Application Of The $(\frac{G'}{G})$-expansion method For The Integrable Sixth-Order Drinfeld-Sokolov-Satsuma-Hirota Equation

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Abstract: In this paper, a generalized $(\frac{G'}{G})$-expansion method is used to seek more general exact solutions of the integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota equation. As a result, the traveling wave solutions with three arbitrary functions are obtained including hyperbolic function solutions, trigonometric function solutions and rational solutions. The method appears to be easier and faster by means of some mathematical software.

Key–Words: $(\frac{G'}{G})$-expansion method, Traveling wave solutions, integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota equation, nonlinear heat conduction equation, exact solution, evolution equation, nonlinear equation

1 Introduction

During the past four decades or so searching for explicit solutions of nonlinear evolution equations (NLEEs) by using various different methods has been the main goal for many researchers, and many powerful methods for constructing exact solutions of nonlinear evolution equations have been established and developed. Some of these approaches are the homogeneous balance method [8,9], the hyperbolic tangent expansion method [10,11], the trial function method [12], the tanh-method [13-15], the non-linear transform method [16], the inverse scattering transform [17], the Backlund transform [18,19], the Hirotas bilinear method [20,21], the generalized Riccati equation [22,23], the Weierstrass elliptic function method [24], the theta function method [25-27], the sineCcosine method [28], the Jacobi elliptic function expansion [29,30], the complex hyperbolic function method [31-33], the truncated Painleve expansion [34], the F-expansion method [35], the rank analysis method [36], the exp-function expansion method [37] and so on. Yet there is no unified method that can be used to deal with all types of nonlinear evolution equations.

Recently a so-called $(\frac{G'}{G})$-expansion method has drawn a lot of attention. The method was presented by Mingliang Wang in [38] at first. The main merits of the $(\frac{G'}{G})$-expansion method over the other methods are that it gives more general solutions with some free parameters and it handles NLEEs in a direct manner with no requirement for initial/boundary condition or initial trial function at the outset. The method was soon been applied to other non-linear problems by several authors [39-42].

In this paper we will apply the $(\frac{G'}{G})$-expansion method to some nonlinear problems. In Section 2, we describe the universe process of the $(\frac{G'}{G})$-expansion method. In section 3 and 4, we will obtain the travelling wave solutions of the integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota equation by the method respectively. In section 5, we will give some conclusions on the $(\frac{G'}{G})$-expansion method.

2 Description of the $(\frac{G'}{G})$-expansion method

In this section we describe the $(\frac{G'}{G})$-expansion method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables $x,\ t$, is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{2.1}$$

or in three independent variables $x,\ y$ and t, is given by

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{yt}, u_{xx}, u_{yy}, ...) = 0, \tag{2.2}$$

where $u = u(x,t)$ or $u = u(x,y,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ or $u = u(x,y,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following, we will give the main steps of the $(\frac{G'}{G})$-expansion method.

Step 1. Suppose that

$$u(x,\ t) = u(\xi), \quad \xi = \xi(x,t) \tag{2.3}$$

or

$$u(x,\ y,\ t) = u(\xi), \quad \xi = \xi(x,y,t) \tag{2.4}$$

The traveling wave variable (2.3) or (2.4) permits us reducing (2.1) or (2.2) to an ODE for $u = u(\xi)$

$$P(u,\ u', u'',\ ...) = 0. \tag{2.5}$$

Step 2. Suppose that the solution of (2.5) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \alpha_m(\frac{G'}{G})^m +\ ... \tag{2.6}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \tag{2.7}$$

$\alpha_m, ..., \lambda$ and μ are constants to be determined later, $\alpha_m \neq 0$. The unwritten part in (2.6) is also a polynomial in $(\frac{G'}{G})$, the degree of which is generally equal to or less than $m - 1$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (2.5).

Step 3. Substituting (2.6) into (2.5) and using second order LODE (2.7), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of (2.5) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_m, ..., \lambda$ and μ.

Step 4. Assuming that the constants $\alpha_m, ..., \lambda$ and μ can be obtained by solving the algebraic equations in Step 3. Since the general solutions of the second order LODE (2.7) have been well known for us, then substituting $\alpha_m, ...$ and the general solutions of (2.7) into (2.6) we have traveling wave solutions of the nonlinear evolution equation (2.1) or (2.2).

3 Application Of The $(\frac{G'}{G})$-Expansion Method For The Integrable Sixth-order Drinfeld-Sokolov-Satsuma-Hirota Equation

We begin with the integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota equation:

$$w_t - 6ww_x + w_{xxx} - 6v_x = 0 \tag{3.1}$$

$$v_t - 2v_{xxx} + 6wv_x = 0 \tag{3.2}$$

In order to obtain the travelling wave solutions of (3.1) and (3.2), we suppose that

$$w(x,\ t) = w(\xi),\ \ v(x,\ t) = v(\xi),\ \ \xi = x - ct \tag{3.3}$$

c is a constant that to be determined later.

By using the wave variable (3.3), Eq.(3.1) and Eq.(3.2) can be converted into ODEs:

$$-cw' - 6ww' + w''' - 6v' = 0 \tag{3.4}$$

$$-cv' - 2v''' + 6wv' = 0 \tag{3.5}$$

Suppose that the solution of (3.4) and (3.5) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$w(\xi) = \sum_{i=0}^{m} a_i(\frac{G'}{G})^i, \quad v(\xi) = \sum_{i=0}^{n} b_i(\frac{G'}{G})^i \tag{3.6}$$

a_i, b_i are constants, and $G = G(\xi)$ satisfies the second order LODE in the form:

$$G'' + \lambda G' + \mu G = 0 \tag{3.7}$$

where λ and μ are constants.

Balancing the order of w''' and v' in Eq.(3.4) and the order of v''' and wv' in Eq.(3.5), we obtain $m+3 = n+1$, $n+3 = m+n+1 \Rightarrow m = 2, n = 4$. So Eq.(3.6) can be rewritten as

$$w(\xi)=a_2(\frac{G'}{G})^2+a_1(\frac{G'}{G})+a_0,\ a_2\neq 0 \quad (3.8)$$

$$v(\xi)=b_4(\frac{G'}{G})^4+b_3(\frac{G'}{G})^3+b_2(\frac{G'}{G})^2$$
$$+b_1(\frac{G'}{G})+b_0,\ b_4\neq 0 \quad (3.9)$$

$a_2,\ a_1,\ a_0,\ b_4,\ b_3,\ b_2,\ b_1,\ b_0$ are constants to be determined later.

Then we can obtain

$$w'(\xi)=-2a_2(\frac{G'}{G})^3+(-a_1-2a_2\lambda)(\frac{G'}{G})^2$$
$$+(-a_1\lambda-2a_2\mu)(\frac{G'}{G})-a_1\mu$$

$$w''(\xi)=6a_2(\frac{G'}{G})^4+(2a_1+10a_2\lambda)(\frac{G'}{G})^3$$
$$+(8a_2\mu+3a_1\lambda+4a_2\lambda^2)(\frac{G'}{G})^2$$
$$+(6a_2\lambda\mu+2a_1\mu+a_1\lambda^2)(\frac{G'}{G})$$
$$+2a_2\mu^2+a_1\lambda\mu$$

$$w'''(\xi)=-24a_2(\frac{G'}{G})^5+(-54a_2\lambda-6a_1)(\frac{G'}{G})^4$$
$$+(-12a_1\lambda-38a_2\lambda^2-40a_2\mu)(\frac{G'}{G})^3$$
$$+(-52a_2\lambda\mu-7a_1\lambda^2-8a_2\lambda^3-8a_1\mu)(\frac{G'}{G})^2$$
$$+(-14a_2\lambda^2\mu-a_1\lambda^3-16a_2\mu^2-8a_1\lambda\mu)(\frac{G'}{G})$$
$$-a_1\lambda^2\mu-2a_1\mu^2-6a_2\lambda\mu^2$$

$$v'(\xi)=-4b_4(\frac{G'}{G})^5+(-3b_3-4b_4\lambda)(\frac{G'}{G})^4$$
$$+(-2b_2-3b_3\lambda-4b_4\mu)(\frac{G'}{G})^3+(-b_1-2b_2\lambda-3b_3\mu)(\frac{G'}{G})^2$$
$$+(-b_1\lambda-2b_2\mu)(\frac{G'}{G})-b_1\mu$$

$$v''(\xi)=20b_4(\frac{G'}{G})^6+(12b_3+36b_4\lambda)(\frac{G'}{G})^5$$
$$+(6b_2+21b_3\lambda+32b_4\mu+16b_4\lambda^2)(\frac{G'}{G})^4$$
$$+(18b_3\mu+28b_4\lambda\mu+10b_2\lambda+9b_3\lambda^2+2b_1)(\frac{G'}{G})^3$$
$$+(3b_1\lambda+8b_2\mu+4b_2\lambda^2+15b_3\lambda\mu+12b_4\mu^2)(\frac{G'}{G})^2$$
$$+(6b_2\lambda\mu+2b_1\mu+6b_3\mu^2+b_1\lambda^2)\frac{G'}{G})$$
$$+2b_2\mu^2+b_1\lambda\mu$$

$$v'''(\xi)=-120b_4(\frac{G'}{G})^7+(-60b_3-300b_4\lambda))(\frac{G'}{G})^6$$
$$+(-248b_4\mu-24b_2-244b_4\lambda^2-144b_3\lambda)(\frac{G'}{G})^5$$
$$+(-392b_4\lambda\mu-6b_1-114b_3\mu$$
$$-111b_3\lambda^2-54b_2\lambda-64b_4\lambda^3)(\frac{G'}{G})^4$$
$$+(-40b_2\mu-12b_1\lambda-152b_4\mu^2-27b_3\lambda^3$$
$$-168b_3\lambda\mu-38b_2\lambda^2-148b_4\lambda^2\mu)(\frac{G'}{G})^3$$
$$+(-7b_1\lambda^2-108b_4\lambda\mu^2-60b_3\mu^2-52b_2\lambda\mu$$
$$-8b_1\mu-8b_2\lambda^3-57b_3\lambda^2\mu)(\frac{G'}{G})^2$$
$$+(-b_1\lambda^3-36b_3\lambda\mu^2-8b_1\lambda\mu$$
$$-14b_2\lambda^2\mu-16b_2\mu^2-24b_4\mu^3)(\frac{G'}{G})$$
$$-6b_3\mu^3-2b_1\mu^2-b_1\lambda^2\mu-6b_2\lambda\mu^2$$

Substituting Eq.(3.8) and (3.9) into (3.4) and (3.5), collecting all terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

For Eq.(3.4):

$$(\frac{G'}{G})^0: \ ca_1\mu - 2a_1\mu^2 - a_1\lambda^2\mu$$
$$-6a_2\lambda\mu^2 + 6a_0a_1\mu + 6b_1\mu = 0$$

$$(\frac{G'}{G})^1: \ 6a_1^2\mu - 16a_2\mu^2 + 12a_0a_2\mu$$
$$-8a_1\lambda\mu + 2ca_2\mu + ca_1\lambda$$
$$+6a_0a_1\lambda - 14a_2\lambda^2\mu + 6b_1\lambda$$
$$+12b_2\mu - a_1\lambda^3 = 0$$

$$(\frac{G'}{G})^2: \ a_1c + 12a_0a_2\lambda + 6a_1^2\lambda$$
$$+18b_3\mu - 7a_1\lambda^2 + 2ca_2\lambda$$
$$+18a_1a_2\mu - 8a_1\mu - 52a_2\lambda\mu$$
$$+6b_1 + 12b_2\lambda + 6a_1a_0 - 8a_2\lambda^3 = 0$$

$$(\frac{G'}{G})^3: \ 18b_3\lambda + 18a_1a_2\lambda + 6a_1^2$$
$$+12a_2^2\mu - 40a_2\mu + 12a_0a_2$$
$$+12b_2 - 12a_1\lambda - 38a_2\lambda^2$$
$$+24b_4\mu + 2ca_2 = 0$$

$$(\frac{G'}{G})^4: \ 18a_1a_2 - 54a_2\lambda - 6a_1$$
$$+24b_4\lambda + 18b_3 + 12a_2^2\lambda = 0$$

$$(\frac{G'}{G})^5: \ 12a_2^2 - 24a_2 + 24b_4 = 0$$

For Eq.(3.5):

$$(\frac{G'}{G})^0: \ -6a_0b_1\mu + cb_1\mu + 4b_1\mu^2$$
$$+12b_3\mu^3 + 2b_1\lambda^2\mu + 12b_2\lambda\mu^2 = 0$$

$$(\frac{G'}{G})^1: \ 2cb_2\mu - 12a_0b_2\mu + 28b_2\lambda^2\mu$$
$$+48b_4\mu^3 - 6a_1b_1\mu + 16b_1\lambda\mu$$
$$+2b_1\lambda^3 - 6a_0b_1\lambda + cb_1\lambda$$
$$+72b_3\lambda\mu^2 + 32b_2\mu^2 = 0$$

$$(\frac{G'}{G})^2: \ -6a_0b_1 + cb_1 + 2cb_2\lambda$$
$$-12a_0b_2\lambda - 18a_0b_3\mu + 114b_3\lambda^2\mu$$
$$+16b_2\lambda^3 - 6a_2b_1\mu - 12a_1b_2\mu$$
$$+3cb_3\mu + 14b_1\lambda^2 + 16b_1\mu$$
$$+216b_4\lambda\mu^2 + 104b_2\lambda\mu - 6a_1b_1\lambda + 120b_3\mu^2 = 0$$

$$(\frac{G'}{G})^3: \ -24a_0b_4\mu + 304b_4\mu^2 - 12a_0b_2$$
$$+336b_3\lambda\mu + 76b_2\lambda^2 + 296b_4\lambda^2\mu$$
$$-18a_0b_3\lambda - 12a_1b_2\lambda - 6a_1b_1$$
$$+3cb_3\lambda + 24b_1\lambda + 2cb_2$$
$$+4cb_4\mu + 80b_2\mu - 18a_1b_3\mu$$
$$-6a_2b_1\lambda - 12a_2b_2\mu + 54b_3\lambda^3 = 0$$

$$(\frac{G'}{G})^4: \ 108b_2\lambda + 3cb_3 - 24a_0b_4\lambda$$
$$+128b_4\lambda^3 - 24a_1b_4\mu + 784b_4\mu\lambda$$
$$-18a_2b_3\mu - 12a_2b_2\lambda - 18a_1b_3\lambda$$
$$+12b_1 + 4cb_4\lambda + 228b_3\mu$$
$$-12a_1b_2 - 18a_0b_3 + 222b_3\lambda^2 - 6a_2b_1 = 0$$

$$(\frac{G'}{G})^5: \ -24a_1b_4\lambda + 48b_2 - 24a_2b_4\mu$$

$$+496b_4\mu + 288b_3\lambda + 488b_4\lambda^2$$
$$+4cb_4 - 24a_0b_4 - 12a_2b_2$$
$$-18a_1b_3 - 18a_2b_3\lambda = 0$$

$$(\frac{G'}{G})^6 : \ 600b_4\lambda - 24a_2b_4\lambda - 24a_1b_4$$
$$-18a_2b_3 + 120b_3 = 0$$

$$(\frac{G'}{G})^7 : \ 240b_4 - 24a_2b_4 = 0$$

Solving the algebraic equations, we get the results in two cases:

Case 1:

$$a_2 = 10, a_1 = 10\lambda$$
$$a_0 = 7\mu + \frac{3}{4}\lambda^2$$
$$b_4 = -40, b_3 = -80\lambda$$
$$b_2 = -40\lambda^2 - 80\mu$$
$$b_1 = -80\lambda, b_0 = b_0$$
$$c = 14\mu - \frac{7}{2}\lambda^2 \qquad (3.10)$$

where b_0 is an arbitrary constant.

Substituting (3.10) into (3.6), it follows

$$w(\xi) = 10(\frac{G'}{G})^2 + 10\lambda(\frac{G'}{G}) + 7\mu + \frac{3}{4}\lambda^2$$

$$v(\xi) = -40(\frac{G'}{G})^4 - 80\lambda(\frac{G'}{G})^3$$
$$+(-40\lambda^2 - 80\mu)(\frac{G'}{G})^2 - 80\lambda(\frac{G'}{G}) + b_0$$

$$\xi = x - (14\mu - \frac{7}{2}\lambda^2)t \qquad (3.11)$$

where b_0 is an arbitrary constant.

Substituting the general solutions of Eq.(3.7) into the formulae (3.11), we have three types of travelling wave solutions of the integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota equation (3.1)-(3.2) as follows:

When $\lambda^2 - 4\mu > 0$

$$w_1(\xi) = -\frac{5}{2}\lambda^2 + \frac{5}{2}(\lambda^2 - 4\mu).$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2$$
$$+7\mu + \frac{3}{4}\lambda^2$$

$$v_1(\xi) = -\frac{5}{2}(\lambda^2 - 4\mu)^2.$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^4$$
$$+5\lambda^2(\lambda^2 - 4\mu).$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2$$
$$-20\mu(\lambda^2 - 4\mu).$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2$$
$$+40\lambda\sqrt{\lambda^2 - 4\mu}(\mu - 1).$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)$$
$$-\frac{5}{2}\lambda^4 - 20\mu\lambda^2 + 40\lambda^2 + b_0$$

where

$$\xi = x - (14\mu - \frac{7}{2}\lambda^2)t$$

b_0, C_1, C_2 are arbitrary constants.

In particular, when $\lambda > 0, \mu = 0, C_1 \neq 0, C_2 = 0$, we can deduce the soliton solutions of the integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota equation as:

$w_1(\xi) = -\frac{5}{2}\lambda^2 sech^2(\frac{\lambda\xi}{2}) + \frac{3}{4}\lambda^2$

$v_1(\xi) = -\frac{5}{2}\lambda^4 tanh^4(\frac{\lambda\xi}{2}) + 5\lambda^4 tanh^2(\frac{\lambda\xi}{2}) - 40\lambda^2 tanh(\frac{\lambda\xi}{2}) - \frac{5}{2}\lambda^4 + 40\lambda^2 + b_0$

When $\lambda^2 - 4\mu < 0$

$$w_2(\xi) = -\frac{5}{2}\lambda^2 + \frac{5}{2}(4\mu - \lambda^2).$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2$$
$$+7\mu + \frac{3}{4}\lambda^2$$

$$v_2(\xi) = -\frac{5}{2}(4\mu - \lambda^2)^2.$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^4$$
$$+5\lambda^2(4\mu - \lambda^2).$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2$$
$$-20\mu(4\mu - \lambda^2).$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2$$
$$+40\lambda\sqrt{4\mu - \lambda^2}(\mu - 1).$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)$$
$$-\frac{5}{2}\lambda^4 - 20\mu\lambda^2 + 40\lambda^2 + b_0$$

where

$$\xi = x - (14\mu - \frac{7}{2}\lambda^2)t$$

b_0, C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$

$$w_3(\xi) = -\frac{5}{2}\lambda^2 + \frac{10C_2^2}{(C_1 + C_2\xi)^2} + 7\mu + \frac{3}{4}\lambda^2$$

$$v_3(\xi) = -40\left[\frac{2c_2 - c_1\lambda - c_2\lambda\xi}{2(c_1 + c_2\xi)}\right]^4$$
$$-80\lambda\left[\frac{2c_2 - c_1\lambda - c_2\lambda\xi}{2(c_1 + c_2\xi)}\right]^3$$
$$+(-40\lambda^2 - 80\mu)\left[\frac{2c_2 - c_1\lambda - c_2\lambda\xi}{2(c_1 + c_2\xi)}\right]^2$$
$$-80\lambda\left[\frac{2c_2 - c_1\lambda - c_2\lambda\xi}{2(c_1 + c_2\xi)}\right] + b_0$$

where $\xi = x - (14\mu - \frac{7}{2}\lambda^2)t$. b_0, C_1, C_2 are arbitrary constants.

Case 2:

$$a_2 = 10, a_1 = 10\lambda$$
$$a_0 = \frac{19}{3}\mu + \frac{11}{12}\lambda^2$$
$$b_4 = -40, b_3 = -80\lambda$$
$$b_2 = -\frac{160}{3}\lambda^2 - \frac{80}{3}\mu$$
$$b_1 = -\frac{40}{3}\lambda^3 - \frac{80}{3}\lambda\mu$$
$$b_0 = b_0, c = \frac{7}{2}\lambda^2 - 14\mu \tag{3.12}$$

where b_0 is an arbitrary constant.

Substituting (3.12) into (3.6), we get that

$$w(\xi) = 10(\frac{G'}{G})^2 + 10\lambda(\frac{G'}{G}) + \frac{19}{3}\mu + \frac{11}{12}\lambda^2$$

$$v(\xi) = -40(\frac{G'}{G})^4 - 80\lambda(\frac{G'}{G})^3 + (-\frac{160}{3}\lambda^2$$
$$-\frac{80}{3}\mu)(\frac{G'}{G})^2 + (-\frac{40}{3}\lambda^3 - \frac{80}{3}\lambda\mu)(\frac{G'}{G}) + b_0$$

$$\xi = x - (-14\mu + \frac{7}{2}\lambda^2)t \tag{3.13}$$

where b_0 is an arbitrary constant.

Substituting the general solutions of Eq.(3.7) into (3.13), we also have three types of travelling wave solutions of the integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota equation (3.1) as follows:

When $\lambda^2 - 4\mu > 0$

$$w_1(\xi) = -\frac{5}{2}\lambda^2 + \frac{5}{2}(\lambda^2 - 4\mu).$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2$$
$$+\frac{19}{3}\mu + \frac{11}{12}\lambda^2$$

$$v_1(\xi) = -\frac{5}{2}(\lambda^2 - 4\mu)^2.$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^4$$
$$+\frac{5}{3}\lambda^2(\lambda^2 - 4\mu).$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2$$
$$-\frac{20}{3}\mu(\lambda^2 - 4\mu).$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2$$
$$+\frac{5}{6}\lambda^4 + \frac{20}{3}\lambda^2\mu + b_0$$

where

$$\xi = x - (14\mu - \frac{7}{2}\lambda^2)t$$

b_0, C_1, C_2 are arbitrary constants.

In particular, when $\lambda > 0, \mu = 0, C_1 \neq 0, C_2 = 0$, we can deduce the soliton solutions of the integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota equation as:

$$w_1(\xi) = -\frac{5}{2}\lambda^2 sech^2(\frac{\lambda\xi}{2}) + \frac{11}{12}\lambda^2$$

$$v_1(\xi) = -\frac{5}{2}\lambda^4 sech^4(\frac{\lambda\xi}{2}) + \frac{10}{3}\lambda^4 sech^2(\frac{\lambda\xi}{2}) + b_0$$

When $\lambda^2 - 4\mu < 0$

$$w_2(\xi) = -\frac{5}{2}\lambda^2 + \frac{5}{2}(4\mu - \lambda^2).$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2$$
$$+\frac{19}{3}\mu + \frac{11}{12}\lambda^2$$

$$v_2(\xi) = -\frac{5}{2}(4\mu - \lambda^2)^2.$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^4$$
$$+\frac{5}{3}\lambda^2(4\mu - \lambda^2).$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2$$
$$-\frac{20}{3}\mu(4\mu - \lambda^2).$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2$$
$$+\frac{5}{6}\lambda^4 + \frac{20}{3}\lambda^2\mu + b_0$$

where

$$\xi = x - (14\mu - \frac{7}{2}\lambda^2)t$$

b_0, C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$

$$w_3(\xi) = -\frac{5}{2}\lambda^2 + \frac{10C_2^2}{(C_1 + C_2\xi)^2} + \frac{19}{3}\mu + \frac{11}{12}\lambda^2$$

$$v_3(\xi) = -40\left[\frac{2c_2 - c_1\lambda - c_2\lambda\xi}{2(c_1 + c_2\xi)}\right]^4$$

$$-80\lambda\left[\frac{2c_2 - c_1\lambda - c_2\lambda\xi}{2(c_1 + c_2\xi)}\right]^3 + (-\frac{160}{3}\lambda^2 - \frac{80}{3}\mu).$$

$$\left[\frac{2c_2 - c_1\lambda - c_2\lambda\xi}{2(c_1 + c_2\xi)}\right]^2 (-\frac{40}{3}\lambda^3 - \frac{80}{3}\lambda\mu).$$

$$\left[\frac{2c_2 - c_1\lambda - c_2\lambda\xi}{2(c_1 + c_2\xi)}\right] + b_0$$

where

$$\xi = x - (14\mu - \frac{7}{2}\lambda^2)t$$

b_0, C_1, C_2 are arbitrary constants.

Remark: the solutions of the integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota equation mentioned above have not been reported so far by others.

4 Conclusions

The main points of the method are that assuming the solution of the ODE reduced by using the traveling wave variable as well as integrating can be expressed by an m-th degree polynomial in $(\frac{G'}{G})$, where $G = G(\xi)$ is the general solutions of a second order LODE. The positive integer m is determined by the homogeneous balance between the highest order derivatives and nonlinear terms appearing in the reduced ODE, and the coefficients of the polynomial can be obtained by solving a set of simultaneous algebraic equations resulted from the process of using the method. Compared to the methods used before, one can see that this method is direct, concise and effective. Moreover, the method can also be used to many other nonlinear equations.

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