Exact solutions through symmetry reductions for a new integrable equation

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Abstract: In this work we study a new completely integrable equation from the point of view of the theory of symmetry reductions in partial differential equations. This equation has been proposed by Qiao and Liu in [24] and it possesses peak solitons. We obtain the classical symmetries and the classical symmetries of the associated potential system admitted, then, we use the transformations groups to reduce the equations to ordinary differential equations. Physical interpretation of these reductions and some exact solutions are also provided. Among them we obtain a travelling wave with decaying velocity and an smooth soliton solution.

Key–Words: Symmetries, partial differential equation, exact solutions

1 Introduction

The study of integrable equations has arisen lot of attention in the last years. Among the integrable equations the study of peaked and cusped soliton equations has been considered in many papers. In [28, 29, 30], Wadati et al. proposed the cusp soliton, which is a kind of peaked soliton. Recently in [24] Qiao and Liu proposed a new completely integrable equation

$$u_t = \frac{1}{2}\left(\frac{1}{u^2}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{u^2}\right)_x \tag{1}$$

which has no smooth solitons. In [24] the authors proved that the new equation proposed Eq. (1) is completely integrable. It was shown that (1) has bi-Hamiltonian structure, and Lax pair that implies its integrability by the Inverse Scattering Transformation. By considering traveling-wave solutions the authors found one peak soliton solutions and three-peaks solitons solutions. The authors state that no smooth solitons were found for equation (1), although equation (1) is completely integrable. They claim to provide an integrable system with no smooth solitons.

In this work, we study equation (1) from the point of view of the theory of symmetry reductions in partial differential equations. We obtain the classical symmetries admitted by (1) for arbitrary n, then, we use the transformations groups to reduce the equations to ordinary differential equations. Physical interpretation of these reductions and some elementary solutions are also provided.

In this paper, we apply the Lie group method of infinitesimals transformations to the generalized equation

$$u_t = \frac{1}{k}\left(u^n\right)_{xxx} - \frac{1}{k}\left(u^n\right)_x . \tag{2}$$

By using this method we bring out the unexplored invariance properties and similarity reduced ordinary differential equations (ODE's) of the above equation (2). First we obtain a point transformation group which leaves the equation (2) invariant. In order to find all invariant solutions with respect to s-dimensional subalgebras, it is sufficient to construct invariant solutions for the optimal system of order s. The set of invariant solutions obtained in this way is called an *optimal system of invariant solutions*. We only consider one-parameter subgroups. For further details [27]. By using the classical Lie method, we derive exact solutions for the integrable equation. Some of these solutions are smooth soliton solutions.

2 Classical symmetries

In this section we perform Lie symmetry analysis for equation (2). Let us consider a one-parameter Lie group of infinitesimal transformations in (x,t,u) given by

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,t,u) + \mathcal{O}(\varepsilon^2),\\ t^* &= t + \varepsilon\tau(x,t,u) + \mathcal{O}(\varepsilon^2),\\ u^* &= u + \varepsilon\phi(x,t,u) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{3}$$

where ε is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of the equation (1). This yields to the overdetermined, linear system of eleven equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}. \tag{4}$$

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi\frac{\partial u}{\partial x} + \tau\frac{\partial u}{\partial t} - \phi = 0. \tag{5}$$

By solving this system we get that for $n \neq 1$, $\xi = \xi(x)$, $\tau = \tau(t)$,

$$\phi = \frac{u}{n-1}\left(3\,\frac{d\xi}{dx} - \frac{d\tau}{dt}\right)$$

where ξ, τ and n must satisfy the following equations

$$\begin{aligned} n\,(2\,n+1)\,\frac{d^2\xi}{dx^2} &= 0,\\ \frac{d^4\xi}{dx^4} - \frac{d^2\xi}{dx^2} &= 0,\\ 3\,\frac{d^2\xi}{dtdx} - \frac{d^2\tau}{dt^2} &= 0,\\ 8\,n\,\frac{d^3\xi}{dx^3} + \frac{d^3\xi}{dx^3} - 2\,n\,\frac{d\xi}{dx} + 2\,\frac{d\xi}{dx} &= 0. \end{aligned} \tag{6}$$

Solving this system we find that:
If n is arbitrary, *the symmetries that are admitted* by (1) are

$$\mathbf{v}_1 = \frac{\partial}{\partial x}, \quad \mathbf{v}_2 = \frac{\partial}{\partial t}, \quad \mathbf{v}_3 = t\frac{\partial}{\partial t} - \frac{u}{n-1}\frac{\partial}{\partial u}.$$

If $n = -\frac{1}{2}$ *the symmetries that are admitted* by (1) are

$$\mathbf{v}_1, \quad \mathbf{v}_2, \quad \mathbf{v}_3 = t\frac{\partial}{\partial t} + \frac{2u}{3}\frac{\partial}{\partial u},$$

$$\mathbf{v}_4 = e^x\frac{\partial}{\partial x} - 2ue^x\frac{\partial}{\partial u}, \quad \mathbf{v}_5 = e^{-x}\frac{\partial}{\partial x} + 2ue^{-x}\frac{\partial}{\partial u}.$$

Our aim in this paper is to use the theory of symmetry reductions to find traveling-wave solutions for (1).

In order to obtain these solutions, we consider the following reductions arising from the optimal system vector fields.

3 Optimal systems and reductions

In order to construct the one-dimensional optimal system, following Olver, we construct and are shown in the appendix the commutator table (Table 1) and the adjoint table (Table 2) which shows the separate adjoint actions of each element in $\mathbf{v}_i$, $i = 1 \dots 5$, as it acts on all other elements. This construction is done easily by summing the Lie series.

The corresponding generators of the optimal system of subalgebras are

$$a\mathbf{v}_1 + \mathbf{v}_2,$$

$$\mathbf{v}_3,$$

$$\mathbf{v}_1 + b\mathbf{v}_3.$$

where $a \in R$ and $b \in R$ are arbitrary.

In the following, reductions of the equation (1) to ODE's are obtained using the generators of the optimal system.

Reduction 1 By using the generator $\mathbf{v}_1 + \lambda\mathbf{v}_2$ we obtain the similarity variables and similarity solution

$$z = x - \lambda t, \qquad u = h(z), \tag{7}$$

and the ODE E_1

$$\begin{aligned} &\frac{dh}{dz}\,k\,\lambda + h^{n-3}\left(\frac{dh}{dz}\right)^3\,(n-2)\,(n-1)\,n\\ &+3\,h^{n-2}\,\frac{dh}{dz}\,\frac{d^2h}{dz^2}\,(n-1)\,n\\ &+h^{n-1}\,\frac{d^3h}{dz^3}\,n - h^{n-1}\,\frac{dh}{dz}\,n = 0 \end{aligned} \tag{8}$$

for $n = -2$ becomes

$$\begin{aligned} &-h^2\,\frac{d^3h}{dz^3} + 9\,h\,\frac{dh}{dz}\,\frac{d^2h}{dz^2} - 12\,\left(\frac{dh}{dz}\right)^3\\ &+c\,h^5\,\frac{dh}{dz} + h^2\,\frac{dh}{dz} = 0. \end{aligned} \tag{9}$$

Reduction 2 By using the generator $\mathbf{v}_3$ we obtain the similarity variables and similarity solution

$$z = x, \qquad u = t^{1-n}f(x), \tag{10}$$

and the ODE E_1

$$\begin{aligned} &f^n\left(\frac{df}{dx}\right)^3 n^3 + 3\,f^{n+1}\,\frac{df}{dx}\,\frac{d^2f}{dx^2}\,n^2 - 3\,f^n\left(\frac{df}{dx}\right)^3 n^2\\ &+f^4\,k\,n + f^{n+2}\,\frac{d^3f}{dx^3}\,n - 3\,f^{n+1}\,\frac{df}{dx}\,\frac{d^2f}{dx^2}\,n+\\ &2\,f^n\left(\frac{df}{dx}\right)^3 n - f^{n+2}\,\frac{df}{dx}\,n - f^4\,k = 0 \end{aligned} \tag{11}$$

For $n=-2$ becomes

$$f^4\,k+6\,\frac{d^3 f}{dx^3}-\frac{54\,\frac{df}{dx}\,\frac{d^2 f}{dx^2}}{f}+\frac{72\,\left(\frac{df}{dx}\right)^3}{f^2}=6\,\frac{df}{dx}. \quad (12)$$

Reduction 3 By using the generator $\mathbf{v}_1+\mathbf{v}_3$ we obtain the similarity variables and similarity solution

$$z=x-\ln\left(|t|\right), \qquad u=t^{\frac{1}{1-n}}h(x), \qquad (13)$$

and the ODE E_3

$$\begin{aligned}
&h^n\,\left(\frac{dh}{dz}\right)^3\,n^4+3\,h^{n+1}\,\frac{dh}{dz}\,\frac{d^2h}{dz^2}\,n^3\\
&-4\,h^n\,\left(\frac{dh}{dz}\right)^3\,n^3+h^{n+2}\,\frac{d^3h}{dz^3}\,n^2-6\,h^{n+1}\,\frac{dh}{dz}\,\frac{d^2h}{dz^2}\,n^2\\
&+5\,h^n\,\left(\frac{dh}{dz}\right)^3\,n^2-h^{n+2}\,\frac{dh}{dz}\,n^2+h^3\,\frac{dh}{dz}\,k\,n\\
&-h^{n+2}\,\frac{d^3h}{dz^3}\,n+3\,h^{n+1}\,\frac{dh}{dz}\,\frac{d^2h}{dz^2}\,n-2\,h^n\,\left(\frac{dh}{dz}\right)^3\,n\\
&+h^{n+2}\,\frac{dh}{dz}\,n-h^3\,\frac{dh}{dz}\,k+h^4\,k=0
\end{aligned} \qquad (14)$$

For $n=-2$ Eq (14) becomes

$$\begin{aligned}
&-3\,h^3\,\frac{dh}{dz}\,k+h^4\,k+6\,\frac{d^3h}{dz^3}\\
&-\frac{54\,\frac{dh}{dz}\,\frac{d^2h}{dz^2}}{h}+\frac{72\,\left(\frac{dh}{dz}\right)^3}{h^2}-6\,\frac{dh}{dz}=0.
\end{aligned} \qquad (15)$$

4 Potential symmetries

In [**?**] Bluman introduced a method to find a new class of symmetries for a PDE. Suppose a given scalar PDE of second order

$$F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt})=0, \qquad (16)$$

where the subscripts denote the partial derivatives of u, can be written as a conservation law

$$\frac{D}{Dt}f(x,t,u,u_x,u_t)-\frac{D}{Dx}g(x,t,u,u_x,u_t)=0, \qquad (17)$$

for some functions f and g of the indicated arguments. Here $\frac{D}{Dx}$ and $\frac{D}{Dt}$ are total derivative operators Through the conservation law (17) one can introduce an auxiliary potential variable v and form an auxiliary potential system

$$\begin{aligned}
v_x &= f(x,t,u,u_x,u_t),\\
v_t &= g(x,t,u,u_x,u_t).
\end{aligned} \qquad (18)$$

Any Lie group of point transformations

$$\begin{aligned}
&\mathbf{w}=\xi(x,t,u,v)\partial_x+\tau(x,t,u,v)\partial_t\\
&+\phi(x,t,u,v)\partial_u+\psi(x,t,u,v)\partial_v,
\end{aligned} \qquad (19)$$

admitted by (18) yields a nonlocal symmetry *potential symmetry* of the given PDE (17) if and only if the following condition is satisfied

$$\xi_v^2+\tau_v^2+\phi_v^2\neq 0. \qquad (20)$$

Let (2) be the generalized equation, in order to find the potential symmetries we write the equation in the conserved form. From this conserved form the associated auxiliary system (21)

$$\begin{aligned}
v_x &= u\\
v_t &= \frac{1}{k}\left(u^n\right)_{xx}-\frac{1}{k}\left(u^n\right)
\end{aligned} \qquad (21)$$

with potentials as additionals dependent variables is given. If $(u(x),v(x))$ satisfies (21) then $u(x)$ solves (2). The classical method applied to (21) gives rise to the following determining equations

$$\begin{aligned}
n\,u^{n+2}\,\frac{d\xi}{dv}-n^2\,u^2\,\frac{d\xi}{dt}-n^2\,\frac{d^2\phi}{dv^2}\,u^{n+2}&\\
+n\,\frac{d\tau}{dt}\,u^{n+1}-n\,\frac{d\psi}{dv}\,u^{n+1}-2\,n^2\,\frac{d^2\phi}{dvdx}\,u^{n+1}&\\
-n^2\,\frac{d^2\phi}{dx^2}\,u^n+n^2\,\phi\,u^n+n^2\,\frac{d\psi}{dt}\,u&=0\\
-n\,u\,\frac{d\xi}{dx}-n\,u^2\,\frac{d\xi}{dv}+n\,\frac{d\psi}{dv}\,u+n\,\frac{d\psi}{dx}-n\,\phi&=0\\
2\,n\,u\,\frac{d\xi}{dx}+n\,u^2\,\frac{d\xi}{dv}-n\,\frac{d\tau}{dt}\,u&\\
+n\,\frac{d\psi}{dv}\,u-n\,\frac{d\phi}{du}\,u-n^2\,\phi+n\,\phi&=0\\
2\,n^2\,u\,\frac{d\xi}{dx}-2\,n\,u\,\frac{d\xi}{dx}+n\,u^n\,\frac{d\xi}{dv}&\\
+n^2\,u^2\,\frac{d\xi}{dv}-n\,u^2\,\frac{d\xi}{dv}-n\,\frac{d^2\phi}{du^2}\,u^n&\\
-n^2\,\frac{d\tau}{dt}\,u+n\,\frac{d\tau}{dt}\,u+n^2\,\frac{d\psi}{dv}\,u&\\
-n\,\frac{d\psi}{dv}\,u-2\,n^2\,\frac{d\phi}{du}\,u+2\,n\,\frac{d\phi}{du}\,u&=0\\
-u^n\,\frac{d^2\xi}{dx^2}-u^{n+2}\,\frac{d^2\xi}{dv^2}-2\,u^{n+1}\,\frac{d^2\xi}{dvdx}&\\
+2\,\frac{d^2\phi}{dudv}\,u^{n+1}+\frac{d\phi}{dv}\,u^n+2\,\frac{d^2\phi}{dudx}\,u^n&\\
+2\,n\,\frac{d\phi}{dv}\,u^2-2\,\frac{d\phi}{dv}\,u^2&\\
+2\,n\,\frac{d\phi}{dx}\,u-2\,\frac{d\phi}{dx}\,u&=0
\end{aligned} \qquad (22)$$

By solving this system we get *only* recover the classical symmetry of (2)

$$\xi=k_1,\quad \tau=k_2,\quad \psi=k_4,\quad \phi=0$$

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5 Some travelling wave solutions

In the following we present some explicit solutions of the second order ODE's as well as the corresponding travelling wave solutions of the new integrable equation $n = -2$. We also discuss some interpretation of the similarity variables in the above reductions.

The most interesting particular case corresponds to reduction (7). In this reduction the similarity variable and similarity solution are respectively given by $z = x - \lambda t$, $u = h$ so that $u(x,t) = h(x - \lambda t)$. Consequently the corresponding solutions are travelling wave solutions for any arbitrary constant n. Due to the interest of this type of solutions, we study further reductions for the associated ODE. First of all, we see that this equation can be trivially integrated once. Dividing (54) by h^5 and integrating once with respect to z we have

$$2\,h^4\,\mathrm{k}_1 - 2\,h\,\frac{d^2h}{dz^2} + 6\,\left(\frac{dh}{dz}\right)^2 + 2\,c\,h^5 - h^2 = 0.$$

Now we can see that making the change of variables $h = y^{-\frac{1}{2}}$ we get

$$\frac{d^2y}{dz^2} - y + \frac{2\,c}{\sqrt{y}} + 2\,\mathrm{k}_1 = 0$$

Multiplying by $\frac{dy}{dz}$ and integrating once with respect to z we get

$$\left(\frac{dy}{dz}\right)^2 - y^2 + 4\,\mathrm{k}_1\,y + 8\,c\,\sqrt{y} + k_2 = 0.$$

Making the change of variables $y = \alpha^{\frac{4}{3}}$ we get

$$\frac{9\,c}{2} + \left(\frac{d\alpha}{dz}\right)^2 - \frac{9\,\alpha^2}{16} = 0.$$

Derivating once with respect to z we get the linear equation

$$16\,\frac{d^2\alpha}{dz^2} - 9\,\alpha = 0,$$

from its general solution and unmaking the changes of variables we get the following solutions

$$h = -\frac{1}{2\,\sqrt[3]{\lambda}\,\left(\sinh\left(\frac{3\,z}{4} + \frac{3\,\mathrm{k}_3}{4}\right)\right)^{2/3}} \qquad h = \frac{1}{\sqrt[3]{2}\,k^{2/3}\,\sqrt[3]{\mathrm{k}_2}\,\left(\sin\left(\frac{3\,z}{4} + \frac{3\,\mathrm{k}_3}{4}\right)\right)^{2/3}}. \tag{23}$$

Making the change of variables $y = \alpha^{\frac{2}{3}}$ we get

$$h = \frac{\sqrt[3]{\mathrm{k}_2}\,e^{\frac{z}{2}}}{\sqrt[3]{{\mathrm{k}_2}^2\,e^{3\,z} + 4\,\lambda\,\mathrm{k}_2\,e^{\frac{3\,z}{2}} + 4\,\lambda^2}}. \tag{24}$$

The corresponding travelling-wave solutions of (1) are

$$u = -\frac{1}{2\,\sqrt[3]{\lambda}\,\left(\sinh\left(\frac{3\,(x-\lambda t)}{4} + \frac{3\,\mathrm{k}_3}{4}\right)\right)^{2/3}} \qquad u = \frac{1}{\sqrt[3]{2}\,k^{2/3}\,\sqrt[3]{\mathrm{k}_2}\,\left(\sin\left(\frac{3\,(x-\lambda t)}{4} + \frac{3\,\mathrm{k}_3}{4}\right)\right)^{2/3}} \tag{25}$$

$$u = \frac{\sqrt[3]{\mathrm{k}_2}\,e^{\frac{z}{2}}}{\sqrt[3]{{\mathrm{k}_2}^2\,e^{3\,(x-\lambda t)} + 4\,\lambda\,\mathrm{k}_2\,e^{\frac{3\,(x-\lambda t)}{2}} + 4\,\lambda^2}} \tag{26}$$

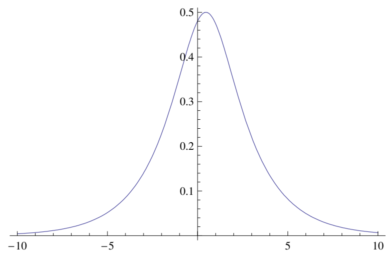

Figure 1: Solution (57) with $k_2 = 1$, $\lambda = 1$.

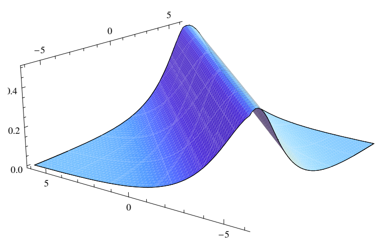

Figure 2: Solution (58) with $k_2 = 1$, $\lambda = 1$.

For reduction (13) we have

$$u = t^{\frac{1}{3}}\,h(x - \log\,(\mathrm{t})). \tag{27}$$

This solution describes a travelling wave with decaying velocity.

In the following we present some explicit solutions of the second order ODE (54). We search the values of parameter n for which (54) admits solutions in terms of the Jacobi elliptic functions. We also present the corresponding travelling wave solutions of the corresponding PDE.

By making the change of variables $y = \alpha^{\frac{1}{n}}$ equation (54) becomes

$$\alpha^{\frac{1}{n}-1}\,\frac{d\alpha}{dz}\,k\,\lambda + \frac{d^3\alpha}{dz^3}\,n - \frac{d\alpha}{dz}\,n = 0. \tag{28}$$

Integrating once with respect to z we get

$$\alpha^{\frac{1}{n}}\,k\,\lambda + \mathrm{k1} + \frac{d^2\alpha}{dz^2} - \alpha = 0. \tag{29}$$

We are now considering the equation

$$h_{zz} + bh^3 + ch^2 + dh + e = 0. \qquad (30)$$

Equation (30) with $b \neq 0$ admits solutions in terms of the Jacobi elliptic functions $sn(kz,p)$, $cn(kz,p)$ when k p b c d and e satisfy some conditions.

Equation (30) with $b = 0$ admits solutions in terms of the Jacobi elliptic functions $sn^2(kz,p)$, $cn^2(kz,p)$ when k, p, c, d and e satisfy some conditions.

We now search for solutions of (30) of the form

$$h = \mathrm{a}_2\, y^2 + \mathrm{a}_1\, y + \frac{\mathrm{b}_1}{y} + \frac{\mathrm{b}_2}{y^2} + \mathrm{a}_0 \qquad (31)$$

where $y = y(z)$ is any of the Jacobi elliptic functions. Substituting (53) into (30), and collecting the coefficients of y we obtain a system of algebraic equations for a_0; a_1; a_2; b_1; b_2; b; c; d; e; k; and p. Solving this system gives, for $y = sn(kz,p)$ the following sets of solutions

We now search for solutions of (30) of the form

$$h = \mathrm{a}_2\, y^2 + \mathrm{a}_1\, y + \frac{\mathrm{b}_1}{y} + \frac{\mathrm{b}_2}{y^2} + \mathrm{a}_0 \qquad (32)$$

where $y = y(kz,p)$ is any of the Jacobi elliptic functions. Substituting (53) into (30), and collecting the coefficients of y we obtain a system of algebraic equations for a0; a1; a2; b1; b2; b; c; d; e; k; and p. Solving this system gives the following sets of solutions

Solution 1

$$h = \mathrm{b}_1\, \sqrt{p}\, \mathrm{sn}\ (k\, z, p) + \frac{\mathrm{b}_1}{\mathrm{sn}\ (k\, z, p)} + \mathrm{a}_0 \qquad (33)$$

Where $a_2 = b_2 = 0$ and the remaining coefficients are related by

$$\begin{aligned}
&b = -\frac{2\, k^2}{\mathrm{b_1}^2} \\
&c = \frac{6\, \mathrm{a}_0\, k^2}{\mathrm{b_1}^2} \\
&d = -\frac{4\, \mathrm{a_0}^3\, k^2 + \mathrm{b_1}^2\, e}{\mathrm{a}_0\, \mathrm{b_1}^2} \\
&a_1^2 = \mathrm{b_1}^2\, p \\
&k^2 = -\frac{\mathrm{b_1}^2\, e\, \sqrt{p}}{\mathrm{a}_0\, \mathrm{b_1}^2\, p^{3/2} + 6\, \mathrm{a}_0\, \mathrm{b_1}^2\, p + \left(\mathrm{a}_0\, \mathrm{b1}^2 - 2\, \mathrm{a_0}^3\right)\, \sqrt{p}} \\
&b_1 \neq 0
\end{aligned}$$

Solution (33) becomes

$$\begin{aligned}
h &= \mathrm{b}_1\, \tanh{(k\, z)} + \frac{\mathrm{b}_1}{\tanh{(k\, z)}} + \mathrm{a}_0 \quad if \quad p = 1 \\
h &= \mathrm{b}_1\, \sin{(k\, z)} + \frac{\mathrm{b}_1}{\sin{(k\, z)}} + \mathrm{a}_0 \qquad if \quad p = 0
\end{aligned}$$

Solution 2

$$h = \mathrm{a}_1\, \mathrm{sn}\ (k\, z, p) + \mathrm{a}_0 \qquad (34)$$

Where $a_2 = b_2 = b_1 = 0$ and the remaining coefficients are related by

$$\begin{aligned}
&b = -\frac{2\, k^2\, p}{\mathrm{a_1}^2} c = \frac{6\, \mathrm{a}_0\, k^2\, p}{\mathrm{a_1}^2} \\
&d = \frac{\left(\mathrm{a_1}^2 - 6\, \mathrm{a_0}^2\right)\, k^2\, p + \mathrm{a_1}^2\, k^2}{\mathrm{a_1}^2} \\
&k^2 = -\frac{\mathrm{a_1}^2\, e}{\left(\mathrm{a}_0\, \mathrm{a_1}^2 - 2\, \mathrm{a_0}^3\right)\, p + \mathrm{a}_0\, \mathrm{a_1}^2}
\end{aligned}$$

Solution (34) becomes

$$\begin{aligned}
h &= \mathrm{a}_1\, \tanh{(k\, z)} + \mathrm{a}_0 \quad p = 1 \\
h &= \mathrm{a}_1\, \sin{(k\, z)} + \mathrm{a}_0 \qquad p = 0
\end{aligned} \qquad (35)$$

Solution 3

$$h = \frac{\mathrm{b1}}{\mathrm{sn}\ (k\, z, p)} + \mathrm{a}_0 \qquad (36)$$

Where $a_2 = b_2 = a_1 = 0$ and the remaining coefficients are related by

$$\begin{aligned}
&b = -\frac{2\, k^2}{\mathrm{b_1}^2} \\
&c = \frac{6\, \mathrm{a}_0\, k^2}{\mathrm{b_1}^2} \\
&d = -\frac{4\, \mathrm{a_0}^3\, k^2 + \mathrm{b_1}^2\, e}{\mathrm{a}_0\, \mathrm{b_1}^2} \\
&k^2 = -\frac{\mathrm{b_1}^2\, e}{\mathrm{a}_0\, \mathrm{b_1}^2\, p + \mathrm{a}_0\, \mathrm{b_1}^2 - 2\, \mathrm{a_0}^3} \\
&b_1 \neq 0
\end{aligned}$$

Solution (36) for becomes

$$\begin{aligned}
h &= \frac{\mathrm{b}_1}{\tanh{(k\, z)}} + \mathrm{a}_0 \quad p = 1 \\
h &= \frac{\mathrm{b}_1}{\sin{(k\, z)}} + \mathrm{a}_0 \qquad p = 0
\end{aligned} \qquad (37)$$

Solution 4

$$h = a_1 \mathrm{cn}\ (kz, p) + \frac{b_1}{\mathrm{cn}(kz,p)} + a_0 \qquad (38)$$

where $a_2 = b_2 = 0$ and the remaining coefficients are related by

$$\begin{aligned}
&b = \frac{2\,k^2\,p}{\mathrm{a_1}^2}\\
&c = -\frac{6\,\mathrm{a_0}\,k^2\,p}{\mathrm{a_1}^2}\\
&d = -\frac{\left(6\,\mathrm{a_1}\,\mathrm{b_1} + 2\,\mathrm{a_1}^2 - 6\,\mathrm{a_0}^2\right)\,k^2\,p - \mathrm{a_1}^2\,k^2}{\mathrm{a_1}^2}\\
&k^2 = \frac{\mathrm{a_1}^2\,e}{\left(6\,\mathrm{a_0}\,\mathrm{a_1}\,\mathrm{b_1} + 2\,\mathrm{a_0}\,\mathrm{a_1}^2 - 2\,\mathrm{a_0}^3\right)\,p - \mathrm{a_0}\,\mathrm{a_1}^2}\\
&b_1^2 = \frac{\mathrm{a_1}^2\,(p-1)}{p}\\
&p \neq 0
\end{aligned}$$

Solution (39) for $p = 1$ becomes

$$h = \mathrm{a_1}\,\mathrm{sech}\,\left(k\,z\right) + \mathrm{a_0}$$

Solution 5

$$h = \mathrm{a_1}\,\mathrm{cn}\,\left(k\,z, p\right) + \mathrm{a_0} \tag{39}$$

where $a_2 = b_2 = b_1 = 0$ and the remaining coefficients are related by

$$\begin{aligned}
&b = \frac{2\,k^2\,p}{\mathrm{a_1}^2}\\
&c = -\frac{6\,\mathrm{a_0}\,k^2\,p}{\mathrm{a1}^2}\\
&d = -\frac{\left(2\,\mathrm{a_1}^2 - 6\,\mathrm{a_0}^2\right)\,k^2\,p - \mathrm{a_1}^2\,k^2}{\mathrm{a_1}^2}\\
&e = \frac{\left(2\,\mathrm{a_0}\,\mathrm{a_1}^2 - 2\,\mathrm{a_0}^3\right)\,k^2\,p - \mathrm{a_0}\,\mathrm{a_1}^2\,k^2}{\mathrm{a_1}^2}\\
&f = \frac{\left(\mathrm{a_1}^4 - 2\,\mathrm{a_0}^2\,\mathrm{a_1}^2 + \mathrm{a_0}^4\right)\,k^2\,p + \left(\mathrm{a_0}^2\,\mathrm{a_1}^2 - \mathrm{a_1}^4\right)\,k^2}{\mathrm{a_1}^2}
\end{aligned}$$

Solution (39) for $p = 1$ becomes

$$h = \mathrm{a_1}\,\mathrm{sech}\,\left(k\,z\right) + \mathrm{a_0}$$

and for $p = 0$ becomes

$$h = \mathrm{a_1}\,\cos\left(k\,z\right) + \mathrm{a_0}$$

Solution 6

$$h = \frac{b_1}{\mathrm{cn}(kz, p)} + a_0 \tag{40}$$

where $a_2 = b_2 = a_1 = 0$ and the remaining coefficients are related by

$$\begin{aligned}
&b = \frac{2\,k^2\,(p-1)}{\mathrm{b_1}^2}\\
&c = -\frac{6\,\mathrm{a_0}\,k^2\,(p-1)}{\mathrm{b_1}^2}\\
&d = -\frac{k^2\,\left(2\,\mathrm{b_1}^2\,p - 6\,\mathrm{a_0}^2\,p - \mathrm{b_1}^2 + 6\,\mathrm{a_0}^2\right)}{\mathrm{b_1}^2}\\
&e = \frac{\mathrm{a_0}\,k^2\,\left(2\,\mathrm{b_1}^2\,p - 2\,\mathrm{a_0}^2\,p - \mathrm{b_1}^2 + 2\,\mathrm{a_0}^2\right)}{\mathrm{b_1}^2}\\
&f = \frac{\left(\mathrm{b_1} - \mathrm{a_0}\right)\,\left(\mathrm{b_1} + \mathrm{a_0}\right)\,k^2\,\left(\mathrm{b_1}^2\,p - \mathrm{a_0}^2\,p + \mathrm{a_0}^2\right)}{\mathrm{b_1}^2}
\end{aligned}$$

Solution (40) for $p = 1$ and $p = 0$ becomes

$$\begin{aligned}
&h = \frac{\mathrm{b_1}}{sech\,(k\,z)} + \mathrm{a_0} \quad p = 1\\
&h = \frac{\mathrm{b_1}}{\cos\,(k\,z)} + \mathrm{a_0} \quad p = 0
\end{aligned} \tag{41}$$

Solution 7 Solution

$$h = \mathrm{a_2}\,\mathrm{sn}^{\,2}\,(k\,z, p) + \frac{\mathrm{b_2}}{\mathrm{sn}^{\,2}\,(k\,z, p)} + \mathrm{a_0} \tag{42}$$

Where $b = a_1 = b_1 = 0$ and the remaining coefficients are related by

$$\begin{aligned}
&a_2 = \mathrm{b_2}\,p\\
&c = -\frac{6\,k^2}{\mathrm{b_2}}\\
&d = \frac{4\,k^2\,(\mathrm{b_2}\,p + \mathrm{b_2} + 3\,\mathrm{a_0})}{\mathrm{b_2}}\\
&e = \frac{2\,k^2\,\left(4\,\mathrm{b_2}^2\,p - 2\,\mathrm{a_0}\,\mathrm{b_2}\,p - 2\,\mathrm{a_0}\,\mathrm{b_2} - 3\,\mathrm{a_0}^2\right)}{\mathrm{b_2}}\\
&f = -\frac{4\,k^2\,\left(\mathrm{b_2}\,p + \mathrm{b_2} + \mathrm{a_0}\right)\,\left(4\,\mathrm{b_2}^2\,p - \mathrm{a_0}^2\right)}{\mathrm{b_2}}
\end{aligned}$$

Solution (42) for $p = 1$ and $p = 0$ becomes

$$\begin{aligned}
&h = \mathrm{a_2}\,\tanh^2(k\,z) + \frac{\mathrm{b_2}}{\tanh^2(k\,z)} + \mathrm{a_0} \quad if \quad p = 1\\
&h = \mathrm{a_2}\,\sin^2(k\,z) + \frac{\mathrm{b_2}}{\sin^2(k\,z)} + \mathrm{a_0} \quad if \quad p = 0
\end{aligned}$$

Solution 8

$$h = \mathrm{a_2}\,\mathrm{cn}^{\,2}\,(k\,z, p) + \frac{\mathrm{b_2}}{\mathrm{cn}^{\,2}\,(k\,z, p)} + \mathrm{a_0} \tag{43}$$

Where $b = a_1 = b_1 = 0$ and the remaining coefficients are related by

$$
\begin{aligned}
b_2 &= \frac{\mathrm{a}_2\,(p-1)}{p}\\
c &= \frac{6\,k^2\,p}{\mathrm{a}_2}\\
d &= -\frac{4\,k^2\,(2\,\mathrm{a}_2\,p+3\,\mathrm{a}_0\,p-\mathrm{a}_2)}{\mathrm{a}2}\\
e &= -\frac{2\,k^2\,(4\,\mathrm{a}_2{}^2\,p-4\,\mathrm{a}_0\,\mathrm{a}_2\,p-3\,\mathrm{a}_0{}^2\,p-4\,\mathrm{a}_2{}^2+2\,\mathrm{a}_0\,\mathrm{a}_2)}{\mathrm{a}_2}\\
f &= \frac{4\,k^2\,(2\,\mathrm{a}_2\,p+\mathrm{a}_0\,p-\mathrm{a}_2)\,(4\,\mathrm{a}_2{}^2\,p-\mathrm{a}_0{}^2\,p-4\,\mathrm{a}_2{}^2)}{\mathrm{a}_2\,p}
\end{aligned}
$$

Solution (43) for $p = 1$ becomes

$$h = \mathrm{a}_2\,\mathrm{sech}^2\,(k\,z) + \mathrm{a}_0$$

Solution 9

$$h = \mathrm{a}_2\,\mathrm{sn}^{\,2}(k\,z,p) + \mathrm{a}_0 \tag{44}$$

Where $b = a_1 = b_1 = 0$ and the remaining coefficients are related by

$$
\begin{aligned}
c &= -\frac{6\,k^2\,p}{\mathrm{a}_2}\\
d &= \frac{(4\,\mathrm{a}_2+12\,\mathrm{a}_0)\,k^2\,p+4\,\mathrm{a}_2\,k^2}{\mathrm{a}_2}\\
e &= -\frac{(4\,\mathrm{a}_0\,\mathrm{a}_2+6\,\mathrm{a}_0{}^2)\,k^2\,p+(2\,\mathrm{a}_2{}^2+4\,\mathrm{a}_0\,\mathrm{a}_2)\,k^2}{\mathrm{a}2}\\
f &= \frac{(4\,\mathrm{a}_0{}^2\,\mathrm{a}_2+4\,\mathrm{a}_0{}^3)\,k^2\,p+(4\,\mathrm{a}_0\,\mathrm{a}_2{}^2+4\,\mathrm{a}_0{}^2\,\mathrm{a}_2)\,k^2}{\mathrm{a}_2}\\
a_2 &\neq 0
\end{aligned}
$$

Solution (44) for $p = 1$ and $p = 0$ becomes

$$
\begin{aligned}
h &= \mathrm{a}_2\,\tanh^2(k\,z) + \mathrm{a}_0 \quad p = 1\\
h &= \mathrm{a}_2\,\sin^2(k\,z) + \mathrm{a}_0 \quad p = 0
\end{aligned}
\tag{45}
$$

Solution 10

$$h = \frac{\mathrm{b}_2}{\mathrm{sn}^{\,2}(k\,z,p)} + \mathrm{a}_0 \tag{46}$$

Where $b = a_1 = b_1 = 0$ and the remaining coefficients are related by

$$
\begin{aligned}
c &= -\frac{6\,k^2}{\mathrm{b}_2}\\
d &= \frac{4\,k^2\,(\mathrm{b}_2\,p+\mathrm{b}_2+3\,\mathrm{a}_0)}{\mathrm{b}_2}\\
e &= -\frac{2\,k^2\,\left(\mathrm{b}_2{}^2\,p+2\,\mathrm{a}_0\,\mathrm{b}_2\,p+2\,\mathrm{a}_0\,\mathrm{b}_2+3\,\mathrm{a}_0{}^2\right)}{\mathrm{b}_2}\\
f &= \frac{4\,\mathrm{a}_0\,(\mathrm{b}_2+\mathrm{a}_0)\,k^2\,(\mathrm{b}_2\,p+\mathrm{a}_0)}{\mathrm{b}_2}
\end{aligned}
$$

Solution (46) for $p = 1$ and $p = 0$ becomes

$$
\begin{aligned}
h &= \frac{\mathrm{b}_2}{\tanh^2(k\,z)} + \mathrm{a}_0 \quad p = 1\\
h &= \frac{\mathrm{b}_2}{\sin^2(k\,z)} + \mathrm{a}_0 \quad p = 0
\end{aligned}
\tag{47}
$$

Solution 11

$$h = \mathrm{a}_2\,\mathrm{cn}^{\,2}(k\,z,p) + \mathrm{a}_0 \tag{48}$$

Where $b = a_1 = b_1 = 0$ and the remaining coefficients are related by

$$
\begin{aligned}
c &= \frac{6\,k^2\,p}{\mathrm{a}_2}\\
d &= -\frac{4\,k^2\,(2\,\mathrm{a}_2\,p+3\,\mathrm{a}_0\,p-\mathrm{a}_2)}{\mathrm{a}_2}\\
e &= \frac{2\,k^2\,(\mathrm{a}_2{}^2\,p+4\,\mathrm{a}_0\,\mathrm{a}_2\,p+3\,\mathrm{a}_0{}^2\,p-\mathrm{a}2^2-2\,\mathrm{a}0\,\mathrm{a}2)}{\mathrm{a}_2}\\
f &= -\frac{4\,\mathrm{a}_0\,(\mathrm{a}_2+\mathrm{a}_0)\,k^2\,(\mathrm{a}_2\,p+\mathrm{a}_0\,p-\mathrm{a}_2)}{\mathrm{a}_2}\\
a_2 &\neq 0
\end{aligned}
$$

Solution (48) for $p = 1$ becomes

$$h = \mathrm{a}_2\,\mathrm{sech}^2\,(k\,z) + \mathrm{a}_0$$

and for $p = 0$ becomes

$$h = \mathrm{a}_2\,\cos^2(k\,z) + \mathrm{a}_0$$

Solution 12

$$h = \frac{\mathrm{b}_2}{\mathrm{cn}^{\,2}(k\,z,p)} + \mathrm{a}_0 \tag{49}$$

Where $b = a_1 = b_1 = 0$ and the remaining coefficients are related by

$$
\begin{aligned}
c &= \frac{6\,k^2\,(p-1)}{\mathrm{b}_2}\\
d &= -\frac{4\,k^2\,(2\,\mathrm{b}_2\,p+3\,\mathrm{a}_0\,p-\mathrm{b}_2-3\,\mathrm{a}_0)}{\mathrm{b}_2}\\
e &= \frac{2\,k^2\,\left(\mathrm{b}_2{}^2\,p+4\,\mathrm{a}_0\,\mathrm{b}_2\,p+3\,\mathrm{a}_0{}^2\,p-2\,\mathrm{a}_0\,\mathrm{b}_2-3\,\mathrm{a}_0{}^2\right)}{\mathrm{b}_2}\\
f &= -\frac{4\,\mathrm{a}_0\,(\mathrm{b}_2+\mathrm{a}_0)\,k^2\,(\mathrm{b}_2\,p+\mathrm{a}_0\,p-\mathrm{a}_0)}{\mathrm{b}_2}
\end{aligned}
$$

Solution (49) for $p = 1$ and $p = 0$ becomes

$$
\begin{aligned}
h &= \frac{\mathrm{b}_2}{sech^2(k\,z)} + \mathrm{a}_0 \quad p = 1\\
h &= \frac{\mathrm{b}_2}{\cos^2(k\,z)} + \mathrm{a}_0 \quad p = 0
\end{aligned}
\tag{50}
$$

Theorem Suppose that y is a solution of the following ODE

$$\frac{d^2h}{dz^2}+\mathrm{n}_1\,h^3+\mathrm{n}_2\,h^2+\mathrm{n}_3\,h+\mathrm{n}_4=0 \tag{51}$$

or

$$\left(\frac{dh}{dz}\right)^2+\frac{\mathrm{n}_1\,h^4}{2}+\frac{2\,\mathrm{n}_2\,h^3}{3}+\mathrm{n}_3\,h^2+2\,\mathrm{n}_4\,h+\mathrm{n}_5=0 \tag{52}$$

Then

$$h=\mathrm{a}_2\,y^2+\mathrm{a}_1\,y+\frac{\mathrm{b}_1}{y}+\frac{\mathrm{b}_2}{y^2}+\mathrm{a}_0 \tag{53}$$

is a solution of the ODE

$$\frac{d^2h}{dz^2}+b\,h^3+c\,h^2+d\,h+e=0 \tag{54}$$

$$\left(\frac{dh}{dz}\right)^2+\frac{b\,h^4}{2}+\frac{2\,c\,h^3}{3}+d\,h^2+2\,e\,h+f=0 \tag{55}$$

with $a_2=b_2=0$ and where the remaining coefficients are related by

$$n_1=\mathrm{a}_1{}^2\,b$$

$$n_2=\mathrm{a}_1\,c+3\,\mathrm{a}_0\,\mathrm{a}_1\,b$$

$$n_3=\frac{\mathrm{a}_1{}^2\,c}{6\,\mathrm{a}_2}$$

$$n_4=\frac{\mathrm{b}_1\,c+3\,\mathrm{a}_0\,b\,\mathrm{b}_1}{3}$$

$$n_5=\frac{b\,\mathrm{b}_1{}^2}{2}$$

$$e=-\frac{3\,\mathrm{a}_0\,d+\left(4\,\mathrm{a}_1\,\mathrm{b}_1+3\,\mathrm{a}_0{}^2\right)\,c+12\,\mathrm{a}_0\,\mathrm{a}_1\,b\,\mathrm{b}_1+3\,\mathrm{a}_0{}^3\,b}{3}. \tag{56}$$

We observe that equation (54) is a particular case of equation (30) for $n=\frac{1}{3}$ where $b=k\lambda$, $c=0$, $d=-1$, $e=k_1$ and for $n=\frac{1}{2}$ where $b=0$, $c=k\lambda$, $d=-1$, $e=k_1$. We obtain the following: Equation (54) admits any solution $h=\alpha^3$ where α is any solution of equation (30) with $n=\frac{1}{3}$, $b=k\lambda$, $c=0$, $d=-1$ and $e=k_1$.

Equation (54) admits any solution $h=\alpha^2$ where α is any solution of equation (30) with $n=\frac{1}{3}$, $b=0$, $c=k\lambda$, $d=-1$ and $e=k_1$.

In the following we present two examples:

Example 1 equation (54) with $n=\frac{1}{3}$ admits the solution

$$h=\frac{2\,\sqrt{6}\,\mathrm{sech}^{\,3}z\,\left(\frac{1}{k\,\lambda}\right)^{3/2}}{\sqrt{3}} \tag{57}$$

The corresponding travelling-wave solutions of (1)

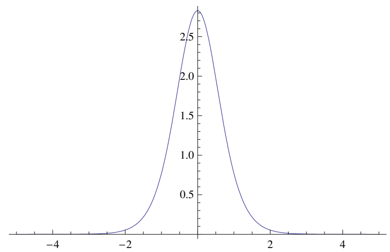

Figure 3: Solution (57) with $k_2=1$, $\lambda=1$.

are

$$u=\frac{2\,\sqrt{6}\,\mathrm{sech}^{\,3}(x-\lambda t)\,\left(\frac{1}{k\,\lambda}\right)^{3/2}}{\sqrt{3}} \tag{58}$$

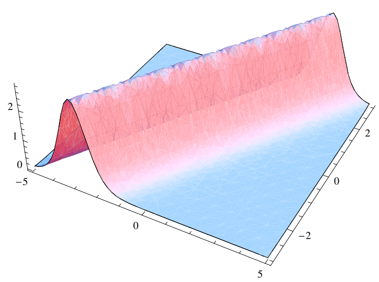

Figure 4: Solution (58) with $k_2=1$, $\lambda=1$.

Example 2 equation (54) with $n=\frac{1}{2}$ admits the solution

$$h=\frac{225\,\mathrm{sn}^{\,4}\left(z,-\frac{5}{4}\right)}{4\,k^2\,\lambda^2} \tag{59}$$

The corresponding travelling-wave solutions of (1)

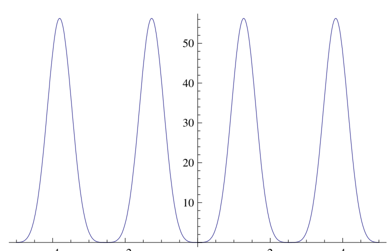

Figure 5: Solution (59) with $k_2=1$, $\lambda=1$.

are

$$u=\frac{225\,\mathrm{sn}^{\,4}\left((x-\lambda t),-\frac{5}{4}\right)}{4\,k^2\,\lambda^2} \tag{60}$$

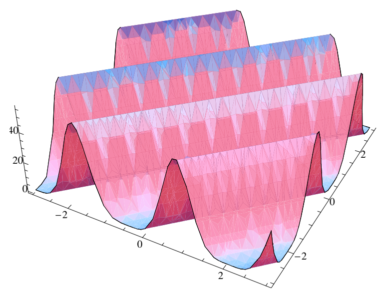

Figure 6: Solution (60) with $k_2 = 1$, $\lambda = 1$.

6 Conclusions

In this work we have discussed symmetry reductions for the generalized equation (2). For $n = -2$ Eq. (2) becomes Eq.(1) which is a new integrable equation introduced in [24]. By using the classical Lie method, we obtained reductions to ODE's and some exact solutions. We apply Lie classical method to the associated potential system (21), but we do not get any potential symmetry, moreover we loose some classical symmetries of (2). We obtain travelling waves with decaying velocity and we exhibit an smooth soliton solution.

7 Acknowledgments

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8 Appendix A

Table 1: Commutator table for the Lie algebra

(a)	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_4$	$-\mathbf{v}_5$
$\mathbf{v}_2$	0	0	$\mathbf{v}_2$	0	0
$\mathbf{v}_3$	0	$-\mathbf{v}_2$	0	0	0
$\mathbf{v}_4$	$-\mathbf{v}_4$	0	0	0	$-2\mathbf{v}_1$
$\mathbf{v}_5$	$\mathbf{v}_5$	0	0	$2\mathbf{v}_1$	0

Table 2: Adjoint table .

Ad	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$e^{\epsilon}\mathbf{v}_4$	$e^{-\epsilon}\mathbf{v}_5$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3-\epsilon\mathbf{v}_2$	$\mathbf{v}_4$	$\mathbf{v}_5$
$\mathbf{v}_3$	$\mathbf{v}_1$	$e^{-\epsilon}\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$
$\mathbf{v}_4$	$\mathbf{v}_1+\epsilon\mathbf{v}_4$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5+2\epsilon\mathbf{v}_1$
$\mathbf{v}_5$	$\mathbf{v}_1-\epsilon\mathbf{v}_5$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4-2\epsilon\mathbf{v}_1$	$\mathbf{v}_5$