Classical potential symmetries of the $K(m,n)$ equation with generalized evolution term

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Abstract: We consider a $K(m,n)$ equation with generalized evolution term which is of considerable interest in mathematical physics. We classify the nonlocal symmetries, which are known as potential symmetries, for this equation. It turns out that potential symmetries exist only if the parameters n, m and l satisfy certain relationship.

Key–Words: Potential symmetries, Partial differential equation, Exact Solutions

1 Introduction

We consider the $K(m,n)$ equation with generalized evolution term

$$(u^l)_t + au^m u_x + b(u^n)_{xxx} = 0, \qquad (1)$$

where the first term is the generalized evolution term, while the second term represents the nonlinear term and the third term is the dispersion term. This equation is the generalized form of the Korteweg-de Vries (KdV) equation. The case $l = m = n = b = 1$ and $a = -6$ leads to the KdV equation that was derived by Korteweg and de Vries (1895) which described weakly nonlinear shallow water waves. This equation was found to have solitary wave solutions, vindicating the observations made 51 years earlier of a solitary channel wave by Russell in Aug. 1834. The classical KdV equation has been studied extensively [1, 12, 14, 22], in particular, by means of the inverse scattering method, by applying the direct method that involves no group theoretical techniques and the Bäcklund transformation has been determined [23].

Rosenau and Hyman [24] studied the role of nonlinear dispersive in the formation of patterns in liquid drops of the nonlinear dispersive equations (1) for $l = a = b = 1$, $m > 0, 1 < n \leq 3$. They also introduced a class of solitary wave solutions with compact support, i.e. the absence of infinite wings or the absence of infinite tails, called *compactons*. In addition to compactons, Rosenau [25] proved that the nonlinear dispersive equations $K(m,n)$

$$u_t \pm a(u^m)_x + (u^n)_{xxx} = 0, \qquad a \quad \text{const.},$$

which exhibits a number of remarkable dispersive effects, can support both; kinks and solitons with an infinite slope(s), periodic waves and dark solitons with cusp(s) all being manifestations of nonlinear dispersion in action. For $n < 0$ the enhanced dispersion at the tail may generate algebraically decaying patterns. Other solitary-wave solutions of K(m,n) equations were also found by Rosenau in [26, 27].

In [9] we carried out a classification of the symmetries of equation (1) with $a, b \in R^*$ and $l, m, n \in Z^+$ by using classical symmetries. In [10] Bruzon and Gandarias obtained traveling wave solutions of the equation (1) with $a, b \in R^*$ and $l, m, n \in Z^+$. They gave a catalogue of new exact solutions and a set of solitons, kinks, antikinks and compactons.

There is no existing general theory for solving nonlinear partial differential equations (PDE's). Due to the great advance in computation in the last years a great progress is being made in the development of methods and their applications to nonlinear PDE's for finding exact solutions. For instance, classical Lie method [5, 6, 17, 18], nonclassical potential symmetries method [15, 16], simplest equation method [20], (G'/G)-expansion method [7, 8, 11], extended simplest equation method [21], among others.

Local symmetries admitted by a PDE are useful for finding invariant solutions. These solutions are obtained by using group invariants to reduce the number of independent variables. The fundamental basis of the technique is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. The machinery of the Lie group theory provides a systematic method to search for these special group invariant solutions. For

PDEs with two independent variables, as it is equation (1), a single group reduction transforms the PDE into ODEs, which are generally easier to solve than the original PDE.

An obvious limitation of group-theoretic methods based on local symmetries, in their utility for particular PDEs, is that many of these equations does not have local symmetries. It turns out that PDEs can admit nonlocal symmetries whose infinitesimal generators depend on the integrals of the dependent variables in some specific manner. It also happens that if a nonlinear scalar PDE does not admit an infinite-parameter Lie group of contact transformations, it is not linearizable by an invertible contact transformation.

In [3, 4] Bluman introduced a method to find a new class of symmetries for a PDE. By writing a given PDE, denoted by R$\{x,t,u\}$ in a conserved form a related system denoted by S$\{x,t,u,v\}$ as additional dependent variables is obtained. If $u(x,t)$, $v(x,t)$ satisfies $S\{x,t,u,v\}$, then $u(x,t)$ solves $R\{x,t,u\}$ and $v(x,t)$ solves an integrated related equation $T\{x,t,v\}$. Any Lie group of point transformations admitted by S$\{x,t,u,v\}$ induces a symmetry for R$\{x,t,u\}$; when at least one of the generators of the group depends explicitly of the potential, then the corresponding symmetry is neither a point nor a Lie-Bäcklund symmetry. These symmetries of R$\{x,t,u\}$ are called *potential* symmetries.

The nature of potential symmetries allows one to extend the use of point symmetries to such nonlocal symmetries. In particular:

1. Invariant solutions of $S\{x,t,u,v\}$, respectively $T\{x,t,v\}$, yield solutions of $R\{x,t,u\}$ which are not invariant solutions for any local symmetry admitted by $R\{x,t,u\}$.

2. If $R\{x,t,u\}$ admits a potential symmetry leading to the linearization of $S\{x,t,u,v\}$, respectively $T\{x,t,v\}$, then $R\{x,t,u\}$ is linearized by a noninvertible mapping.

Suppose $S\{x,t,u,v\}$ admits a local Lie group of transformations with the infinitesimal generator

$$\begin{aligned} X_S = \quad & \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t \\ & +\psi(x,t,u,v)\partial_u + \varphi(x,t,u,v)\partial_v. \end{aligned} \tag{2}$$

this group maps any solution of Sx,t,u,v to another solution of Sx,t,u,v and hence induces a mapping of any solution of Rx,t,u to another solution of $R\{x,t,u\}$. Thus, (2) defines a symmetry group of $R\{x,t,u\}$. If

$$\xi_v^2 + \tau_v^2 + \psi_v^2 \neq 0$$

then (2) yields a nonlocal symmetry of $R\{x,t,u\}$, such a nonlocal symmetry is called a potential symmetry of $R\{x,t,u\}$, otherwise X_S projects onto a point symmetry of $R\{x,t,u\}$.

The purpose of the present paper is to obtain potential symmetries of equation (1). That is, to determine the values of the parameters l,m,n,a and b, with $l,m,n,a,b \neq 0$ and $m \neq -1$, for which the equation admits nonlocal symmetries.

2 Classical Lie Method

We consider the PDE:

$$\Delta \equiv \Delta(x,u,\mathbf{u}^{(1)}(x),\ldots,\mathbf{u}^{(n)}(x)) = 0,$$

where $x = (x_1,\ldots,x_p)$ are the independent variables, $u = u(x)$ is the dependent variable and $\mathbf{u}^{(l)}(x)$ denotes the set of all the partial derivatives of order l of u. We require that the PDE would be invariant under the group

$$\begin{aligned} x^* = \quad & x + \epsilon\xi(x,u) + \mathcal{O}(\epsilon^2), \\ t^* = \quad & t + \epsilon\tau(x,u) + \mathcal{O}(\epsilon^2), \\ u^* = \quad & u + \epsilon\eta(x,u) + \mathcal{O}(\epsilon^2), \end{aligned}$$

with infinitesimal generator:

$$V = \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u}.$$

By Criterion of Invariance we require that

$$\mathrm{pr}^{(n)}V(\Delta) = 0 \qquad \text{when} \qquad \Delta = 0,$$

where

$$\mathrm{pr}^{(n)}V = V + \sum_J \eta^J(x,u^{(n)})\frac{\partial}{\partial u_J},$$

$$J = (j_1,\cdots,j_k), \quad \text{with} \quad 1 \leq j_k \leq p, \quad 1 \leq k \leq n,$$

$$\eta^J(x,u^{(n)}) = D_J\left(\eta - \sum_{i=1}^{p}\xi_i u_i\right) + \sum_{i=1}^{p}\xi_i u_{J,i},$$

$$u_i = \frac{\partial u}{\partial x^i} \qquad \text{and} \quad u_{J,i} = \frac{\partial u_J}{\partial x^i}.$$

We obtain an overdetermined, linear system of equations for the infinitesimals $\xi_i(x,u)$ and $\eta(x,t,u)$.

The similarity variables are found by solving the invariant surface condition:

$$\frac{\partial u}{\partial x_p} = \eta(x,u) - \sum_{i=1}^{p-1} \xi_i(x,u)\frac{\partial u}{\partial x_i}$$

The Similarity variables

$$\begin{cases} z = z(x), \\ u = U(x, h(z)). \end{cases}$$

reduce the PDE

$$\Delta(x, u, \mathbf{u}^{(1)}(x), \ldots, \mathbf{u}^{(n)}(x)) = 0$$

into an ODE

$$\Delta(z, h, \mathbf{h}^{(1)}(z), \ldots, \mathbf{h}^{(n)}(z)) = 0.$$

3 Classical Symmetries of Eq. (1)

To apply the classical method to Eq. (1) with $a, b \neq 0$ and $l, m, n \in \mathbb{R}^+$ we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$x^* = x + \epsilon\xi(x,t,u) + O(\epsilon^2),$$

$$t^* = t + \epsilon\tau(x,t,u) + O(\epsilon^2),$$

$$u^* = u + \epsilon\eta(x,t,u) + O(\epsilon^2),$$

where ϵ is the group parameter. We require that this transformation leaves invariant the set of solutions of (1). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u. \quad (3)$$

Invariance of Eq. (1) under a Lie group of point transformations with infinitesimal generator (3) leads to a set of nineteen determining equations. Solving this system we obtain that if $l \neq n$ $\xi = \xi(x,t)$, $\tau = \tau(t)$ and $\eta = \dfrac{\tau_t - 3\xi_x}{l-n}u$ where ξ and τ are related by the following conditions:

$$\xi_{xx}\,(n-1)\;n\;(l+2\,n) = 0,$$

$$\xi_{xx}\;(l+2\,n) = 0,$$

$$\tau_{tt}\,l\,u^l - 3\,\xi_{tx}\,l\,u^l - 3\,b\,\xi_{xxxx}\,n\,u^n - 3\,a\,\xi_{xx} = 0,$$

$$\begin{aligned}\tau_t\,l^2\,u^l - \xi_t\,n\,l\,u^l + b\,\xi_{xxx}\,n\,l\,u^n + 8\,b\,\xi_{xxx}\,n^2\,u^n \\ -2\,a\,\xi_x\,l\,u^{m+1} + a\,\tau_t\,n\,u^{m+1} - a\,\xi_x\,n\,u^{m+1} \\ -a\,\tau_t\,m\,u^{m+1} + 3\,a\,\xi_x\,m\,u^{m+1} - a\,\tau_t\,u^{m+1} \\ +3\,a\,\xi_x\,u^{m+1} = 0.\end{aligned}$$

The solutions of this system depend on the parameters of equation (1).
If a, b, n and m are arbitrary constants with $a, b, m, n \in R^*$ the symmetries admitted by (1) are the group of space and time translations, which are defined by the infinitesimal generators

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t,$$

and the generator

$$\mathbf{v}_3 = \frac{1}{2}(n-m-1)x\partial_x + \frac{1}{2}(n-3m-1)t\partial_t + u\partial_u.$$

In the following we given the infinitesimal generator for which Eq.(1) have extra symmetries:

1. If $l = n = 2, m = 1$,

$$\xi = \tfrac{k_1}{3}(x + at) + k_3,$$

$$\tau = k_1 t + k_2,$$

$$\eta = \frac{f(x,t)}{u} + k_4 u.$$

   where f satisfy $2bf_{xxx} + af_x + 2f_t = 0$

2. If $l = n, m = 2n - 1$,

$$\xi = \tfrac{k_1}{3}x + k_3 t + k_4,$$

$$\tau = k_1 t + k_2,$$

$$\eta = \tfrac{k_3}{au^{n-1}} - \tfrac{2k_1 u}{3n}.$$

4 Classical Potential Symmetries

In [4] Bluman introduced a method to find a new class of symmetries for a PDE. Suppose a given scalar PDE of second order

$$F(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0, \quad (4)$$

where the subscripts denote the partial derivatives of u, can be written as a conservation law

$$\frac{D}{Dt}f(x,t,u,u_x,u_t)-\frac{D}{Dx}g(x,t,u,u_x,u_t)=0, \quad (5)$$

for some functions f and g of the indicated arguments. Here $\frac{D}{Dx}$ and $\frac{D}{Dt}$ are total derivative operators defined by

$$\begin{aligned}\frac{D}{Dx} &= \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}+\dots,\\ \frac{D}{Dt} &= \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{xt}\frac{\partial}{\partial u_x}+u_{tt}\frac{\partial}{\partial u_t}+\dots.\end{aligned}$$

Through the conservation law (5) one can introduce an auxiliary potential variable v and form an auxiliary potential system (system approach)

$$\begin{aligned} v_x &= f(x,t,u,u_x,u_t),\\ v_t &= g(x,t,u,u_x,u_t). \end{aligned} \quad (6)$$

For many physical equations one can eliminate u from the potential system (6) and form an auxiliary integrated or potential equation (integrated equation approach)

$$G(x,t,v,v_x,v_t,v_{xx},v_{xt},v_{tt})=0, \quad (7)$$

for some function G of the indicated arguments. Any Lie group of point transformations (2)

$$\begin{aligned} X_S = &\ \xi(x,t,u,v)\partial_x+\tau(x,t,u,v)\partial_t\\ &+\psi(x,t,u,v)\partial_u+\varphi(x,t,u,v)\partial_v. \end{aligned}$$

admitted by (6) yields a nonlocal symmetry *potential symmetry* of the given PDE (5) if and only if the following condition is satisfied

$$\xi_v^2+\tau_v^2+\phi_v^2\neq 0. \quad (8)$$

5 Classical Potential Symmetries of Eq. (1)

In order to find potential symmetries of (1) with $a,b,l,n\neq 0$ and $m\neq -1$ we write the equation in a conserved form and the associated auxiliary system is given by

$$\begin{cases} v_x=u^l,\\ v_t=-\frac{a}{m+1}u^{m+1}-b\left(u^n\right)_{xx}. \end{cases} \quad (9)$$

A Lie point symmetry admitted by $S(x,t,u,v)$ is a symmetry characterized by an infinitesimal transformation of the form

$$\begin{aligned} x^* &= x_i+\epsilon\xi(x,t,u,v)+\mathcal{O}(\epsilon^2),\\ t^* &= t+\epsilon\tau(x,t,u,v)+\mathcal{O}(\epsilon^2),\\ u^* &= u+\epsilon\eta(x,t,u,v)+\mathcal{O}(\epsilon^2)\\ v^* &= v+\epsilon\varphi(x,t,u,v)+\mathcal{O}(\epsilon^2) \end{aligned} \quad (10)$$

admitted by system (9). In the present work, we will present the point symmetries of (9) and we will study that symmetries induce potential symmetries of equation (1). These symmetries are such that the condition (8)

$$\xi_v^2+\tau_v^2+\psi_v^2\neq 0$$

is satisfied. If the above relation does not hold, then the point symmetries of (9) project into point symmetries of (1). System (9) admit Lie symmetries if and only if

$$\begin{aligned} &\mathrm{pr}^{(2)}X(v_x-u^l)=0,\\ &\mathrm{pr}^{(2)}X(v_t+\tfrac{a}{m+1}u^{m+1}+b\left(u^n\right)_{xx})=0, \end{aligned}$$

where $\mathrm{pr}^{(2)}V$ is the second extended generator of

$$\begin{aligned} X_S = &\ \xi(x,t,u,v)\partial_x+\tau(x,t,u,v)\partial_t\\ &+\psi(x,t,u,v)\partial_u+\varphi(x,t,u,v)\partial_v. \end{aligned}$$

In other words, we require that the infinitesimal generator leaves invariant the set of solutions of (9). This yields to an overdetermined system of fourteen equations for the infinitesimals $\xi(x,t,u,v)$, $\tau(x,t,u,v)$, $\psi(x,t,u,v)$ and $\varphi(x,t,u,v)$. From this system we obtain that $\xi=\xi(x,t,v)$, $\tau=\tau(t)$, $\varphi=\varphi(x,t,v)$ and

$$\psi=-\frac{\xi_v\,u^{2l+1}+\left(\xi_x-\varphi_v\right)\,u^{l+1}-\varphi_x\,u}{lu^l}$$

where ξ, τ and φ must satisfy the following four equations

$$\xi_v(2l+n)\,u^{2l}+\varphi_x(n-l)=0,$$

$$\begin{aligned}(2l+n)\xi_v\,u^{2l}+(l-n)\varphi_v\,u^l-l\tau_t\,u^l+2l\xi_x\,u^l\\ +\xi_x n\,u^l+(l-n)\varphi_x=0,\end{aligned}$$

$$\begin{aligned}4\,l\xi_{vv}\,u^{3l}+2\,\xi_{vv}\,n\,u^{3l}-l\varphi_{vv}\,u^{2l}+5l\xi_{vx}\,u^{2\,l}\\ -2\,n\varphi_{vv}u^{2l}+4\,\xi_{vx}\,n\,u^{2l}+l\varphi_{vx}\,u^l+l\xi_{xx}\,u^l\\ -4\,n\,\varphi_{vx}\,u^l+2\,\xi_{xx}\,n\,u^l+2\,l\varphi_{xx}-2\,n\,\varphi_{xx}=0,\end{aligned}$$

$$\begin{aligned}b\,\xi_{vvv}m\,n\,u^{4l+n}+b\,\xi_{vvv}\,n\,u^{4l+n}\\ -bm\,n\varphi_{vvv}\,u^{3l+n}-b\,n\varphi_{vvv}\,u^{3l+n}\\ +3\,b\,\xi_{vvx}m\,n\,u^{3l+n}+3\,b\,\xi_{vvx}\,n\,u^{3l+n}\\ -3\,bm\,n\,\varphi_{vvx}\,u^{2l+n}-3\,b\,n\,\varphi_{vvx}\,u^{2l+n}\\ +3\,b\,\xi_{vxx}m\,n\,u^{2l+n}+3\,b\,\xi_{vxx}\,n\,u^{2l+n}\\ -3\,a\,\xi_v l\,u^{2l+k+1}-a\,\xi_v\,n\,u^{2l+k+1}\\ +a\,\xi_v m\,u^{2l+k+1}+a\,\xi_v\,u^{2l+k+1}\\ -3\,bm\,n\,\varphi_{vxx}\,u^{l+n}-3\,b\,n\,\varphi_{vxx}\,u^{l+n}\\ +b\,\xi_{xxx}m\,n\,u^{l+n}+b\,\xi_{xxx}\,n\,u^{l+n}\\ -2\,a\,\xi_x l\,u^{l+k+1}+a\,n\,\varphi_v\,u^{l+k+1}\\ -am\,\varphi_v\,u^{l+k+1}-a\,\varphi_v\,u^{l+k+1}-a\,\xi_x\,n\,u^{l+k+1}\\ +a\,\xi_x m\,u^{l+k+1}+a\,\xi_x\,u^{l+k+1}+\xi_t ml\,u^{2l}\\ +\xi_t l\,u^{2l}-m\,\varphi_t l\,u^l-\varphi_t l\,u^l-bm\,n\,\varphi_{xxx}\,u^n\\ -b\,n\,\varphi_{xxx}\,u^n-a\,\varphi_x l\,u^{k+1}+a\,n\,\varphi_x\,u^{k+1}\\ -am\,\varphi_x\,u^{m+1}-a\,\varphi_x\,u^{m+1}=0.\end{aligned} \tag{11}$$

From system (11) we should consider three cases:

Case 1: The parameters m, n, l are arbitrary constants. From system (11) we obtain the infinitesimals:

$$\xi=k_1 \qquad \tau=k_2$$

$$\psi=0 \qquad \varphi=k_3$$

It is no potential symmetry of the equation (1) because the condition (8) is not satisfied.

Case 2: If $n\neq -2l+3(m+1)$ from system (11) we obtain the infinitesimals:

$$\xi=\tfrac{k_2(n-m-1)}{2l+n-3(m+1)}x+k_5$$

$$\tau=k_2t+k_3$$

$$\psi=\tfrac{2k_2}{2l+n-3(m+1)}u$$

$$\varphi=\tfrac{k_2(2l+n-(m+1))}{2l+n-3(m+1)}v+k_4$$

It is no potential symmetry of the equation (1) because the condition (8) is not satisfied.

Case 3: If $l=n$ from system (11) we obtain

$$\xi=\tfrac{1}{3}\tau_t x+\alpha(t),$$

$$\tau=\tau(t),$$

$$\psi=\tfrac{\beta_x(x,t)}{nu^{n-1}}+\tfrac{\delta(t)u}{n}-\tfrac{\tau_t u}{3n},$$

$$\varphi=k_1v+\beta(x,t),$$

where $\tau(t)$, $\alpha(t)$, $\beta(x,t)$ and $\delta(t)$ must satisfy the equation

$$\begin{aligned}\tau_{tt}\,m\,n\,u^{2\,n}\,x+\tau_{tt}\,n\,u^{2\,n}\,x+3\,a\,k_1\,n\,u^{n+m+1}\\ -3\,a\tau_t\,n\,u^{n+m+1}-3\,a\,k\,k_1\,u^{n+m+1}\\ -3\,a\,k_1\,u^{n+m+1}+a\,\tau_t\,k\,u^{n+m+1}+a\,\tau_t\,u^{n+m+1}\\ +3\,m\,n\,\alpha_t\,u^{2\,n}+3\,n\,\alpha_t\,u^{2\,n}-3\,b\,m\,n\,\beta_{xxx}\,u^n\\ -3\,b\,n\,\beta_{xxx}\,u^n-3\,m\,n\,\beta_t\,u^n-3\,n\,\beta_t\,u^n\\ -3\,a\,m\,\beta_x\,u^{m+1}-3\,a\,\beta_x\,u^{m+1}=0.\end{aligned} \tag{12}$$

From system (12) we obtain the following solutions

- $$\begin{aligned}\xi&=\tfrac{k_1}{3}x+k_3,\\ \tau&=k_1t+k_2,\\ \psi&=\tfrac{k_1}{3n}u,\\ \varphi&=\tfrac{2}{3}k_1v+k_4\end{aligned} \tag{13}$$

- $$\begin{aligned}\xi&=\tfrac{(k+1)\,k_1\,x+2\,a\,k_1\,t+3(k+1)\,k_4}{3\,k+3},\\ \tau&=k_1t+k_2,\\ \psi&=\tfrac{k_1}{3n}u,\\ \varphi&=\tfrac{2}{3}k_1v+k_4\end{aligned} \tag{14}$$

The infinitesimal generators (13) and (14) are not potential symmetries of the equation (1) because the condition (8) is not satisfied.

Case 4: If $l=1$, $n=-2$ and $m=1$

$$\begin{aligned}\xi&=k_1x+f(v),\\ \tau&=k_3t+k_4,\\ \psi&=-\alpha_v u^2+(\tfrac{k_3}{3}-k_1)u,\\ \varphi&=\tfrac{k_3}{3}v+k_2\end{aligned} \tag{15}$$

where the function $f(v)$ satisfy the linear equation $f_{vvv} + f + k_1 - k_3 = 0$.

The infinitesimal generators (15) are potential symmetries of the equation (1) because the condition (8) is satisfied.

Case 5: If $n = -2l$ and $l = m + 1$ from system (11) we obtain the following symmetries:

$$X_1 = \partial_t, \qquad X_2 = \partial_v,$$

$$X_3 = x\partial_x + t\partial_t - \frac{2}{3(m+1)}u\partial_u + \frac{1}{3}v\partial_v,$$

$$X_\infty = f(t,v)\partial_x - \frac{1}{m+1}u^{m+2}f_v(t,v)\partial_u,$$

where the function $f(t,v)$ satisfy the linear equation

$$2bf_{vvv} - f_t = 0.$$

Consequently, we can state that: *the equation* (1) *admits potential symmetries if* $n = -2l$ *and* $l = m + 1$ *or* $l = 1$, $n = -2$ *and* $m = 1$.

5.1 Symbolic manipulations programs

The procedure to obtain the classical potential symmetries is entirely algorithmic, it involves a large amount of algebra and of auxiliary calculations. Some symbolic manipulation programs have been developed to simplify the calculations. We use the MACSYMA program symmgrp.max [13] and the MATHEMATICA software.

To use symmgrp.max, we have to convert (9) into the appropiate MACSYMA syntax: x[1] and x[2] represent the independent variables x and t, respectively, u[1] and u[2] represent the dependent variables u and v, respectively. The system (9) is rewritten as

$$u[2,[0,1]] + (a/(k+1)) * u[1]^{(k+1)} + b * (n * u[1]^(n-1) * u[1,[2,0]] + (n-1) * n * u[1]^(n-2) * (u[1,[1,0]])^2) = 0;$$

$$u[2,[1,0]] - (u[1])^s = 0;$$

where $u[1,[0,1]]$ represents v_t, $u[2,[1,0]]$ represents u_{xx}, etc.
The infinitesimals ξ, τ, η and φ asre represented by eta1, eta2, phi1 and phi2, respectively. The program symmgrp.max automatically computes the determining equations for the infinitesimals

```
kill(all);

derivabbrev:true;

batchload("symmgrp.max");

/* KLMN- CLASSICAL POTENTIAL*/

batch("Klmnpc.dat.txt");

symmetry(1,0,0);

printeqn(lode);

save("lodegnlh.lsp",lode);

for j thru q do (x[j]:=concat(x,j));

for j thru q do (u[j]:=concat(u,j));

ev(lode)$

gnlhode:ev(%,x1=x,x2=t,u1=u,u2=v);

grind:true$

stringout("gnlhode",gnlhode);

closefile();
```

The fiel klmnpc.cas in turn batches the file klmnpc.dat which contains data about (9)

```
p:2$

q:2$

m:2$

parameters:[a,b,n,m]$

warnings:true$

sublisteqs:[all]$

subst_deriv_of_vi:true$

info_given:true$

highest_derivatives:all$

depends([eta1,eta2,phi1,phi2],[x[1],x[2],u[1],u[2]]);

eta2:1;

vt:phi2-eta1*u[2,[1,0]];
```

e1:$vt + (a/(k+1)) * u[1]^{(k+1)} + b * (n * u[1]^{(n-1)} * u[1,[2,0]] + (n-1) * n * u[1]^{(n-2)} * (u[1,[1,0]])^2)$;

e2:$u[2,[1,0]] - (u[1])^s$;

```
v1:u[1,[2,0]];

v2:u[2,[1,0]];
```

We note that $u[2,[0,1]]$ (i.e. v_t) has been eliminated using the invariant surface conditions.

6 Similarity solutions

As in the case of Lie point symmetries, potential symmetries may be used to derive similarity transformations (solutions). Such transformations reduce the number of independent variables of a system of PDEs. In particular, we use the potential symmetries obtained in Section 5, which are Lie symmetries of system (9), to derive similarity solutions for system (9). In order to find similarity solutions for system (9) we need to solve the invariant surface conditions

$$\xi u_x + \tau u_t = \psi, \qquad \xi v_x + \tau v_t = \varphi,$$

where ξ, τ, ψ and φ are the infinitesimal of the transformation (10). The similarity solutions can be found by solving the corresponding characteristic equations

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\psi} = \frac{dv}{\varphi}. \tag{16}$$

We present some examples of similarity solutions which are produced by symmetries that were obtained in Section 5.

Example 1. We use the symmetry $X_1 + X_\infty = \lambda\partial_t + f(t,v)\partial_x - \frac{1}{m+1}u^{m+2}f_v(t,v)\partial_u$. We set $f(t,v) = \mu$. Substituting into equations (16) we get

$$z = \mu x - \lambda t, \qquad u(x,t) = h(z), \qquad v = w(z). \tag{17}$$

Similarity transformation (17) reduces system (9) to nonlinear system

$$\mu w' - h^l = 0, \tag{18}$$

$$\begin{aligned} -\lambda w' + b\,\mu^2\,n^2\,h^{n-2}\,(h')^2 \\ +b\,\mu^2\,n\,h^{n-1}\,h'' - b\,\mu^2\,n\,h^{n-2}\,(h')^2 \\ +\frac{a\,h^{m+1}}{m+1} = 0. \end{aligned} \tag{19}$$

System (18)-(19) is reduced for $n = -2l$ and $l = m+1$ into nonlinear ODE

$$h'' - \frac{(2\,m+3)}{h}\,(h')^2 + \frac{h^{3\,m+4}\,(m\,\lambda+\lambda-a\,\mu)}{2\,b\,(m+1)^2\,\mu^3} = 0. \tag{20}$$

Let $g = h'$. Since $h'' = gg'$ we get

$$\frac{(\lambda-a)}{2\,b\,(m+1)\,\mu^3}h^{3\,m+4} - \frac{2\,g^2\,(m+1)}{h} - \frac{g^2}{h} + g\,g' = 0. \tag{21}$$

Equation (21) is a Bernoulli equation and this equation has the general solution

$$g^2 = \frac{(m\,\lambda+\lambda-a\,\mu)}{b\,(m+1)^3\,\mu^3}h^{3\,m+5} + 2\,c\,h^{4\,m+6}, \tag{22}$$

where c is a constant of integration. Back to the function h we have

$$(h')^2 = \frac{(m\,\lambda+\lambda-a\,\mu)}{b\,(m+1)^3\,\mu^3}h^{3\,m+5} + 2\,c\,h^{4\,m+6}. \tag{23}$$

The general solution of equation (23) is

$$h(z) = \text{InverseFunction}[-2[2\text{Hypergeometric2}\\ F_1(a_1,a_2,a_3,a_4)]\#1H_1/(4m+4)H_2\&][-z,C_1][[2]]$$

where Hypergeometric2F1[a,b,c,z] is the hypergeometric function ${}_2F_1(a,b;c;z)$,

$$a_1 = \tfrac{1}{2}, a_2 = \tfrac{4m+4}{2(m+1)},$$

$$a_3 = \tfrac{4m+4}{2(m+1)} + 1,$$

$$a_4 = -\frac{\#1^{-m-1}(\lambda(m+1)-a\mu)}{2bc(m+1)^3\mu^3},$$

$$H_1 = \sqrt{\frac{\#1^{-m-1}(\lambda(m+1)-a\mu)}{2bc(m+1)^3\mu^3} + 1},$$

$$H_2 = \sqrt{\frac{\#1^{3m+5}(\lambda(m+1)-a\mu)}{b(m+1)^3\mu^3} + 2c\#1^{4m+6}}.$$

This case was not studied in [9, 10].

Example 2. Substituting (18) into (19) we obtain the equation

$$h'' - \frac{h^{l-n+1}\,\lambda}{b\,\mu^3\,n} + \frac{(h_z)^2\,(n-1)}{h} + \frac{a\,h^{-n+m+2}}{b\,(m+1)\,\mu^2\,n} = 0. \tag{24}$$

For $l = n = m$, we obtain that $h(z) = \text{sech}^2(z)$ is a solution of equation (24).
From (17) we obtain that

$$u(x,t) = \text{sech}^2(\mu x - \lambda t) \tag{25}$$

is a solution of equation (1). In Figure 1 we plot solution (25) with $\mu = \lambda = 1$ which describes a soliton solution. The solitons are defined as localized waves that propagate without change of its shape and velocity properties, and are stable against mutual collisions. The existence of solitary wave solutions implies perfect balance between nonlinearity and dispersion which usually requires rather specific conditions and cannot be established in general [19].

Example 3. From (24) if $l = n$, for $\alpha = 1$, $\beta = 6$: $m = n - \frac{4}{3}$, $a = \frac{(3n-1)(6n-1)\lambda}{18\mu n}$, $b = -\frac{\lambda}{36\mu^3 n^2}$, by (17) we obtain that

$$u(x,t) = \begin{cases} \sin^6(\mu x - \lambda t) & |x-t| \le 4\pi, \\ 0 & |x-t| > 4\pi \end{cases} \tag{26}$$

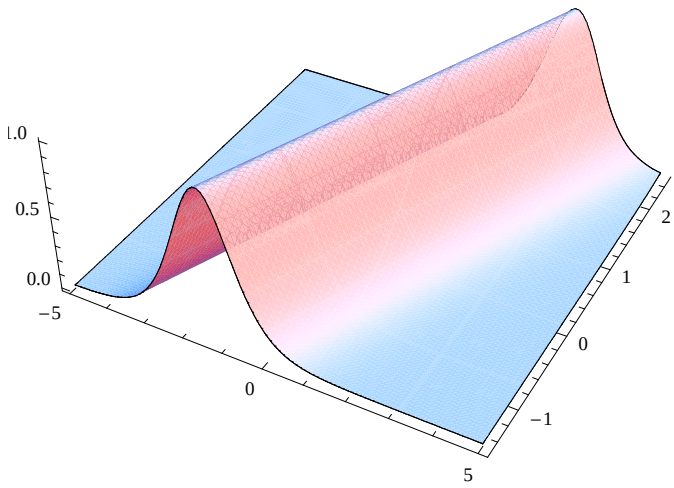

Figure 1: Solution (25) for $\mu = 1$ and $\lambda = 1$

is a solution of equation (1). In Figure 2 we plot solution (26) with $\mu = \frac{1}{4}$ and $\lambda = \frac{1}{4}$ which describes a compacton solution.

Example 4. From (24) we obtain that

$$u(x,t) = \frac{1}{4}\tanh(\mu x - \lambda t) \tag{27}$$

is a solution of equation (1) with $l = n = 2$, $m = 3$, $a = \frac{48\lambda}{\mu}$ and $b = -\frac{\lambda}{8\mu^3}$. In Figure 3 we plot solution (27) with $\mu = 1$ and $\lambda = -\frac{1}{2}$ which describes a kink solution.

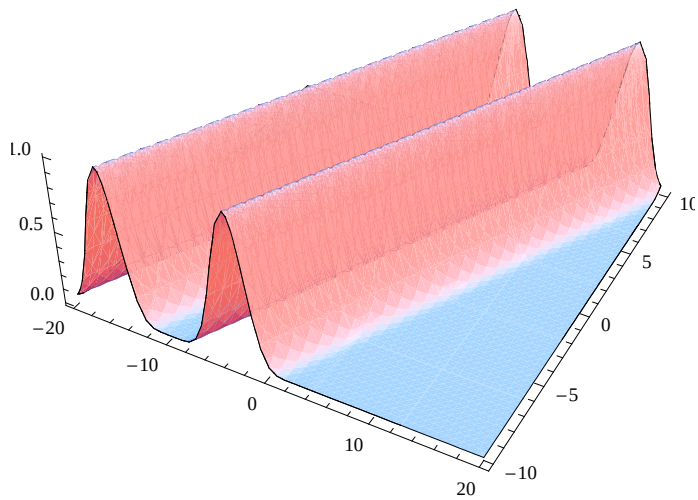

Figure 2: Solution (26) for $\mu = \frac{1}{4}$ and $\lambda = \frac{1}{4}$

7 Conclusions

In this paper we have seen a classification of potential symmetries of a $K(m,n)$ equation with generalized evolution term, depending on the values of the constants a, b, n, l and m. We have proven that the equation (1) admits potential symmetries if $n = -2l$ and $l = m + 1$. Consequently the equation (1) studied in [9] does not admit potential symmetries.

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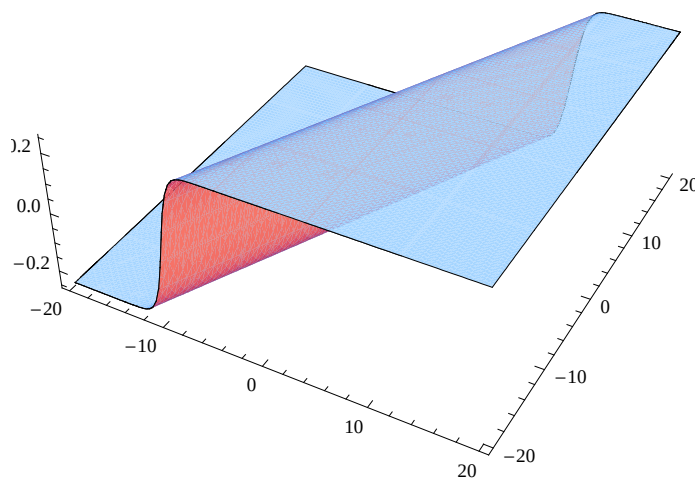

Figure 3: Solution (27) for $\mu = 1$ and $\lambda = -\frac{1}{2}$

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