Estimations and predictions using record statistics from the modified Weibull model

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Abstract: - This paper develops Bayesian and non-Bayesian analysis in the context of record statistics values from the modified Weibull distribution. We obtained non-Bayes estimators using MLE and Bayes estimators using the general entropy loss functions. This was done with respect to the conjugate prior for the shape parameter. Finally, Bayesian predictive density function, which is necessary to obtain bounds for predictive interval of future record, is derived. The results may be of interest in a situation where only record values are stored.

Key-Words: - Bayesian and non-Bayesian estimation, Bayesian prediction, Entropy loss function, Modified Weibull distribution, Record values, Record statistics.

1 Introduction

The Weibull distribution is one of the most popular widely used models of failure time in life testing and reliability theory. The Weibull distribution has been shown to be useful for modeling and analysis of life time data in medical, biological and engineering sciences.

Applications of the Weibull distribution in various fields are given in Zaharim et al [36], Gotoh et al [11], Shamilov et al [27], Vicen-Bueno et al [35], Niola et al [20], Green et al.[12].

A great deal of research has been done on estimating the parameters of the Weibull distribution using both classical and Bayesian techniques, and a very good summary of this work can be found in Johnson et al. [15].

Recently, Hossain and Zimmer [14] have discussed some comparisons of estimation methods for Weibull parameters using complete and censored

samples, Carrasco et al. [10] proposes a regression model considering the modified Weibull distribution and Carrasco et al [9] studies a four parameter generalization of the Weibull distribution.
This new distribution has a number of well-known lifetime special sub-models, such as the Weibull, extreme value, exponentiated Weibull, generalized Rayleigh and modified Weibull distributions, among others.
Record values and the associated statistics are of interest and importance in many areas of real life applications involving data relating to industry, economics, lifetesting, meteorology, hydrology, seismology, athletic events and mining.
Many authors have studied records and associated statistics. Among others are Resnick [26], Nagaraja [19], Ahsanullah [2], [3], Arnold et al. [4], [5], Gulati and Padgett [13], Raqab and Ahsanullah [25], Raqab [24], Sultan [32], Preda et al. [23].
The objective of this paper is to obtain and compare several techniques of estimation based on record statistics for the three unknown parameter of the modified Weibull distribution and the survival time parameters, namely the hazard and reliability functions. Section 2 contains some preliminaries.
In Section 3 we give the maximum likelihood estimators and in Section 4, the Bayes estimators of the parameters, the reliability and hazard functions are derived based on upper record values using the conjugate prior on the scale parameter and discretizing the shape parameter to a finite number of values. The estimates are obtained using general entropy loss functions.
In Section 5, we provide Bayesian prediction interval for the future record. A numerical example simulated is used for illustration, and comparison is also given in Section 6.
We conclude with a brief summary of the results in Section 7.

2 Preliminaries

Let $X_1, X_2, X_3 \ldots$ a sequence of independent and identically distributed (iid) random variables with cdf $F(x)$ and pdf $f(x)$.
Setting $Y_n = \max(X_1, X_2, X_3, \ldots, X_n)$, $n \geq 1$, we say that X_j is an upper record and denoted by $X_{U(j)}$ if $Y_j > Y_{j-1}$, $j > 1$.
Assuming that $X_{U(1)}, X_{U(2)}, X_{U(3)}, \ldots, X_{U(n)}$ are the first n upper record values arising from a sequence $\{X_i\}$ of iid modified Weibull variables with pdf

$$f(x) = \alpha x^{\beta-1}(\beta + \lambda x) e^{\lambda x - \alpha x^{\beta} e^{\lambda x}} \quad (1)$$

$$x \geq 0,\ \alpha, \beta, \lambda > 0,$$

and cdf

$$F(x) = 1 - e^{-\alpha x^{\beta} e^{\lambda x}} \quad (2)$$

$$x \geq 0,\ \alpha, \beta, \lambda > 0$$

where α is the scale parameter and β, λ are the shape parameters.
The reliability function $R(t)$, and the hazard (instantaneous failure rate) function $H(t)$ at mission time t for the modified exponential distribution are respectively given by

$$R(t) = e^{-\alpha t^{\beta} e^{\lambda t}},\ t > 0, \quad (3)$$

and

$$H(t) = \alpha t^{\beta-1}(\beta + \lambda t) \cdot e^{\lambda t},\ t > 0. \quad (4)$$

It is remarkable that most of the Bayesian inference procedures have been developed with the usual squared-error loss function, which is symmetrical and associates equal importance to the losses due to overestimation and underestimation of equal magnitude.
However, such a restriction may be impractical in most situations of practical importance. For example, in the estimation of reliability and failure rate functions, an overestimation is usually much more serious than an underestimation.
In this case, the use of symmetrical loss function might be inappropriate as also emphasized by Basu and Ebrahimi in [6].
A useful asymmetric loss known as the LINEX loss function (linear-exponential) was introduced by Varian in [34] and has been widely used by several authors such Basu and Ebrahimi [6], Calabria and Pulcini [7], Soliman [31], Singh et al [28], and Ahmadi et al. [1].
Despite the flexibility and popularity of the Linex loss function for the location parameter estimation, it appears to be unsuitable for the scale parameter and other quantities (c.f. Basu and Ebrahimi [6], Parsian and Sanjari Farsipour [21]).
Keeping these points in mind, Basu and Ebrahimi in [6] defined a modified Linex loss. A suitable alternative to the modified Linex loss is the General Entropy loss proposed in Calabria and Pulcini [8]. The General Entropy loss (GEL) is defined as:

$$L_{BE}(\phi^*, \phi) \propto \left(\frac{\phi^*}{\phi}\right)^c - c \log\left(\frac{\phi^*}{\phi}\right) - 1 \quad (5)$$

where ϕ^* is an estimate of parameter ϕ.
The Bayes estimate ϕ^*_{BE} of under general entropy loss (GEL) is given as

$$\phi^*_{BE} = \left[E_{\phi}\{\phi^{-c}\}\right]^{-\frac{1}{c}} \quad (6)$$

provided that $E_\phi\{\phi^{-c}\}$ exists and is finite. It can be shown that, when *c=1*, the Bayes estimate (6) coincides with the Bayes estimate under the weighted squared-error loss function. Similarly, when *c*=-1 the Bayes estimate (6) coincides with the Bayes estimate under squared error loss function.

3 Non-Bayesian estimation (MLE)

The joint density function of the first n upper record values $x \equiv (x_{U(1)}, x_{U(2)}, \ldots, x_{U(n)})$ is given by

$$f_{1,2,\ldots,n}(x_{U(1)}, x_{U(2)}, \ldots, x_{U(n)}) = f(x_{U(n)})\prod_{i=1}^{n-1} H(x_{U(i)}) =$$

$$= f(x_{U(n)})\prod_{i=1}^{n-1}\frac{f(x_{U(i)})}{1-F(x_{U(i)})}$$

$$0 \le x_{U(1)} < x_{U(2)} < \ldots < x_{U(n)} < \infty$$

where $f(x)$, and $F(x)$ are given, respectively, by (1) and (2) after replacing *x* by $x_{U(i)}$. The likelihood function based on the *n* upper record values *x* is given by

$$\ell(\alpha,\beta,\lambda \mid x) = \alpha^n \cdot e^{-\alpha x_{U(n)}^{\beta} e^{\lambda x_{U(n)}}} \cdot$$

$$\cdot \prod_{i=1}^{n}\{x_{U(i)}^{\beta-1}(\beta+\lambda x_{U(i)}) \cdot \exp(\lambda x_{U(i)})\} \qquad (7)$$

and the log-likelihood function may be written as

$$L(\alpha,\beta,\lambda \mid x) = n\ln\alpha - \alpha x_{U(n)}^{\beta}\exp(\lambda x_{U(n)}) + \lambda \cdot \sum_{i=1}^{n} x_{U(i)} +$$

$$+(\beta-1)\sum_{i=1}^{n}\ln(x_{U(i)}) + \sum_{i=1}^{n}\ln\left(\beta+\lambda x_{U(i)}\right) \qquad (8)$$

Assuming that the shape parameters β and λ are known, the maximum likelihood estimator (MLE), $\hat{\alpha}_{ML}$ of the scale parameter α can be shown by using (8) to be

$$\hat{\alpha}_{ML} = \frac{n}{x_{U(n)}^{\beta}} e^{-\lambda x_{U(n)}}. \qquad (9)$$

If only the shape parameter λ is known, the MLEs of the scale parameter α and the shape parameter β, $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$, can be obtained as solutions of equations:

$$\begin{cases} \dfrac{n}{\alpha} - x_{U(n)}^{\beta} e^{\lambda x_{U(n)}} = 0 \\ -\alpha x_{U(n)}^{\beta} e^{\lambda x_{U(n)}} \ln x_{U(n)} + \sum_{i=1}^{n} \ln x_{U(i)} + \\ \qquad + \sum_{i=1}^{n} \dfrac{1}{\beta+\lambda x_{U(i)}} = 0 \end{cases} \qquad (10)$$

which maybe solve using for example, Newton-Raphson iteration scheme.

If $0 < \lambda < \dfrac{\sum_{i=1}^{n}\frac{1}{x_{U(i)}}}{n\ln x_{U(n)} - \sum_{i=1}^{n}\ln x_{U(i)}}$ then the maximum likelihood estimate of β, $\hat{\beta}_{ML}$ is the (unique) solution of the equation in β obtained by eliminating α in (10). Then the maximum likelihood estimate of α, $\hat{\alpha}_{ML}$, is

$$\hat{\alpha}_{ML} = \frac{n}{x_{U(n)}^{\hat{\beta}_{ML}}} e^{-\lambda x_{U(n)}} \qquad (11)$$

If only the shape parameter β is known, the MLEs of the scale parameter α and the shape parameter λ, $\hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$, can be obtained as solutions of equations:

$$\begin{cases} \dfrac{n}{\alpha} - x_{U(n)}^{\beta} e^{\lambda x_{U(n)}} = 0 \\ -\alpha x_{U(n)}^{\beta+1} e^{\lambda x_{U(n)}} + \sum_{i=1}^{n} x_{U(i)} + \\ \qquad + \sum_{i=1}^{n} \dfrac{x_{U(i)}}{\beta+\lambda x_{U(i)}} = 0 \end{cases} \qquad (12)$$

which maybe solve using a iteration scheme. Then, if $0 < \beta < \dfrac{\sum_{i=1}^{n}\frac{1}{x_{U(i)}}}{\sum_{i=1}^{n}\frac{1}{x_{U(i)}} - nx_{U(n)}}$, we obtain the MLE of λ, $\hat{\lambda}_{ML}$, the (unique) solution of the equation in λ obtained by eliminating α in (12). Then the maximum likelihood estimate of α, $\hat{\alpha}_{ML}$, is

$$\hat{\alpha}_{ML} = \frac{n}{x_{U(n)}^{\beta}} e^{-\hat{\lambda}_{ML} x_{U(n)}} \qquad (13)$$

If the three parameters α, β and λ are unknown, using the first likelihood equation (of α), we obtain α, and, after replacing α in (8), we get

$$\widetilde{L}(\beta,\lambda) = n\ln n - n\lambda x_{U(n)} - n\beta\ln(x_{U(n)}) + \lambda \cdot \sum_{i=1}^{n} x_{U(i)} +$$

$$-n + (\beta-1)\sum_{i=1}^{n}\ln x_{U(i)} + \sum_{i=1}^{n}\ln\left(\beta+\lambda x_{U(i)}\right) \qquad (14)$$

The Hessian of $\widetilde{L}$ is negative defined matrix and then $\widetilde{L}$ is a concave application on the admissible region (β>0, λ>0).

Then, the Newton-Raphson algorithm converges to the global optimum, assuming that it does not go outside the admissible region. The Newton-Raphson algorithm requires initial parameter estimates. Different types of initialization are discussed in Ronning (1989) and Wicker (2008).

However, in this case we can use the likelihood equations for this $\widetilde{L}$. After some transformations, we get

$$\sum_{i=1}^{n}\left(n+\lambda\left(x_{U(i)}\left(n\ln x_{U(n)}-\sum_{j=1}^{n}\ln x_{U(j)}\right)-\left(nx_{U(n)}-\sum_{j=1}^{n}x_{U(j)}\right)\right)\right)^{-1}=1 \qquad (15)$$

which, again, maybe solve using a iteration scheme. We note that this equation has a solution (unique) only if

$$\frac{\sum_{i=1}^{n}x_{U(i)}}{n}\left(n\ln x_{U(n)}-\sum_{i=1}^{n}\ln x_{U(i)}\right)>1$$

and

$$\frac{\sum_{i=1}^{n}\frac{1}{x_{U(i)}}}{n}\left(nx_{U(n)}-\sum_{i=1}^{n}x_{U(i)}\right)>n\ln x_{U(n)}-\sum_{i=1}^{n}\ln x_{U(i)}$$

So first, we get $\hat{\lambda}_{ML}$, the MLE of λ, and then $\hat{\beta}_{ML}$ and $\hat{\alpha}_{ML}$, the MLEs of α and β

$$\hat{\beta}_{ML}=\frac{n-\hat{\lambda}_{ML}\left(nx_{U(n)}-\sum_{i=1}^{n}x_{U(i)}\right)}{n\ln x_{U(n)}-\sum_{i=1}^{n}\ln x_{U(i)}} \qquad (16)$$

$$\hat{\alpha}_{ML}=\frac{n}{x_{U(n)}^{\hat{\beta}_{ML}}}e^{-\hat{\lambda}_{ML}x_{U(n)}} \qquad (17)$$

Finally, the corresponding MLE's $\hat{R}_{ML}(t)$, and $\hat{H}_{ML}(t)$ of $R(t)$ and $H(t)$ are given by (3) and (4) after replacing α, β and λ by $\hat{\alpha}_{ML}$, $\hat{\beta}_{ML}$ and $\hat{\lambda}_{ML}$, respectively.

4 Bayes estimation

In this section, considering the general entropy loss function, we estimate α, β and λ, and *R(t)* and *H(t)*.

4.1 Known shape parameters λ and β

Under the assumption that the shape parameter λ is known, we assume a gamma $\gamma(a,b)$ conjugate prior for α as

$$\pi(\alpha)=\frac{b^{a}\alpha^{a-1}e^{-b\alpha}}{\Gamma(a)},\ \alpha>0,\ a,b>0 \qquad (18)$$

Applying Bayes theorem, we obtain from (6), the likelihood function, and (11), the prior density, the posterior density of α in the form

$$\pi^{*}(\alpha\,|\,x)=\frac{\nu^{(n+a)}\alpha^{(n+a-1)}e^{-\alpha\nu}}{\Gamma(n+a)}$$

$$\nu=b+x_{U(n)}^{\beta}\cdot e^{\lambda x_{U(n)}} \qquad (19)$$

where $\Gamma(\cdot)$ is gamma function.

Theorem 1: *If shape parameters λ and β are known, under the general entropy loss function, the Bayes estimators for α, R(t) and H(t) are given by*

$$\widetilde{\alpha}_{BE}=\frac{1}{\nu}\left(\frac{\Gamma(n+a-c)}{\Gamma(n+a)}\right)^{-\frac{1}{c}}$$

$$\widetilde{R}_{BE}(t)=\left(1-\frac{ct^{\beta}e^{\lambda t}}{\nu}\right)^{\frac{n+a}{c}}$$

and respectively

$$\widetilde{H}_{BE}(t)=\frac{t^{\beta-1}(\beta+\lambda t)e^{\lambda t}}{\nu}\left(\frac{\Gamma(n+a-c)}{\Gamma(n+a)}\right)^{-\frac{1}{c}}$$

Proof: The Bayes estimator for α, $\widetilde{\alpha}_{BE}$, under the general entropy loss function, is:

$$\widetilde{\alpha}_{BE}=\left(\int_{0}^{\infty}\alpha^{-c}\pi^{*}(\alpha\,|\,x)d\alpha\right)^{-\frac{1}{c}}=$$

$$=\left(\int_{0}^{\infty}\alpha^{-c}\frac{\nu^{(n+a)}\alpha^{(n+a-1)}e^{-\alpha\nu}}{\Gamma(n+a)}d\alpha\right)^{-\frac{1}{c}}=\frac{1}{\nu}\left(\frac{\Gamma(n+a-c)}{\Gamma(n+a)}\right)^{-\frac{1}{c}}$$

Similarly, the Bayes estimators for *R(t)* and *H(t)* are given, respectively, by

$$\widetilde{R}_{BE}(t)=\left(\int_{0}^{\infty}\left(R(t)\right)^{-c}\pi^{*}(\alpha\,|\,x)d\alpha\right)^{-\frac{1}{c}}=$$

$$=\left(\int_{0}^{\infty}\left(e^{-\alpha t^{\beta}e^{\lambda t}}\right)^{-c}\frac{\nu^{(n+a)}\alpha^{(n+a-1)}e^{-\alpha\nu}}{\Gamma(n+a)}d\alpha\right)^{-\frac{1}{c}}=$$

$$=\left(1-\frac{ct^{\beta}e^{\lambda t}}{\nu}\right)^{\frac{n+a}{c}}$$

and

$$\widetilde{H}_{BE}(t)=\left(\int_{0}^{\infty}\left(H(t)\right)^{-c}\pi^{*}(\alpha\,|\,x)d\alpha\right)^{-\frac{1}{c}}=$$

$$=\left(\int_{0}^{\infty}\left(\alpha t^{\beta-1}(\beta+\lambda t)e^{\lambda t}\right)^{-c}\frac{\nu^{(n+a)}\alpha^{(n+a-1)}e^{-\alpha\nu}}{\Gamma(n+a)}d\alpha\right)^{-\frac{1}{c}}=$$

$$=\frac{t^{\beta-1}(\beta+\lambda t)e^{\lambda t}}{\nu}\left(\frac{\Gamma(n+a-c)}{\Gamma(n+a)}\right)^{-\frac{1}{c}}\ \square.$$

4.2 Known shape parameter λ

It is well known that, for the Bayes estimators, the performance depends on the form of the prior distribution and the loss function assumed. Under the assumption that both the parameters α and β are unknown, no analogous reduction via sufficiency is possible for the likelihood corresponding to a sample of records from the modified Weibull density (1). Also, specifying a general joint prior for α and β leads to computational complexities. In trying to solve this problem and simplify the Bayesian analysis, we use Soland's method. Soland in [30] considered a family of joint prior distributions that places continuous distributions on the scale parameter and discrete distributions on the shape parameter.

We assume that the shape parameter β is restricted to a finite number of values $\beta_1, \beta_2, ..., \beta_k$ with respective prior probabilities $\eta_1, \eta_2, ..., \eta_k$ such that $0 \le \eta_j \le 1$ and $\sum_{j=1}^{k} \eta_j = 1$ [i.e. $P(\beta = \beta_j) = \eta_j$].

Further, suppose that conditional upon , $\beta = \beta_j$ has a natural conjugate prior with distribution having a gamma (a_j, b_j) with density

$$\pi(\alpha \mid \beta = \beta_j) = \frac{b_j^{a_j} \alpha^{a_j - 1} e^{-b_j \alpha}}{\Gamma(a_j)}, \qquad (20)$$

$$\alpha > 0, a_j, b_j > 0$$

where a_j and b_j are chosen so as to reflect prior beliefs on α given that $\beta = \beta_j$. Then given the set of the first n upper record values x, the conditional posterior pdf of α is given by

$$\pi^*(\alpha \mid \beta = \beta_j, \mathrm{x}) = \frac{B_j^{A_j} \alpha^{A_j - 1} e^{-B_j \alpha}}{\Gamma(A_j)}, \qquad (21)$$

$$\alpha > 0, A_j, B_j > 0$$

which is a gamma (A_j, B_j), where

$$A_j = a_j + n \text{ and } B_j = b_j + x_{U(n)}^{\beta_j} e^{\lambda x_{U(n)}} \qquad (22)$$

The marginal posterior probability distribution of β_j obtained by applying the discrete version of Bayes' theorem, is given by

$$P_{j(\beta)} = A_{(\beta)} \int_0^{\infty} \frac{\eta_j b_j^{a_j} \alpha^{n + a_j - 1} u_{j(\beta)}}{\Gamma(a_j)} e^{-\alpha\left(b_j + x_{U(n)}^{\beta_j} e^{\lambda x_{U(n)}}\right)} d\alpha$$

$$P_{j(\beta)} = A_{(\beta)} \frac{\eta_j b_j^{a_j} u_{j(\beta)} \Gamma(A_j)}{\Gamma(a_j) B_j^{A_j}} \qquad (23)$$

where $A_{(\beta)}$ is a normalized constant given by

$$\left(A_{(\beta)}\right)^{-1} = \sum_{j=1}^{k} \frac{\eta_j b_j^{a_j} u_{j(\beta)} \Gamma(A_j)}{\Gamma(a_j) B_j^{A_j}}$$

and $u_{j(\beta)} = \prod_{i=1}^{n} x_{U(i)}^{\beta_j - 1} \left(\beta_j + \lambda x_{U(i)}\right) e^{\lambda x_{U(i)}}$.

Under the general entropy loss function (5), the Bayes estimator of ϕ_{BE}^* a function $\phi(a, b)$ is given by (6).

Theorem 2: *If shape parameters λ is known, under the general entropy loss function, the Bayes estimators for α, β, R(t) and H(t) are given by*

$$\tilde{\alpha}_{BE} = \left(\sum_{j=1}^{k} P_{j(\beta)} B_j^c \frac{\Gamma(A_j - c)}{\Gamma(A_j)} \right)^{-\frac{1}{c}}$$

$$\tilde{\beta}_{BE} = \left(\sum_{j=1}^{k} P_{j(\beta)} \beta_j^{-c} \right)^{-\frac{1}{c}}$$

$$\tilde{R}_{BE}(t) = \left(\sum_{j=1}^{k} P_{j(\beta)} \left(1 - \frac{ct^{\beta_j} e^{\lambda t}}{B_j} \right)^{-A_j} \right)^{-\frac{1}{c}}$$

and respectively

$$\tilde{H}_{BE}(t) = \left(\sum_{j=1}^{k} P_{j(\beta)} B_j^c \left(t^{\beta_j - 1} (\beta_j + \lambda t) e^{\lambda t} \right)^{-c} \frac{\Gamma(A_j - c)}{\Gamma(A_j)} \right)^{-\frac{1}{c}}$$

Proof: The Bayes estimator for the scale parameter α is given by

$$\tilde{\alpha}_{BE} = \left(\int_0^{\infty} \sum_{j=1}^{k} P_{j(\beta)} \alpha^{-c} \pi^*(\alpha \mid x) d\alpha \right)^{-\frac{1}{c}} =$$

$$= \left(\int_0^{\infty} \sum_{j=1}^{k} P_{j(\beta)} \alpha^{-c} \frac{B_j^{A_j} \alpha^{A_j - 1} e^{-B_j \alpha}}{\Gamma(A_j)} \right)^{-\frac{1}{c}} =$$

$$= \left(\sum_{j=1}^{k} P_{j(\beta)} B_j^c \frac{\Gamma(A_j - c)}{\Gamma(A_j)} \right)^{-\frac{1}{c}}$$

and the Bayes estimator for β is given by

$$\tilde{\beta}_{BE} = \left(\sum_{j=1}^{k} P_{j(\beta)} \beta_j^{-c} \right)^{-\frac{1}{c}}.$$

Similarly, the Bayes estimator for the reliability function *R(t)* is given by

$$\tilde{R}_{BE}(t) = \left(\int_0^{\infty} \sum_{j=1}^{k} P_{j(\beta)} R(t)^{-c} \pi^*(\alpha \mid x) d\alpha \right)^{-\frac{1}{c}} =$$

$$= \left(\int_0^{\infty} \sum_{j=1}^{k} P_{j(\beta)} \left(e^{-\alpha t^{\beta_j} e^{\lambda t}} \right)^{-c} \frac{B_j^{A_j} \alpha^{A_j - 1} e^{-B_j \alpha}}{\Gamma(A_j)} d\alpha \right)^{-\frac{1}{c}} =$$

$$= \left(\sum_{j=1}^{k} P_{j(\beta)} \left(1 - \frac{ct^{\beta_j} e^{\lambda t}}{B_j} \right)^{-A_j} \right)^{-\frac{1}{c}}$$

where *R(t)* is given in (3). The Bayes estimator for the hazard function *H(t)* is obtained as

$$\widetilde{H}_{BE}(t)=\left(\int_0^\infty \sum_{j=1}^k P_{j(\beta)} H(t)^{-c}\pi^*(\alpha\mid x)d\alpha\right)^{-\frac{1}{c}}=$$

$$=\left(\int_0^\infty \sum_{j=1}^k P_{j(\beta)}\left(\alpha t^{\beta_j-1}(\beta_j+\lambda t)e^{\lambda t}\right)\right)^{-c}\frac{B_j^{A_j}\alpha^{A_j-1}e^{-B_j\alpha}}{\Gamma(A_j)}d\alpha\Bigg)^{-\frac{1}{c}}$$

$$=\left(\sum_{j=1}^k P_{j(\beta)} B_j^c\left(t^{\beta_j-1}(\beta_j+\lambda t)e^{\lambda t}\right)^{-c}\frac{\Gamma(A_j-c)}{\Gamma(A_j)}\right)^{-\frac{1}{c}}.\square$$

To implement the calculations in this section, it is first necessary to elicit the values of (β_j,η_j) and the hyperparameters (a_j,b_j) in the conjugate prior (20), for j=1,2,...,k. The former pairs of values are fairly straightforward to specify, but for (a_j,b_j) it is necessary to condition prior beliefs about α on each β_j in turn and this can be difficult in practice.

An alternative method for obtaining the values (a_j,b_j) can be based on the expected value of the reliability function *R(t)* conditional on $\beta=\beta_j$, which is given using (20) by

$$E_{\alpha|\lambda_j}[R(t)\mid\beta=\beta_j]=\int_0^\infty\left(e^{-\alpha t^{\beta_j}e^{\lambda t}}\right)^{-c}\frac{b_j^{a_j}\alpha^{a_j-1}e^{-b_j\alpha}}{\Gamma(a_j)}d\alpha$$

$$E_{\alpha|\lambda_j}[R(t)\mid\beta=\beta_j]=\left(1+\frac{t^{\beta_j}\cdot e^{\lambda t}}{b_j}\right)^{-a_j} \qquad (24)$$

Now, suppose that prior beliefs about the lifetime distribution enable one to specify two values $\left((R(t_1),t_1),(R(t_2),t_2)\right)$. Thus, for these two prior values $R(t_1)=t_1$ and $R(t_2)=t_2$, the values of a_j and b_j for each value β_j, can be obtained numerically from (24). If there are no prior beliefs, a nonparametric procedure can be used to estimate the corresponding two different values of *R(t)*, see Martz [16].

The case of known shape parameter β is similar with the case of known shape parameter λ.

4.3 Unknown scale parameter α and shape parameters β and λ

We assume that the shape parameters β and λ are restricted to a finite number of values $\beta_1,\beta_2,...,\beta_k$ and respective $\lambda_1,\lambda_2,...,\lambda_m$ with prior probabilities $\eta_1,\eta_2,...,\eta_k$ and $\xi_1,\xi_2,...,\xi_m$ such that $0\le\eta_j,\xi_i\le 1$, $\sum_{j=1}^k\eta_j=1$ and $\sum_{i=1}^m\xi_i=1$ [i.e. $P(\beta=\beta_j)=\eta_j$ and $P(\lambda=\lambda_i)=\xi_i$]. Further, suppose that conditional upon , $\beta=\beta_j$ and $\lambda=\lambda_i$ has a natural conjugate prior with distribution having a gamma (g_{ij},h_{ij}) with density

$$\pi(\alpha\mid\beta=\beta_j,\lambda=\lambda_i)=\frac{h_{ij}^{g_{ij}}\alpha^{g_{ij}-1}e^{-h_{ij}\alpha}}{\Gamma(g_{ij})}, \qquad (25)$$

$$\alpha>0,\ g_{ij},h_{ij}>0$$

where g_{ij} and h_{ij} are chosen so as to reflect prior beliefs on α given that $\beta=\beta_j$ and $\lambda=\lambda_i$, $i=\overline{1,m}$, $j=\overline{1,k}$. Then given the set of the first n upper record values *x*, the conditional posterior pdf of α is given by

$$\pi^*(\alpha\mid\beta=\beta_j,\lambda=\lambda_i,x)=\frac{H_{ij}^{G_{ij}}\alpha^{G_{ij}-1}e^{-H_{ij}\alpha}}{\Gamma(G_{ij})}, \qquad (26)$$

$$\alpha>0,\ G_{ij},H_{ij}>0$$

which is a gamma (G_{ij},H_{ij}), where

$$G_{ij}=g_{ij}+n \text{ and } H_{ij}=h_{ij}+x_{U(n)}^{\beta_j}e^{\lambda_i x_{U(n)}} \qquad (27)$$

The marginal posterior probability distribution of β_j and λ_i obtained by applying the discrete version of Bayes' theorem, is given by

$$P_{ij(\beta\lambda)}=A_{(\beta\lambda)}\frac{\eta_j\xi_i h_{ij}^{g_{ij}}u_{ij(\beta\lambda)}\Gamma(G_{ij})}{\Gamma(g_{ij})H_{ij}^{G_{ij}}} \qquad (28)$$

where $A_{(\beta\lambda)}$ is a normalized constant given by

$$\left(A_{(\beta\lambda)}\right)^{-1}=\sum_{j=1}^k\sum_{i=1}^m\frac{\eta_j\xi_i h_{ij}^{g_{ij}}u_{ij(\beta\lambda)}\Gamma(G_{ij})}{\Gamma(g_{ij})H_{ij}^{G_{ij}}}$$

and $u_{ij(\beta\lambda)}=\prod_{z=1}^n x_{U(z)}^{\beta_j-1}\left(\beta_j+\lambda_i x_{U(z)}\right)e^{\lambda_i x_{U(z)}}$.

The Bayes estimator $\widetilde{\alpha}_{BE}$, $\widetilde{\beta}_{BE}$ and $\widetilde{\lambda}_{BE}$ of α, β and λ respectively, under the general entropy loss function are obtained by using the posterior pdfs (26) and (28).

Using the line from Theorem 2 we get:

Theorem 3: *If all parameters are unknown, under the general entropy loss function, the Bayes estimators for α, β, λ, R(t) and H(t) are given by*

$$\widetilde{\alpha}_{BE}=\left(\sum_{j=1}^k\sum_{i=1}^m P_{ij(\beta\lambda)}H_{ij}\frac{\Gamma(G_{ij}-c)}{\Gamma(G_{ij})}\right)^{-\frac{1}{c}}$$

$$\widetilde{\beta}_{BE}=\left(\sum_{j=1}^k\sum_{i=1}^m P_{ij(\beta\lambda)}\beta_j^{-c}\right)^{-\frac{1}{c}}$$

$$\widetilde{\lambda}_{BE}=\left(\sum_{j=1}^k\sum_{i=1}^m P_{ij(\beta\lambda)}\lambda_i^{-c}\right)^{-\frac{1}{c}}$$

$$\widetilde{R}_{BE}(t)=\left(\sum_{j=1}^{k}\sum_{i=1}^{m}P_{ij(\beta\lambda)}\left(1-\frac{ct^{\beta_j}e^{\lambda_i t}}{H_{ij}}\right)^{-G_{ij}}\right)^{-\frac{1}{c}}$$

and respectively

$$\widetilde{H}_{BE}(t)=\left(\sum_{j=1}^{k}\sum_{i=1}^{m}P_{ij(\beta)}H_{ij}^{c}\left(t^{\beta_j-1}(\beta_j+\lambda_i t)e^{\lambda_i t}\right)^{-c}\cdot\right.$$

$$\left.\cdot\frac{\Gamma(G_{ij}-c)}{\Gamma(G_{ij})}\right)^{-\frac{1}{c}}.$$

To implement the calculations in this section, it is first necessary to elicit the values of (β_j,η_j), (λ_i,ξ_i) and the hyperparameters (g_{ij},h_{ij}) in the conjugate prior (25), for $i=\overline{1,m}$, $j=\overline{1,k}$. The former pairs of values are fairly straightforward to specify, but for (g_{ij},h_{ij}) it is necessary to condition prior beliefs about α on each β_j and β_i in turn, and this can be difficult in practice.

An alternative method for obtaining the values (g_{ij},h_{ij}) can be based on the expected value of the reliability function *R(t)* conditional on $\beta=\beta_j$ and $\lambda=\lambda_i$ which is given using (25) by

$$E_{\alpha|\beta_j\lambda_j}[R(t)\mid\beta=\beta_j,\lambda=\lambda_i]=\left(1+\frac{t^{\beta_j}\cdot e^{\lambda_i t}}{h_{ij}}\right)^{-g_{ij}} \qquad (29)$$

Now, suppose that prior beliefs about the lifetime distribution enable one to specify two values $((R(t_1),t_1),(R(t_2),t_2))$. Thus, for these two prior values $R(t_1)=t_1$ and $R(t_2)=t_2$, the values of g_{ij} and h_{ij} for each value β_j and λ_i can be obtained numerically from (29). If there are no prior beliefs, a nonparametric procedure can be used to estimate the corresponding two different values of *R(t)*, see Martz [16].

5 Bayesian prediction

In the context of prediction of the future record observations, the prediction intervals provide bounds to contain the results of a future record, based upon the results of the previous record observed from the same sample. This section is devoted for deriving the Bayes predictive density function, which is necessary to obtain bounds for predictive interval.

Suppose that we observe only the first n upper record observations $x\equiv\left(x_{U(1)},x_{U(2)},...,x_{U(n)}\right)$ and the goal is to obtain the Bayes predictive interval for the *s*-th future upper record, where $1\le n<s$.

Let $Y=X_{U(s)}$ be *L* the *s*-th upper record value. The conditional density function of *Y* for given $x_n=x_{U(n)}$ is given, Ahsanullah [3], by

$$f(y\mid x_n;\theta)=\frac{[\omega(y)-\omega(x_n)]^{s-n-1}}{\Gamma(s-n)}\frac{f(y)}{1-F(x_n)} \qquad (30)$$

where $\omega(\cdot)=-\ln[1-F(\cdot)]$.

Upon using the modified Weibull distribution, with pdf given by (1), the conditional density function (30) is given by

$$f(y|x_n;\alpha,\beta,\lambda)=\frac{[\alpha\gamma(y)]^{s-n-1}\alpha y^{\beta-1}(\beta+\lambda x)e^{\lambda y-\alpha\gamma(y)}}{\Gamma(s-n)} \qquad (31)$$

where $\gamma(y)=y^{\beta}e^{\lambda y}-x_n^{\beta}e^{\lambda x_n}$.

The Bayes predictive density function of y given the observed record *x* is given by

$$f(y|x)=\int_0^{\infty}f(y|x_n;\alpha,\beta,\lambda)\sum_{j=1}^{k}\sum_{i=1}^{m}P_{ij(\beta\lambda)}\cdot$$

$$\cdot\pi^{*}(\alpha\mid\beta=\beta_j,\lambda=\lambda_i)\mathrm{d}\alpha \qquad (32)$$

From (31), (25) and (27) into (32), we get

$$f(y|x)=\sum_{j=1}^{k}\sum_{i=1}^{m}P_{ij(\beta\lambda)}\frac{H_{ij}^{G_{ij}}y^{\beta_j-1}e^{\lambda_j y}(\beta_j+\lambda_i y)}{B(g_{ij}+n,s-n)}\cdot$$

$$\cdot\frac{[\gamma_{ij}(y)]^{s-n-1}}{w_{ij}(y)^{g_{ij}+s}} \qquad ....(33)$$

where H_{ij} and G_{ij} are as given by (26), $B(\cdot,\cdot)$ is the beta function of the second kind, and

$$\gamma_{ij}(y)=y^{\beta_j}e^{\lambda_i y}-x_n^{\beta_j}e^{\lambda_i x_n} \text{ and } w_{ij}=h_{ij}-y^{\beta_j}e^{\lambda_i y} \qquad (34)$$

It follows that the lower and upper 100τ% prediction bounds for $Y=X_{U(s)}$, given the past record values *x*, can be derived using the predictive survival function defined by

$$f(Y\ge y_0|x)=\int_{y_0}^{\infty}f(y|x)dy$$

i.e.

$$f(Y\ge y_0|x)=\sum_{j=1}^{k}\sum_{i=1}^{m}P_{ij(\beta\lambda)}\frac{P_{ij(\beta\lambda)}}{B(g_{ij}+n,s-n)}\cdot$$

$$\cdot IncB(s-n,g_{ij}+n,\delta_{ij}) \qquad (35)$$

$$\delta_{ij}=\frac{y_0^{\beta_j}e^{\lambda_i y_0}-x_n^{\beta_j}e^{\lambda_i x_n}}{H_{ij}}$$

and $IncB(z_1,z_2,\xi)$ is the incomplete beta function.

Iterative numerical methods are required to obtain the lower and upper 100τ% prediction bounds for $Y=X_{U(s)}$ by finding from (34), using $\Pr[LL(x)<Y<UL(x)]=\tau$ where *LL(x)* and *UL(x)* are the lower and upper limits, respectively, satisfying

$$\Pr[Y > LL(x) \mid x] = \frac{1+\tau}{2} \text{ and}$$

$$\Pr[Y > UL(x) \mid x] = \frac{1-\tau}{2} \qquad (36)$$

It is often important to predict the first unobserved record value $X_{U(n+1)}$; the predictive survival function for $Y_{n+1} = X_{U(n+1)}$ is given from (34) by setting $s=n+1$ as

$$f(Y_{n+1} \geq y_0 \mid x) = \sum_{j=1}^{k}\sum_{i=1}^{m} P_{ij(\beta\lambda)}\left(1+\delta_{ij}\right)^{-(g_{ij}+n)}$$

Iterative numerical methods are also required to obtain prediction bounds for Y_{n+1}.

6 Applications

To illustrate the estimation techniques developed in this paper, we consider the real data set which was used in Tong (1990).

The data are daily precipitation recorded in Hveravellir (mm) between 1 Jan 1972 and 31 Dec 1974.

We consider the following eight upper record values observed in dataset:

8.1, 9.2, 19.4, 42.2, 54, 60.3, 77.7, 79.3

The hyperparameters of the gamma prior (29) and the values of λ_i and β_j are derived by the following steps:

1. based on the above seven upper records, we estimate two values of the reliability function using a nonparametric procedure
$$R\left(t_i = X_{U(i)}\right) = \frac{n-i+0.625}{n+0.25},$$
with (n = 7), see Martz and Waller (1982). So, we assume that the reliability for times $t_1 = 9.2$, and $t_2 = 42.2$ are, respectively, $R(t_1) = 0.803$, and $R(t_2) = 0.561$;
2. concerning the value of the MLE of the parameters β and λ obtained from (15) and (16), ($\hat{\beta}_{ML} = 0.872$ and $\hat{\lambda}_{ML} = 0.006$), we assume that β takes the values: 0.6(0.05)1.05 and λ takes the values: 0.005(0.0005)0.0095;
3. the two prior values obtained in step 1 are substituted into (29), where h_{ij} and g_{ij} are solved numerically for each given β_j and λ_i, $i=\overline{1,m}$, $j=\overline{1,k}$ using the Newton-Raphson method.

Tables 1 and 2 gives the values of the hyperparameters and the posterior probabilities derived for each β_j and λ_i. The ML estimates $(\cdot)_{ML}$, and the Bayes estimates $(\cdot)_{BE}$ of α, β, λ, $R(t)$ and $H(t)$, are computed and the results are displayed in Table 3.

Tab 1: Prior information and hyper parameter values.

i	1	2	3	4	5
β	0.6	0.65	0.7	0.75	0.8
λ	0.005	0.0055	0.006	0.0065	0.007
η	0.100	0.100	0.100	0.100	0.100
ξ	0.100	0.100	0.100	0.100	0.100
i	6	7	8	9	10
β	0.85	0.9	0.95	1.0	1.05
λ	0.0075	0.008	0.0085	0.009	0.0095
η	0.100	0.100	0.100	0.100	0.100
ξ	0.100	0.100	0.100	0.100	0.100

g	1	2	3	4	5
1	1.715	1.498	1.330	1.198	1.090
2	1.034	0.954	0.886	0.828	0.777
3	0.749	0.708	0.671	0.638	0.609
4	0.592	0.567	0.544	0.523	0.504
5	0.493	0.476	0.460	0.445	0.432
6	0.424	0.411	0.400	0.389	0.379
7	0.373	0.364	0.355	0.347	0.339
8	0.334	0.327	0.320	0.313	0.307
9	0.303	0.297	0.292	0.286	0.281
10	0.278	0.273	0.269	0.264	0.260
g	6	7	8	9	10
1	1.001	0.927	0.863	0.807	0.759
2	0.732	0.693	0.658	0.627	0.598
3	0.582	0.558	0.536	0.515	0.497
4	0.486	0.469	0.454	0.440	0.427
5	0.419	0.407	0.396	0.385	0.375
6	0.369	0.360	0.352	0.344	0.336
7	0.331	0.324	0.317	0.311	0.305
8	0.301	0.295	0.290	0.284	0.279
9	0.276	0.272	0.267	0.263	0.258
10	0.256	0.252	0.248	0.244	0.241

h	1	2	3	4	5
1	29.065	25.250	22.323	20.005	18.124
2	18.741	17.213	15.916	14.801	13.833
3	14.545	13.685	12.921	12.238	11.623
4	12.333	11.760	11.239	10.761	10.322
5	11.015	10.594	10.203	9.841	9.503
6	10.179	9.848	9.538	9.246	8.971
7	9.637	9.364	9.105	8.861	8.629
8	9.289	9.055	8.833	8.621	8.419
9	9.078	8.873	8.676	8.488	8.308
10	8.970	8.785	8.608	8.437	8.273

h	6	7	8	9	10
1	16.568	15.258	14.141	13.176	12.335
2	12.983	12.232	11.564	10.965	10.425
3	11.068	10.563	10.102	9.680	9.291
4	9.918	9.544	9.197	8.875	8.574
5	9.187	8.892	8.615	8.354	8.109
6	8.712	8.468	8.237	8.018	7.810
7	8.408	8.199	7.999	7.809	7.628
8	8.226	8.041	7.865	7.695	7.533
9	8.135	7.968	7.809	7.655	7.507
10	8.115	7.963	7.816	7.674	7.537

Tab. 2: Posterior probabilities.

P	1	2	3	4	5
1	0.0008	0.0012	0.0017	0.0022	0.0029
2	0.0027	0.0034	0.0042	0.0050	0.0058
3	0.0054	0.0062	0.0070	0.0078	0.0085
4	0.0080	0.0087	0.0094	0.0101	0.0107
5	0.0102	0.0107	0.0113	0.0117	0.0121
6	0.0116	0.0120	0.0123	0.0126	0.0127
7	0.0124	0.0126	0.0127	0.0128	0.0128
8	0.0126	0.0127	0.0126	0.0125	0.0124
9	0.0124	0.0123	0.0121	0.0119	0.0117
10	0.0118	0.0116	0.0113	0.0111	0.0108
P	**6**	**7**	**8**	**9**	**10**
1	0.0036	0.0044	0.0052	0.0060	0.0068
2	0.0066	0.0074	0.0081	0.0088	0.0095
3	0.0092	0.0099	0.0104	0.0109	0.0114
4	0.0112	0.0116	0.0120	0.0122	0.0124
5	0.0124	0.0126	0.0127	0.0128	0.0128
6	0.0128	0.0129	0.0128	0.0127	0.0126
7	0.0127	0.0126	0.0125	0.0122	0.0120
8	0.0122	0.0120	0.0117	0.0115	0.0111
9	0.0114	0.0112	0.0108	0.0105	0.0101
10	0.0105	0.0102	0.0098	0.0095	0.0091

Tab. 3: Estimates of α, β, λ, $R(t)$ and $H(t)$ with t=0.5

		$(\cdot)_{BE}$			
	$(\cdot)_{ML}$	c=-1	c=-0.5	c=0.5	c=1
α	0.1078	0.1073	0.0988	0.0829	0.0758
β	0.8728	0.8562	0.8513	0.8413	0.8362
λ	0.0062	0.0073	0.0073	0.0071	0.0071
$R(t)$	0.9427	0.9406	0.9402	0.9394	0.9389
$H(t)$	0.0517	0.0985	0.0918	0.0792	0.0733

Based on the eight record values with the corresponding hyperparameter values obtained in Tables 1 and 2, and using the results in (35) and (36), the lower and the upper 95% prediction bounds for the next record values $X_{U(9)}$ are 79.48 and 110.78.

7 Conclusion

Based on the set of the upper record values, the present paper proposes classical and Bayesian approaches to estimate the three unknown parameters as well as the reliability and hazard functions for the modified Weibull model. We also considered the problem of predicting future record in a Bayesian setting. The Bayes estimators are obtained using general entropy loss functions.

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