are carried out under the sense of classical solutions, and advantage of this is that behaviors of the solution at infinity can be reflected more concretely; (2) an extremum principle of the harmonic function on the exterior domain is proposed, for the first time, in this paper; (3) section 3 establishes the boundary integral equations (BIEs) which are equivalent to the BVP on the exterior domain; (4) in section 4, some modifications for the first kind of Fredholm integral equation are constructed to ensure the existence and uniqueness of the solution.

2 The sufficient and necessary condition to exterior boundary value problem

We assume that Ω is a bounded domain with the boundary Γ in R^2, $\Omega_c = R^2 - (\Omega \cup \Gamma)$ is open complement of Ω.

Lemma 1 For Laplace's equation $\Delta\omega = \dfrac{\partial^2\omega}{\partial^2 x_1} + \dfrac{\partial^2\omega}{\partial^2 x_2} = 0$, define the following nondegenerate transformation

$\begin{cases} x_1 = \varphi(y_1, y_2) \\ x_2 = \psi(y_1, y_2) \end{cases}$,which satisfies the conditions

$\dfrac{\partial\varphi}{\partial y_1} = \dfrac{\partial\psi}{\partial y_1}$, $\dfrac{\partial\varphi}{\partial y_2} = -\dfrac{\partial\psi}{\partial y_2}$,Then we have $\Delta\omega = \dfrac{\partial^2\omega}{\partial^2 y_1} + \dfrac{\partial^2\omega}{\partial^2 y_2} = 0$.

Theorem 2 Assume $H \subset R^2 - \{0\}$, and Let $\widetilde{H}$, $\widetilde{\widetilde{H}}$ be the two sets satisfying

$$\widetilde{H} \subset \{\boldsymbol{y} \mid \boldsymbol{y} = \boldsymbol{x} / |\boldsymbol{x}|^2, \boldsymbol{x} \in H\},$$
$$\widetilde{\widetilde{H}} \subset \{\boldsymbol{y} \mid \boldsymbol{y} = \boldsymbol{x} / |\boldsymbol{x}|^2, \boldsymbol{x} \in \widetilde{H}\}$$

where $\boldsymbol{x} = (x_1, x_2)$ and $|\boldsymbol{x}|^2 = x_1^2 + x_2^2$,

then we have $H = \widetilde{\widetilde{H}}$.

Proof. To any $\boldsymbol{z} \in \widetilde{\widetilde{H}}$, there is $\boldsymbol{y} \in \widetilde{H}$ in such a way that $\boldsymbol{z} = \boldsymbol{y} / |\boldsymbol{y}|^2$; then to $\boldsymbol{y} \in \widetilde{H}$, there is $\boldsymbol{x} \in H$ such that $\boldsymbol{y} = \boldsymbol{x} / |\boldsymbol{x}|^2$. Consequently, we have $\boldsymbol{z} = \boldsymbol{y} / |\boldsymbol{y}|^2 = \boldsymbol{x} / |\boldsymbol{x}|^2 / \left|\boldsymbol{x} / |\boldsymbol{x}|^2\right|^2 = \boldsymbol{x} \in H$

Conversely, to any $\boldsymbol{x} \in H$, there exist $\boldsymbol{y} = \boldsymbol{x} / |\boldsymbol{x}|^2$, so $\boldsymbol{z} = \boldsymbol{y} / |\boldsymbol{y}|^2$ belongs to $\widetilde{\widetilde{H}}$. Thus, we get

$$\boldsymbol{x} = \boldsymbol{x} / |\boldsymbol{x}|^2 / \left|\boldsymbol{x} / |\boldsymbol{x}|^2\right|^2 = \boldsymbol{y} / |\boldsymbol{y}|^2 = \boldsymbol{z} \in \widetilde{\widetilde{H}},$$

and completes the proof.

Definition 1 If $u(\boldsymbol{x})$ is a function in $H \subset R^2 - \{0\}$, we can construct a new function $\tilde{u}(\boldsymbol{y})$ defined as follows $\tilde{u}(\boldsymbol{y}) = u(\boldsymbol{y} / |\boldsymbol{y}|^2)$, $\boldsymbol{y} \in \widetilde{H}$

where $\widetilde{H} = \{(y_1, y_2) \mid y_k = x_k / |\boldsymbol{x}|^2, \boldsymbol{x} \in H, k = 1,2\}$.

Theorem 3 If $u(\boldsymbol{x})$ is the harmonic function in $H \subset R^2 - \{0\}$, then we have $\tilde{u}(\boldsymbol{x})$ is the harmonic function in $\widetilde{H}$.

Proof. Define a transformation as follows

$x_k = y_k / |\boldsymbol{y}|^2, \quad k = 1,2$

One can checks easily that the above transformation is nondegenerate. Thus we can apply Lemma 1 to obtain that $\tilde{u}(\boldsymbol{y})$ is the harmonic function in $\widetilde{H}$. This completes the proof.

Theorem 4 Assume $u(\boldsymbol{x})$ is the harmonic function in $\Omega - \{\boldsymbol{y}\}$. $\boldsymbol{y}$ is removable singular point of u, if $|u(\boldsymbol{x})| = o(\log|\boldsymbol{x} - \boldsymbol{y}|)$ as $\boldsymbol{x} \to \boldsymbol{y}$.

Proof. Without the generality, suppose $\boldsymbol{y}=0$ and $\overline{B_1}$ is closed unit circle belongs to Ω, then u is continuous on $\partial\overline{B_1}$. Hence there is $\upsilon\in C(\overline{B_1})$ such that

$$\begin{cases}\Delta\upsilon=0 & in\ B_1\\ \upsilon=u, & on\ \partial B_1\end{cases}$$

To any $\varepsilon>0$ and $0<\delta<1$, we construct the following function in $\overline{B_1}-B_\delta$

$$V(\boldsymbol{x})=u(\boldsymbol{x})-\upsilon(\boldsymbol{x})+\varepsilon\log|\boldsymbol{x}|$$

It is easily verified that $V(\boldsymbol{x})$ is harmonic in $B_1-\overline{B}_\delta$, continuous in $\overline{B_1-\overline{B}_\delta}$, and equal to zero on ∂B_1. By applying $|u(\boldsymbol{x})|=o(\log|\boldsymbol{x}-\boldsymbol{y}|)$ to $V(x)$, we have $V(\boldsymbol{x})<0$ for sufficiently small δ on ∂B_δ. According to the extremum principle, $V(\boldsymbol{x})<0$ holds in $B_1-\{0\}$, then let $\varepsilon\to 0$, in $B_1-\{0\}$, we obtain $u-\upsilon\le 0$. In a similar way, $u-\upsilon\ge 0$ is proved. This implies $u=\upsilon$ in $B_1-\{0\}$, that is to say, $\boldsymbol{y}=0$ is removable singular point.

The proof is complete.

By applying theorem 3 and theorem 4, we have the following corollary.

Corollary Without the generality, assume $0\notin\overline{\Omega}_c$ and u is the harmonic function in Ω_c. There exists a constant C such that $\lim\limits_{x\to\infty}u(\boldsymbol{x})=C$, if $|u(\boldsymbol{x})|=o(\log|\boldsymbol{x}|)$ as $\boldsymbol{x}\to\infty$.

Theorem 5 (the extremum principle in infinite domain) Without the generality, let $0\notin\overline{\Omega}_c$ and $u(\boldsymbol{x})$ is the harmonic function but not a constant in Ω_c. There does not exist the value of $u(\boldsymbol{x})$ to reach its supremum and infimum in Ω_c if, and only if, $|u(\boldsymbol{x})|\le M$ (M is a constant) as $\boldsymbol{x}\to\infty$.

Proof. (Necessity). It is obvious for the proof of necessity.

(Sufficiency). Since $0\notin\overline{\Omega}_c$ and $u(\boldsymbol{x})$ is the harmonic function in Ω_c, the theorem 3 shows that $\tilde{u}(\boldsymbol{y})$ is harmonic in $\tilde{\Omega}_c$. Besides, $u(\boldsymbol{x})$ is a bounded function in Ω_c, thus $\tilde{u}(\boldsymbol{y})$ is bounded in the neighborhood of origin O, that is to say, $|\tilde{u}(\boldsymbol{y})|=o(\log|\boldsymbol{y}|)$ as $\boldsymbol{y}\to 0$. According to the theorem 4, there is a constant C such that $\lim\limits_{y\to 0}\tilde{u}(\boldsymbol{y})=C$, namely, origin 0 is the harmonic point of $\tilde{u}(\boldsymbol{y})$. By the extremum principle in finite domain, there are the following relational expressions

$\tilde{u}(\boldsymbol{y})<\sup\limits_{y\in\tilde{\Omega}_c\cup\tilde{\Gamma}}\tilde{u}(\boldsymbol{y})$ and $\sup\limits_{y\in\tilde{\Omega}_c\cup\tilde{\Gamma}}\tilde{u}(\boldsymbol{y})=\sup\limits_{y\in\tilde{\Gamma}}\tilde{u}(\boldsymbol{y})$

for all $\boldsymbol{y}\in\tilde{\Omega}_c$

According the definition 1, we have $\tilde{u}(\boldsymbol{y})=u(\boldsymbol{x}),\quad \boldsymbol{y}\in\tilde{\Omega}_c, \boldsymbol{x}\in\Omega_c$

and $\tilde{u}(\boldsymbol{y})=u(\boldsymbol{x}),\quad \boldsymbol{y}\in\tilde{\Gamma}, \boldsymbol{x}\in\Gamma\ (\boldsymbol{x}=\boldsymbol{y}/|\boldsymbol{y}|^2)$

Therefore, the following equations are established

$$\sup_{y\in\tilde{\Omega}_c\cup\tilde{\Gamma}}\tilde{u}(\boldsymbol{y})=\sup_{x\in\Omega_c\cup\Gamma}u(\boldsymbol{x})$$

and $\sup\limits_{y\in\tilde{\Gamma}}\tilde{u}(\boldsymbol{y})=\sup\limits_{x\in\Gamma}u(\boldsymbol{x})$

This proves the $\sup\limits_{x\in\Omega_c\cup\Gamma}u(\boldsymbol{x})=\sup\limits_{x\in\Gamma}u(\boldsymbol{x})$ and completes the proof.

Let us consider Dirichlet exterior BVP as follows

$$\Delta u(\boldsymbol{x})=0,\ \boldsymbol{x}\in\Omega_c \tag{1}$$

$$u(\boldsymbol{x})=u_0(\boldsymbol{x}),\ \boldsymbol{x}\in\Gamma \tag{2}$$

$$|u_c(\boldsymbol{x})|\le M \tag{3}$$

($\boldsymbol{x}$ is large enough)

where M is a constant.

Theorem 6 Let Γ be a piecewise smooth curve, $0\notin\tilde{\Omega}_c\cup\Gamma$ and $u_0(\boldsymbol{x})$ is a continuous functions on Γ, there exists a unique solution of problem (1)-(2) if and only if (3) holds.

Proof. (Sufficiency). If $\lim\limits_{x\to\infty}u(\boldsymbol{x})=C$ holds, and then consider the BVP as follows

$$\nabla^2 u=0,\quad in\ \tilde{\Omega}_c\cup\{0\} \tag{4}$$

$$\upsilon(\boldsymbol{y})=u_0(\boldsymbol{y}/|\boldsymbol{y}|^2),\ on\ \tilde{\Gamma} \tag{5}$$

Since $\tilde{\Omega}_c \cup \{0\}$ is a finite domain with boundary $\tilde{\Gamma}$, and $\upsilon(\boldsymbol{y})$ is a continuous function on $\tilde{\Gamma}$. Obviously, there exists a unique solution to the above problem, and it is well known in mathematical physical equation.

Suppose $\tilde{\upsilon}(\boldsymbol{x})$ (definition 1) is a function in $\Omega_c \cup \Gamma = \tilde{\tilde{\Omega}}_c \cup \tilde{\tilde{\Gamma}}$. According to both theorem 2 and theorem 3, $\tilde{\upsilon}(\boldsymbol{x})$ is a harmonic functions in $\tilde{\tilde{\Omega}}_c = \Omega_c$, so $\tilde{\upsilon}(\boldsymbol{x})$ satisfies (1). On $\tilde{\tilde{\Gamma}} = \Gamma$, we have

$$\tilde{\upsilon}(\boldsymbol{x}) = \upsilon(\boldsymbol{y}) = \upsilon(\boldsymbol{x}/|\boldsymbol{x}|^2) = u_0(\boldsymbol{x})$$

That is to say, $\tilde{\upsilon}(\boldsymbol{x})$ satisfies (2). Thus, $u(\boldsymbol{x}) = \tilde{\upsilon}(\boldsymbol{x})$ is the solution of the boundary value problem (1)-(2).

Necessity of the theorem6 will be proved later.

Remark: Let Γ be a piece smooth curve, $u_0(\boldsymbol{x})$ is a continuous function which only contains a finite number of discontinuity points of the first kind, then theorem 6 still is true.

Theorem 7 There exist the special harmonic function $\hat{u}$ in Ω_c such that

$$\begin{cases} \Delta\hat{u}(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega_c \\ \hat{u}(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Gamma \\ \hat{u}(\boldsymbol{x}) = O(\ln|\boldsymbol{x}|), & \boldsymbol{x} \to \infty \end{cases} \tag{6}$$

Proof. Assume $U(\boldsymbol{y}) = \dfrac{1}{2\pi|\Gamma|}\int_\Gamma \ln|\boldsymbol{x}-\boldsymbol{y}|\, d\Gamma_x$ and $U_0 = U|_\Gamma$, then consider the following exterior BVP

$$\nabla^2\upsilon(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Omega_c$$

$$\upsilon(\boldsymbol{x}) = -U_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma$$

$|\upsilon(\boldsymbol{x})| \le M$, ($\boldsymbol{x}$ is large enough)where M is a constant.

There exists a unique solution $\upsilon(\boldsymbol{x})$ for this problem, and if we suppose $\hat{u}(\boldsymbol{x}) = U(\boldsymbol{x}) + \upsilon(\boldsymbol{x})$, then $\hat{u}(\boldsymbol{x})$ is the special harmonic function satisfying the conditions (6). The proof is complete.

Now we will prove the necessity of the theorem 6.

(Necessity). Assume (3) is false. When $\boldsymbol{x}$ approaches ∞, either we have $u(\boldsymbol{x}) = O(\ln|\boldsymbol{x}|)$, or we can find a unbounded sequence $\{\boldsymbol{x}_k\}$ satisfying $\lim_{k\to\infty} u(x_k) = b$ for random real number b. Thus, if $u(\boldsymbol{x})$ is the solution of problem (1)-(2), and l is random real number, then $u(\boldsymbol{x}) + l\hat{u}(\boldsymbol{x})$ is also the solution of that according to theorem 7. But this is a contradiction because there exists a unique solution to the problem (1)-(2). This completes the proof.

Theorem 8[22] Assume $u(\boldsymbol{x})$, $u_c(\boldsymbol{x})$ are the harmonic functions in Ω and Ω_c respectively, and $u_c(\boldsymbol{x}) = O(1)$ at ∞, then we have

$$\int_\Gamma \frac{\partial u}{\partial \boldsymbol{n}} d\Gamma = 0, \quad \int_\Gamma \frac{\partial u}{\partial \boldsymbol{n}_c} d\Gamma = 0.$$

3 Equivalent boundary integral equations

3.1 Equivalent direct boundary integral equations

Theorem 9 Let Γ be a piece smooth curve in plane. Suppose $u(\boldsymbol{x})$ is a harmonic function in Ω_c, and its all the first-order partial derivatives are continuous in Ω_c. If $|u(\boldsymbol{x})| \le M$ (M is a constant) when $\boldsymbol{x}$ is large enough, then $u(\boldsymbol{x})$ can be expressed as follows

$$\int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} d\Gamma_x = 0 \tag{7}$$

$$u(\boldsymbol{y}) = \int_\Gamma [u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} - u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}] d\Gamma_x + C, \quad \boldsymbol{y} \in \Omega_c \tag{8}$$

where $\boldsymbol{n}$ is the unit outward normal vectors of Γ at the point $\boldsymbol{x}$.

Proof. Let $B_R(0)$ including Ω denote a sufficiently large circle with radius R and center at the original point. Setting $\widehat{\Omega} = B_R(0) \cap \Omega_c$ and applying Green theorem on $\widehat{\Omega}$, we can obtain

$$u(\boldsymbol{y}) = \int_\Gamma [u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} - u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x$$
$$+\int_{\partial B_R} [u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} - u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x, \quad \boldsymbol{y} \in \widehat{\Omega} \tag{9}$$

Since $u(\boldsymbol{x})$ is bounded at infinite, and $\lim_{x\to\infty} u(\boldsymbol{x}) = C$ according to theorem 4, thus

$$u(\boldsymbol{x}) = C + o(\frac{1}{R}) \quad and \quad \frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}} = -\frac{1}{2\pi R} + o(\frac{1}{R}) \quad \boldsymbol{x} \in \partial B_R$$

So

$$\lim_{R\to\infty} \int_{\partial B_R} [u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} - u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x = C \tag{10}$$

substituting (10) into (9) and setting $R \to \infty$, we can deduce

$$u(\boldsymbol{y}) = \int_\Gamma [u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} - u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x + C, \quad \boldsymbol{y} \in \Omega_c$$

This proves (8).

Applying Green theorem to harmonic function $u(\boldsymbol{x})$ and $\upsilon(\boldsymbol{x}) \equiv 1$ in $\widehat{\Omega}$, then we have

$$\int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} d\Gamma_x + \int_{\partial B_R} \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} d\Gamma_x = 0$$

The above expression is changed into the following equation as $R \to \infty$.

$\int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} d\Gamma_x = 0$. This proves theorem.

Theorem 10 A harmonic function $u(\boldsymbol{y})$ is defined as follows

$$u(\boldsymbol{y}) = \int_\Gamma [u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} - u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x + C, \quad \boldsymbol{y} \in \Omega_c$$

As $\boldsymbol{y} \to \infty$, $u(\boldsymbol{y}) \to C$ if and only if

$$\int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} d\Gamma_x = 0.$$

Proof. Since

$$2\ln|\boldsymbol{x}-\boldsymbol{y}| = 2\ln|\boldsymbol{y}| + \ln(1 - \frac{2(\boldsymbol{x},\boldsymbol{y})}{|\boldsymbol{y}|^2} + \frac{|\boldsymbol{x}|^2}{|\boldsymbol{y}|^2})$$

$$\frac{\partial}{\partial \boldsymbol{n}} \ln|\boldsymbol{x}-\boldsymbol{y}| = \boldsymbol{n} \bullet \nabla \ln|\boldsymbol{x}-\boldsymbol{y}| = \frac{\cos((\boldsymbol{x},\boldsymbol{y}),\boldsymbol{n})}{|\boldsymbol{x}-\boldsymbol{y}|}$$

then, as $\boldsymbol{y} \to \infty$, we have

$$u(\boldsymbol{y}) = -\frac{1}{2\pi}(\int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} d\Gamma_x)\ln|\boldsymbol{y}| + O(\frac{1}{|\boldsymbol{y}|}) + C$$

Therefore $u(\boldsymbol{y}) \to C$ if, and only if,

$\int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} d\Gamma_x = 0$ as $\boldsymbol{y} \to \infty$.

This proves theorem.

Letting $\boldsymbol{y} \to \Gamma$, according the equations (7) and (8), we can get the following equivalent boundary integral equations

$$\int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} d\Gamma_x = 0 \tag{11}$$

$$ku(\boldsymbol{y}) = \int_\Gamma [u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} - u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_x + C, \quad \boldsymbol{y} \in \Gamma \tag{12}$$

where k equals $\alpha / 2\pi$ for $2D$ and α denotes the interior angle at point $\boldsymbol{y}$ on the boundary [25].

3.2 Equivalent indirect boundary integral equations

Let us consider Dirichlet exterior BVP as Follows

$$\begin{cases} \Delta u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega_c \\ u(\boldsymbol{x}) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Gamma \\ u(\boldsymbol{x}) = O(1), & \boldsymbol{x} \to \infty \end{cases} \tag{13}$$

Let $B_R(0)$ including Ω denote a sufficiently large circle with radius R and center at the original point. Setting $\widehat{\Omega} = B_R(0) \cap \Omega_c$ and applying Green theorem in $\widehat{\Omega}$, we can obtain

$$u_c(\boldsymbol{y})=\int_\Gamma[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u_c(\boldsymbol{x})}{\partial \boldsymbol{n}_c}-u_c(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}]d\Gamma_{\boldsymbol{x}} +\int_{\partial B_R}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u_c(\boldsymbol{x})}{\partial \boldsymbol{n}_c}-u_c(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}]d\Gamma_{\boldsymbol{x}},\quad \boldsymbol{y}\in\hat{\Omega} \tag{14}$$

Since $u_c(\boldsymbol{x})$ is bounded at infinite, and $\lim_{x\to\infty} u_c(\boldsymbol{x})=C$ according to theorem 4, thus

$$u_c(\boldsymbol{x})=C+O(\frac{1}{R}),\ \frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}=-\frac{1}{2\pi R}+o(\frac{1}{R}),$$

$\boldsymbol{x}\in\partial B_R$

So

$$\lim_{R\to\infty}\int_{\partial B_R}[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u_c(\boldsymbol{x})}{\partial \boldsymbol{n}}-u_c(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_{\boldsymbol{x}}=C$$

then we can deduce

$$u_c(\boldsymbol{y})=\int_\Gamma[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u_c(\boldsymbol{x})}{\partial \boldsymbol{n}_c}-u_c(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}]d\Gamma_{\boldsymbol{x}}+C,\quad \boldsymbol{y}\in\Omega_c \tag{15}$$

Performing non-analytic continuation of $u_c(\boldsymbol{x})$ to the finite domain, we have the following harmonic function $u(\boldsymbol{x})$

$$\begin{cases}\Delta u(\boldsymbol{x})=0, & \boldsymbol{x}\in\Omega\\ u(\boldsymbol{x})=u_c(\boldsymbol{x}), & \boldsymbol{x}\in\Gamma\end{cases}$$

Applying Green theorem in Ω, then the following equation exists

$$0=\int_\Gamma[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_{\boldsymbol{x}},\quad \boldsymbol{y}\in\Omega_c \tag{16}$$

By the addition of equations (15) and (16), we can get

$$u_c(\boldsymbol{y})=\int_\Gamma[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u_c(\boldsymbol{x})}{\partial \boldsymbol{n}_c}-u_c(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}]d\Gamma_{\boldsymbol{x}} +\int_\Gamma[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}-u(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}]d\Gamma_{\boldsymbol{x}}+C,\quad \boldsymbol{y}\in\Omega_c \tag{17}$$

According to $\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}=-\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_c}$ and $u(\boldsymbol{x})=u_c(\boldsymbol{x})$, there are the following equation

$$u_c(\boldsymbol{y})=\int_\Gamma \varphi(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma_{\boldsymbol{x}}+C \tag{18}$$

where $\varphi(\boldsymbol{x})=\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}+\frac{\partial u_c(\boldsymbol{x})}{\partial \boldsymbol{n}_c}$, $\boldsymbol{y}\in\Omega_c$

By the theorem 8 we have

$$\int_\Gamma \varphi(\boldsymbol{x})d\Gamma_{\boldsymbol{x}}=0 \tag{19}$$

Therefore the equivalent expression of the harmonic function can be expressed as

$$\begin{cases}\int_\Gamma \varphi(\boldsymbol{x})d\Gamma_{\boldsymbol{x}}=0\\ u_c(\boldsymbol{y})=\int_\Gamma \varphi(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma_{\boldsymbol{x}}+C\end{cases}$$

Letting $\boldsymbol{y}\to\Gamma$, according the expression (18), we can get the following equivalent boundary integral equations

$$u_c(\boldsymbol{y})=\int_\Gamma \varphi(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma_{\boldsymbol{x}}+C,\quad \boldsymbol{y}\in\Gamma \tag{20}$$

$$\frac{\partial u_c(\boldsymbol{y})}{\partial \boldsymbol{n}_y}=k\varphi(\boldsymbol{y})+\int_\Gamma \varphi(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_y}d\Gamma_{\boldsymbol{x}},\quad \boldsymbol{y}\in\Gamma \tag{21}$$

$\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_y}$ in the equation (21) has singularity $O(\frac{1}{r})$. Thus, considering the nonexistence of regular integral on Γ, we can only get Cauchy principal value integrals.

Equivalent indirect boundary integral equations in three typical boundary conditions are given in the table 1 ($\bar{u}_n=\frac{\partial u}{\partial \boldsymbol{n}}$).

Table 1: Equivalent direct boundary integral equations

Boundary value problems	Boundary conditions	Boundary integral equation
Dirichlet	$\bar{u}$	(19), (20)
Neumann	$\bar{u}_n$	(19), (21)
Mix	$\bar{u}$, $\bar{u}_n$	(19), (20), (21)

4 The modification for the first kind Fredholm BIE

4.1 Some discussions on the conventional first kind Fredholm BIE

The conventional first kind Fredholm BIE can be expressed as follows

$$g(\boldsymbol{y})=\int_\Gamma \phi(\boldsymbol{x})\ln|\boldsymbol{x}-\boldsymbol{y}|\,d\Gamma_x,\quad \boldsymbol{x}\in\Gamma \qquad (22)$$

The solution of the above integral equation may not exist, even though $g(\boldsymbol{y})\in C(\Gamma)$, even satisfy more stringent conditions. This situation can be illustrated more fully by the following Laplace's problem

$$\begin{cases}\Delta u=0, & in\ B_1(0)\\ u(\boldsymbol{y})\big|_{\partial B_1(0)}=g(\boldsymbol{y}), & on\ \partial B_1(0)\end{cases}$$

Since $g(\boldsymbol{y})\in C(\Gamma)$, for the above BVP, it is obvious that the solution exists. Considering the integral equation (22) , we have

$g(0)=\int_\Gamma \phi(\boldsymbol{x})\ln|\boldsymbol{x}-0|\,d\Gamma_x=0$.

If $u(0)>0$, this integral equation can not be solved.

In order to make a further discussion on the conventional first kind Fredholm BIE, we take the integral equation (22) as the BIE for Laplace's problem in infinite domain. Now, consider the following two exterior BVPs

$$(\mathrm{I}):\begin{cases}\Delta u=0, & \boldsymbol{x}\in\Omega_c\\ u|_\Gamma=g(\boldsymbol{y}) & \boldsymbol{y}\in\Gamma\\ |u(\boldsymbol{x})|\le M & \boldsymbol{x}\to\infty\end{cases}\qquad (\mathrm{II}):\begin{cases}\Delta u=0, & \boldsymbol{x}\in\Omega_c\\ u|_\Gamma=g(\boldsymbol{y})+K & \boldsymbol{y}\in\Gamma\\ |u(\boldsymbol{x})|\le M & \boldsymbol{x}\to\infty\end{cases}$$

where K , M are constants. Suppose $\phi_1(\boldsymbol{x})$, $\phi_2(\boldsymbol{x})$ are the solutions of integral equations for the problems (I) and (II) respectively, there are

$$\int_\Gamma \phi_1(\boldsymbol{x})\ln|\boldsymbol{x}-\boldsymbol{y}|\,d\Gamma_x=g(\boldsymbol{y}) \qquad (23)$$

$$\int_\Gamma \phi_2(\boldsymbol{x})\ln|\boldsymbol{x}-\boldsymbol{y}|\,d\Gamma_x=g(\boldsymbol{y})+K \qquad (24)$$

which, by subtraction, yield the following expression

$$\int_\Gamma \phi(\boldsymbol{x})\ln|\boldsymbol{x}-\boldsymbol{y}|\,d\Gamma_x\equiv K \quad \text{for any } \boldsymbol{y}\in R^2 \qquad (25)$$

where $\phi(\boldsymbol{x})=\phi_2(\boldsymbol{x})-\phi_1(\boldsymbol{x})$. If $\int_\Gamma \phi(\boldsymbol{x})\,d\Gamma_x=0$, the equation (25) approaches 0 as $\boldsymbol{y}\to\infty$; if $\int_\Gamma \phi(\boldsymbol{x})\,d\Gamma_x\neq 0$, the equation (25) approaches ∞ as $\boldsymbol{y}\to\infty$. Thus, at least one solution of problem (I) and (II) can't obtained by means of the integral equation (22).

4.2 The modification for the conventional first kind Fredholm BIE

Theorem 11 Write the following integral equations

$$\int_\Gamma \phi(\boldsymbol{x})\,d\Gamma_x=0 \qquad (26)$$

$$g(\boldsymbol{y})=\int_\Gamma \phi(\boldsymbol{x})\ln|\boldsymbol{x}-\boldsymbol{y}|\,d\Gamma_x+C \qquad (27)$$

There exists a unique solution $\phi(\boldsymbol{x})$ for these integral equations, if $g(\boldsymbol{x})\in C(\Gamma)$.

Proof: Assume the below BVPs

$$\begin{cases}\Delta u=0, & \boldsymbol{x}\in\Omega\\ u(\boldsymbol{x})|_\Gamma=g(\boldsymbol{x}) & \boldsymbol{x}\in\Gamma\end{cases} \qquad (28)$$

$$\begin{cases}\Delta u_c=0, & \boldsymbol{x}\in\Omega_c\\ u_c(\boldsymbol{x})|_\Gamma=g(\boldsymbol{x}) & \boldsymbol{x}\in\Gamma\\ |u_c(\boldsymbol{x})|\le M & \boldsymbol{x}\to\infty\end{cases} \qquad (29)$$

where (28) and (29) are the interior and exterior BVPs respectively. Since $g(\boldsymbol{x})\in C(\Gamma)$, the solutions of these two problems exist. Now, suppose $u(\boldsymbol{x})$, $u_c(\boldsymbol{x})$ are the solutions of integral equations for the problems (28) and (29) respectively, and n , n_c are the unit outward normal vectors of Γ to the domain Ω and Ω_c respectively. Define $\phi(\boldsymbol{x})=\dfrac{\partial u}{\partial n}+\dfrac{\partial u_c}{\partial n_c}$, it can be easily verified that $\phi(\boldsymbol{x})$ is the solution of the problem (26)-(27). If there are two solutions $(\phi_1(\boldsymbol{x}),C_1),(\phi_2(\boldsymbol{x}),C_2)$ for the integral equation (26)-(27), then we have

$$\int_\Gamma \phi(\boldsymbol{x})\,d\Gamma=0$$

$$\int_\Gamma \phi(\boldsymbol{x})\ln|\boldsymbol{x}-\boldsymbol{y}|d\Gamma_x \equiv C \quad \text{for any } \boldsymbol{y}\in R^2$$

where $\phi(\boldsymbol{x})=\phi_2(\boldsymbol{x})-\phi_1(\boldsymbol{x})$, $C=C_2-C_1$. As $\boldsymbol{y}\to 0$, the integral expression $\int_\Gamma \phi(\boldsymbol{x})\ln|\boldsymbol{x}-\boldsymbol{y}|d\Gamma_x$ tends to zero, that is to say, $C=0$. Hence $\phi(\boldsymbol{x})=0$, and the proof is complete.

5 Discussion and conclusion

5.1 Laplace's exterior boundary value problem

Example 1 Let Ω_c be the open complement of the unit circle with the boundary Γ, consider the following Dirichlet exterior BVP

$$\begin{cases}\Delta u=0, & x\in\Omega_c\\ u(\boldsymbol{x})\equiv 1, & x\in\Gamma\end{cases}$$

In fact, both $u_1(\boldsymbol{x})\equiv 1$ and $u_2(\boldsymbol{x})\equiv \ln|\boldsymbol{x}|+1$, $\forall\, \boldsymbol{x}\in\overline{\Omega}_c$ are all the solutions of the above problem. It shows that the problem may have not unique solution if any conditions ensuring the behavior of the solution at infinite are not given. However, the constraint condition is not arbitrary. If we give the following condition

$|u(\boldsymbol{x})|=O(1/|\boldsymbol{x}|),\ \boldsymbol{x}\to\infty$

then the solution of the above problem does not exist.

5.2 Boundary integral equation of Laplace's exterior boundary value problem

Now we will certify that the CBIEs have no extensive suitability by an example.

Example 2 Suppose Ω is a bounded domain with the boundary Γ, and $\Omega_c=R^2-(\Omega\bigcup\Gamma)$ is its open complement. We shall consider the following two Dirichlet exterior BVPs

$$(\mathrm{I}):\begin{cases}\Delta u(\boldsymbol{x})=0, & \boldsymbol{x}\in\Omega_c\\ u(\boldsymbol{x})=u_0(\boldsymbol{x})+C & \boldsymbol{x}\in\Gamma\\ u(\boldsymbol{x})=O(1) & \boldsymbol{x}\to\infty\end{cases}\qquad (\mathrm{II}):\begin{cases}\Delta u(\boldsymbol{x})=0, & \boldsymbol{x}\in\Omega_c\\ u(\boldsymbol{x})=u_0(\boldsymbol{x}) & \boldsymbol{x}\in\Gamma\\ u(\boldsymbol{x})=O(1) & \boldsymbol{x}\to\infty\end{cases}$$

where C is a constant.

The CBIEs base on the harmonic function in Ω_c can be expressed as

$$u(\boldsymbol{x})=\int_\Gamma[u^*(\boldsymbol{x},\boldsymbol{y})\frac{\partial u(\boldsymbol{y})}{\partial\boldsymbol{n}}-u(\boldsymbol{y})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{n}}]d\Gamma_y,\quad \boldsymbol{x}\in\Omega_c \tag{30}$$

Supposing the solutions u_1 and u_2 to the above two problem can be get by the use of the CBIEs, then $u=u_1-u_2$ is the solution of the following boundary integral equation

$$\begin{cases}\Delta u(\boldsymbol{x})=0, & \boldsymbol{x}\in\Omega_c\\ u(\boldsymbol{x})=C & \boldsymbol{x}\in\Gamma\\ u(\boldsymbol{x})=O(1) & \boldsymbol{x}\to\infty\end{cases}$$

It is obvious that $u(\boldsymbol{x})\equiv C$ ($\boldsymbol{x}\in\Omega_c$) is the solution of the above problem. According to the theorem 9, if $\int_\Gamma\frac{\partial u(\boldsymbol{y})}{\partial\boldsymbol{n}}d\Gamma_y=0$, the equation (30) approaches 0 as $\boldsymbol{x}\to\infty$; if $\int_\Gamma\frac{\partial u(\boldsymbol{y})}{\partial\boldsymbol{n}}d\Gamma_y\neq 0$, the equation (30) approaches ∞ as $\boldsymbol{x}\to\infty$. Thus, at least one solution of problem (Ⅰ) and (Ⅱ) can't obtained by means of the CBIEs.

5.3 Conclusions

For any two boundary value problems that the difference of two boundary functions is a constant, at least one problem can not be solved by CBIEs. What's more, on account of complexity of boundary conditions in projects, it is difficult to judge that the problem can be solved under whatever conditions. In this situation, the EBIEs of the Laplace's exterior BVP problems are developed. The above conclusions also apply to Neumann and mixed BVPs. Besides, we constructed some modifications for the conventional first kind Fredholm integral equation to satisfy the existence and uniqueness of the solution.

Acknowledgement: The research is supported by the National Nature Science Foundation of china (no.10571110) and the Natural Science Foundation of Shandong Province of China (no.2003ZX12).

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