Producer′s Replenishment Policy for an EPQ Model with Rework and Machine Failure Taking Place in Backorder Reloading Time

KUANG-KU CHEN[a], YUAN-SHYI PETER CHIU[b], CHIA-KUAN TING[b,*]

[a]Department of Accounting, College of Management, National Changhua University of Education, Changhua 500, TAIWAN

[b]Department of Industrial Engineering and Management, Chaoyang University of Technology 168 Jifong East Road, Wufong, Taichung 413, TAIWAN

[*]E-mail: djk@cyut.edu.tw (CHIA-KUAN TING) http://www.cyut.edu.tw/~iem/teacher.htm

Abstract: - This paper studies the optimal producer′s replenishment policy for an economic production quantity (EPQ) model with rework and breakdown taking place in backorder replenishing time. A recent published article has studied the lot-sizing problem on an imperfect quality EPQ model. However, another reliability factor - random machine breakdown is inevitable in most real-life production systems. To deal with stochastic machine failures, production planners must practically calculate the mean time between failures (MTBF) and establish the robust plan accordingly, in terms of the optimal production run time that minimizes total production-inventory costs for such an unreliable system. This study extends Chiu's work and assumes that a machine failure takes place in the backorder replenishing stage. Mathematical modeling and cost analysis are employed. The renewal reward theorem is used to cope with variable cycle length. Convexity of the long-run average cost function is proved and an optimal manufacturing lot-size that minimizes the expected overall costs for such an imperfect system is derived. Numerical example is provided to demonstrate its practical usages.

Key-Words: - Optimization, Lot size, Breakdown, Backorder, Production run time, Rework, Inventory

1 Introduction

The economic production quantity (EPQ) model is often employed in the manufacturing sector for determining the optimal replenishment lot-size that minimizes the expected long-run average production inventory costs [1-3]. Disregarding the simplicity of the EPQ model, it is still applied nowadays and it remains the basis for the analyses of more complex systems [3-8]. An implicit assumption of EPQ model is that the manufactured items are of perfect quality. However, in real life production systems, owing to various unpredictable factors, generation of random defective items is inevitable. Many studies have been carried out to address the issue of imperfect quality items in EPQ model [9-16].

Lee and Rosenblatt [10] examined an EPQ model with joint determination of production cycle time and inspection schedules. A relationship that can be used to determine the effectiveness of maintenance by inspection was derived. Cheng [13] formulated inventory as a geometric program and obtained the closed-form optimal solutions for an EOQ model with demand-dependent unit production cost and imperfect production processes. Boone et al. [14] investigated the impact of imperfect processes on the production run time. They proposed a model in an attempt to provide managers with guidelines to choose the appropriate production run times to cope with both the defective items and stoppages occurring due to machine breakdowns. Cheung and Hausman [16] developed an analytical model of preventive maintenance (PM) and safety stock (SS) strategies in a production environment subject to random machine breakdowns. They illustrated the trade-off between investing in the two options (PM and SS) and also provided optimality conditions under which either one or both strategies should be implemented to minimize the associated cost function. Both the deterministic and exponential repairing time distributions are analyzed in their study.

Imperfect quality items, in some circumstances, can be reworked and repaired. For example, the plastic goods in plastic injection molding process, the printed circuit board assembly (PCBA) in PCBA manufacturing, etc. Therefore, overall production-inventory costs can be significantly reduced [17-23]. Jamal et al. [18] examined optimal manufacturing batch size with rework process at a single-stage production system. Chiu [20] studied the optimal lot size for an imperfect quality finite production rate with rework and backlogging.

Out of stock situations may arise occasionally due to the excess demands. These shortages sometimes are allowed; and they are backordered and satisfied by the next replenishment. So the long-run average production- inventory costs can be reduced substantially [17,19,20,22].

Machine failure can be considered as a critical reliability factor that seems to trouble practitioners in their production management. Breakdown may occur unexpectedly even machine is under a well planned preventive maintenance (PM). In order to minimize the overall production-inventory costs, to effectively control and manage the disruption caused by machine breakdown becomes a very important task to most practitioners in production management field. Therefore, determination of the optimal run time for manufacturing systems subject to machine failures has received extensive attention from researchers in the past decades [24-33]. Groenevelt et al. [24] proposed two inventory control policies to deal with machine failures. One assumes that the production of the interrupted lot is not resumed (called no resumption-NR policy) after a breakdown. The other policy considers that the production of the interrupted lot will be immediately resumed (called abort/resume-AR policy) after the breakdown is fixed and if the current on-hand inventory falls below a certain threshold level. Repair time is assumed to be negligible, and effects of machine breakdowns and corrective maintenance on the economic lot sizing decisions were investigated. Giri and Dohi [25] presented the exact formulation of stochastic EMQ model for an unreliable production system. Their EMQ model is formulated based on the net present value (NPV) approach and by taking limitation on the discount rate the traditional long-run average cost model is obtained. They also provided criteria for the existence and uniqueness of the optimal production time and computational results showing that the optimal decision based on the NPV approach is superior to that based on the long-run average cost approach. Chiu et al. [26] considered the optimal run time for EPQ model with scrap, rework and random breakdown. They have proved theorems on conditional convexity of the integrated cost function and on bounds of the production run time. Then, an optimal run time was located by the use of the bisection method based on the intermediate value theorem. Makis and Fung [28] studied effects of machine failures on the optimal lot size as well as on optimal number of inspections. Formulas for the long-run average cost per unit time were obtained. Then optimal production-inspection policy that minimizes the expected total costs was derived. Chiu et al. [33] examined the EMQ model with imperfect rework and random breakdown under AR policy. Backordering was not considered in their proposed model.

This paper extends the work of Chiu [20], by assuming that a machine breakdown takes place in the backorder satisfying time of such an imperfect quality EPQ model. The effects of machine failure on the optimal run time and on the long-run production-inventory costs are studied in this paper. Since little attention was paid to aforementioned area, this study intends to bridge the gap.

2 Modeling and Solution Derivations

Reexamine the EPQ model studied by Chiu [20]. A manufactured product can be made at an annual rate of P and the annual demand of this item is λ units. The production rate P is much larger than the demand rate λ, and the production system may randomly produce x portion of defective items at a rate d, where $d=Px$. All items produced are screened and the inspection cost per item is included in the unit production cost C.

The production rate of perfect quality items must always be greater than or equal to the sum of the demand rate λ and the defective rate d. Hence, the following condition must hold: $(P-d-\lambda)>=0$ or $(1-x-\lambda/P)>=0$. A θ portion of the imperfect quality items is scrap and is discarded when regular production ends. The other $(1-\theta)$ portion of the defective items is reworked at a rate of P_1 immediately after the regular process. Stock-outs are allowed and backordered, they are satisfied by the next replenishment. Further, according to the mean time between failures (MTBF) data, machine breakdown may take place randomly in the backorder-filling time (see Figure 1) and the abort/ resume inventory control policy is adopted in this study. Under such a policy, when a breakdown happens, machine is under corrective maintenance immediately. A constant repair time is assumed and the interrupted lot will be resumed right after the restoration of machine.

It is also assumed that during the setup time, prior to the production uptime, the working status of machine is fully checked and confirmed. Hence, the chance of breakdown in a very short period of time when production begins is small. It is also assumed that due to tight preventive maintenance schedule, the probability of more than one machine breakdown occurrences in a production cycle is very small. However, if it does happen, safety stock will be used to satisfy the demand during machine repairing time. Therefore, multiple machine failures are assumed to have insignificant effect on the proposed model. Figure 1 illustrates the on-hand inventory of perfect quality items in the proposed EPQ.

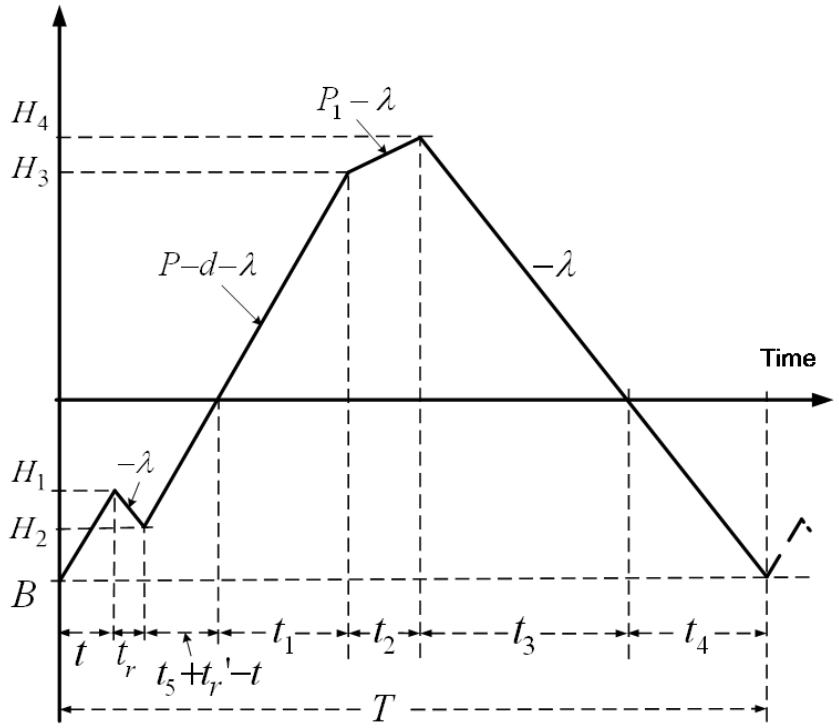

Fig. 1: On-hand inventory level of perfect quality items in EPQ model with rework and machine failure taking place in backorder replenishing time

Cost parameters considered in the proposed model include: unit production cost C, setup cost K, unit holding cost h, repair cost for each defective item reworked C_R, unit holding cost per reworked item h_1, disposal cost per scrap item C_S, unit shortage/ backordered cost b, and cost for repairing and restoring machine M. Other notation used is listed as follows.

T_1 = optimal production uptime - decision variable to be solved for the proposed EPQ model,
t = production time before a random breakdown occurs,
H_1 = the level of backorder quantity when machine breakdown occurs,
t_r = time required for repairing and restoring the machine,
H_2 = the level of backorder quantity when machine is repaired and restored,
t_5 = time required for filling the backorder quantity B (excluding t_r and t_r'),
t_4 = shortage permitted time,
T = the production cycle length,
t_r' = time required for producing sufficient stocks to satisfy the demand during machine repair time t_r,
t_1 = time for piling up stocks during the production uptime in each cycle,
H_3 = the level of perfect quality inventory when regular production process ends,
t_2 = time needed to rework $(1-\theta)$ of the repairable defective items,
H_4 = the maximum level of perfect quality inventory when rework finishes,
t_3 = time required for depleting all available perfect quality on-hand items,
Q = production lot size for each cycle,
B = the maximum backorder level allowed for each cycle,
$TC(T_1,B)$ = total production-inventory costs per cycle,
$TCU(T_1,B)$ = total production-inventory costs per unit time (e.g. annual),
$E[TCU(T_1,B)]$ = the expected production-inventory costs per unit time.

Because t denotes the production time before a breakdown taking place in backorder replenishing period t_5, that is $t < t_5$. Let the maximum machine repair time be a constant and $t_r = g$. In this study, it is conservatively assumed that if a failure of a machine cannot be fixed within a certain allowable amount of time, then a spare machine will be in place to avoid further delay of production. t_r' is time needed to produce required items demanded in t_r.

The following derivation procedure is similar to what was used by past studies [20]. From Figure 1, one can obtain the following: production uptime T_1; t_r'; the cycle length T; the levels of backorder H_1 and H_2; the levels of on-hand perfect quality inventory H_3 and H_4; time for piling up stocks t_1; time for reworking repairable items t_2; time required for depleting all available on-hand items t_3; t_4; time required for satisfying B (i.e. maximum backorder quantity) t_5; and production lot size Q.

$$T_1 = t_5 + t_r' + t_1 = \frac{Q}{P} \tag{1}$$

$$t_r' = \frac{g\lambda}{P-d-\lambda} \tag{2}$$

$$T = \sum_{i=1}^{5} t_i + (t_r + t_r') \tag{3}$$

$$H_1 = B - (P-d-\lambda)t \tag{4}$$

$$H_2 = H_1 + t_r\lambda = \left[B - (P-d-\lambda)t\right] + g\lambda \tag{5}$$

$$H_3 = (P-d-\lambda)t_1 \tag{6}$$

$$H_4 = H_3 + (P_1 - \lambda)t_2 \tag{7}$$

$$t_1 = H_3 / (P-d-\lambda) \tag{8}$$

$$t_2 = (d \cdot T_1)\cdot(1-\theta)/P_1 \tag{9}$$

$$t_3 = H_4 / \lambda \tag{10}$$

$$t_4 = B / \lambda \tag{11}$$

$$t_5 = B / (P-d-\lambda) \tag{12}$$

$$Q = P \cdot T_1 = P \cdot \left[t_5 + t_r' + t_1\right] \tag{13}$$

where $d=Px$.

The total defective items produced (see Figure 2) during the production uptime T_1 can be obtained as shown in Eq. (14) and total scrap items produced (see Figure 3) can be calculated by Eq. (15).

$$d \cdot T_1 = P \cdot x \cdot \left[t_5 + t_r' + t_1\right] = x \cdot Q \tag{14}$$

$$\theta \cdot (d \cdot T_1) = \theta \cdot (P \cdot x \cdot T_1) = \theta \cdot (x \cdot Q) \tag{15}$$

Total cost per cycle $TC(T_1,B)$ is

$$\begin{aligned} TC_1(T_1,B) &= K + M + C\cdot(P\cdot T_1) + C_R\cdot(1-\theta)\cdot(T_1\cdot P\cdot x) \\ &+ Cs\cdot\theta\cdot(T_1\cdot P\cdot x) + h\cdot\left[\frac{H_3}{2}(t_1) + \frac{H_3+H_4}{2}(t_2) + \frac{H_4}{2}(t_3)\right] \\ &+ h_1\cdot\left[\frac{d\cdot T_1\cdot(1-\theta)}{2}\right]\cdot(t_2) \\ &+ b\left[\frac{(B+H_1)}{2}(t) + \frac{(H_1+H_2)}{2}(t_r) + \frac{H_2}{2}(t_5+t_r'-t) + \frac{B}{2}(t_4)\right] \\ &+ h\left[\frac{(d\cdot t)}{2}(t) + (d\cdot t)t_r + \frac{(d\cdot t + d\cdot T_1)}{2}(t_5+t_r'-t+t_1)\right] \end{aligned} \tag{16}$$

Substituting all related parameters from Eqs. (1) to (15) in Eq. (16), one obtains $TC(T_1,B)$ as follows.

$$\begin{aligned} TC_1(T_1,B) &= K + M + P\cdot T_1\cdot\left[C + C_R\cdot x\cdot(1-\theta) + C_S\cdot x\cdot\theta\right] \\ &+ gPt\cdot\left[hx - b(1-x)\right] \\ &+ \frac{(b+h)}{2}\cdot\left\{g\cdot[2B+g\lambda] + \frac{B^2}{\lambda}\right\}\cdot\left(\frac{1-x}{1-x-\lambda/P}\right) \\ &+ \frac{T_1^2P^2x^2(1-\theta)^2\cdot(h_1-h)}{2P_1} - hT_1P(1-x\theta)\left[g+\frac{B}{\lambda}\right] \\ &+ \frac{h\cdot T_1^2P}{2}\left[\frac{P(1-2x\theta+x^2\theta^2)}{\lambda} + 2x\theta - 1\right] \end{aligned} \tag{17}$$

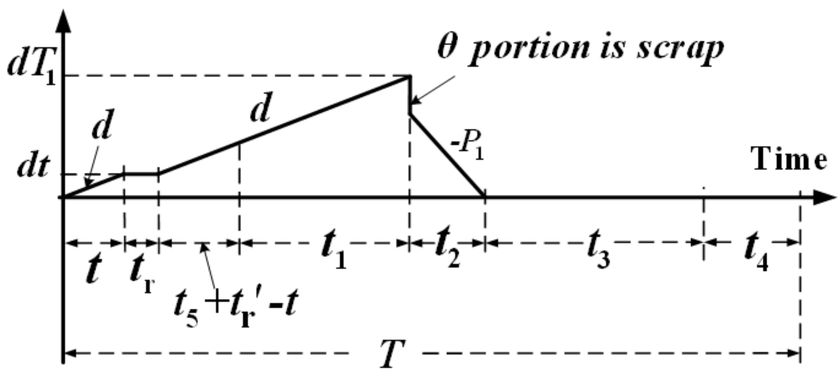

Fig. 2: On-hand inventory of defective items in the proposed EPQ model with rework and machine failure taking place in backorder replenishing time

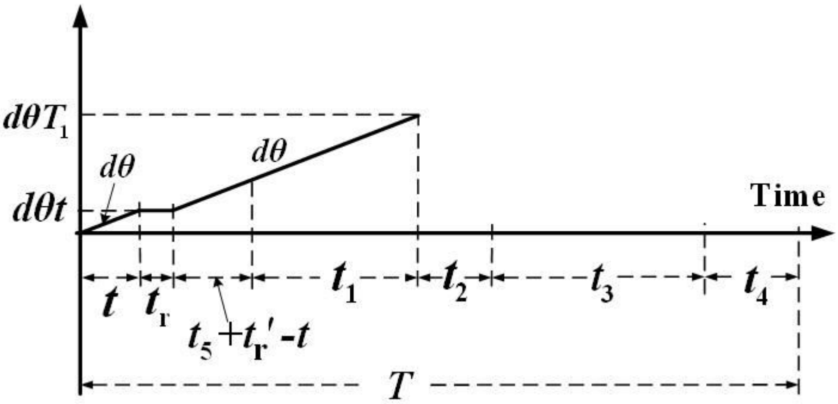

Fig. 3: On-hand inventory of scrap items in the proposed EPQ model with rework and machine failure taking place in backorder replenishing time

Due to the random defective/scrap rates and a uniformly distributed breakdown is assumed to occur in the backorder filling period, the production cycle length is not constant. Thus, to take the randomness of scrap rate and breakdown into account, one can employ the renewal reward theorem in production-inventory cost analysis to cope with the variable cycle length and use the integration of $TC(T_1,B)$ to deal with the random breakdown happening in period t_5. The long-run expected costs per unit time can be calculated as follows.

$$E\left[TCU\left(T_1,B\right)\right]=\frac{E\left[\int_0^{t_5}TC\left(T_1,B\right)f\left(t\right)dt\right]}{E\left[T\right]}=\frac{E\left[\int_0^{t_5}TC\left(T_1,B\right)\left(\frac{1}{t_5}\right)dt\right]}{T_1\cdot P\cdot\left(1-\theta\cdot E\left[x\right]\right)/\lambda} \tag{18}$$

$$\begin{aligned}E\left[TCU\left(T_1,B\right)\right]&=\lambda\left[\frac{\left(K+M\right)}{T_1P}+C\right]\frac{1}{\left(1-E\left[x\right]\theta\right)}-hg\lambda\\&+\lambda\left[C_R(1-\theta)+C_S\theta\right]\frac{E\left[x\right]}{\left(1-E\left[x\right]\theta\right)}+\frac{h}{2}\left[-2B+T_1\left(P-\lambda\right)\right]\frac{1}{\left(1-E\left[x\right]\theta\right)}\\&+h\cdot\theta\cdot\left[B-T_1\left(P-\lambda\right)\right]\frac{E\left[x\right]}{\left(1-E\left[x\right]\theta\right)}\\&+\frac{1}{2T_1P}\left\{g\lambda\left[\left(b+h\right)\left(2B+g\lambda\right)-Bb\right]+\left(b+h\right)B^2\right\}\frac{1}{\left(1-E\left[x\right]\theta\right)}E\left[\frac{1-x}{1-x-\frac{\lambda}{P}}\right]\\&+\frac{T_1P}{2}\left[\frac{\lambda\left(1-\theta\right)^2}{P_1}\left(h_1-h\right)+h\theta^2\right]\frac{E\left[x^2\right]}{\left(1-E\left[x\right]\theta\right)}+\frac{Bhg\lambda}{2T_1P}\frac{1}{\left(1-E\left[x\right]\theta\right)}E\left[\frac{x}{1-x-\frac{\lambda}{P}}\right]\end{aligned} \tag{19}$$

Let

$$E_0=\frac{1}{\left(1-E\left[x\right]\theta\right)};\ E_1=\frac{E\left[x\right]}{\left(1-E\left[x\right]\theta\right)};\ E_2=\frac{E\left[x^2\right]}{\left(1-E\left[x\right]\theta\right)}$$

$$E_3=\frac{1}{\left(1-E\left[x\right]\theta\right)}E\left[\frac{1-x}{1-x-\frac{\lambda}{P}}\right];\ E_4=\frac{1}{\left(1-E\left[x\right]\theta\right)}E\left[\frac{x}{1-x-\frac{\lambda}{P}}\right]$$

then Eq. (19) becomes:

$$\begin{aligned}\therefore\ E\left[TCU\left(T_1,B\right)\right]&=\lambda\left[\frac{\left(K+M\right)}{T_1P}+C\right]E_0+\frac{h}{2}\left[-2B+T_1\left(P-\lambda\right)\right]E_0\\&+\lambda\left[C_R(1-\theta)+C_S\theta\right]E_1-hg\lambda+h\cdot\theta\left[B-T_1\left(P-\lambda\right)\right]E_1\\&+\frac{T_1P}{2}\left[\frac{\lambda\left(1-\theta\right)^2}{P_1}\left(h_1-h\right)+h\cdot\theta^2\right]E_2+\frac{Bhg\lambda}{2T_1P}\cdot E_4\\&+\frac{1}{2}\frac{1}{T_1P}\left\{g\lambda\left[\left(b+h\right)\left(2B+g\lambda\right)-Bb\right]+\left(b+h\right)B^2\right\}\cdot E_3\end{aligned} \tag{20}$$

2.1 Convexity of Cost Function $E[TCU(T_1,B)]$

The optimal replenishment policy can be obtained by minimizing expected cost function $E[TCU(T_1,B)]$. For the proof of convexity of $E[TCU(T_1,B)]$, one can use the Hessian matrix equations [34] and verify the existence of the following:

$$\begin{bmatrix}T_1 & B\end{bmatrix}\cdot\begin{pmatrix}\dfrac{\partial^2E\left[TCU\left(T_1,B\right)\right]}{\partial T_1^2} & \dfrac{\partial^2E\left[TCU\left(T_1,B\right)\right]}{\partial T_1\partial B}\\ \dfrac{\partial^2E\left[TCU\left(T_1,B\right)\right]}{\partial T_1\partial B} & \dfrac{\partial^2E\left[TCU\left(T_1,B\right)\right]}{\partial B^2}\end{pmatrix}\cdot\begin{bmatrix}T_1\\ B\end{bmatrix}>0 \tag{21}$$

The expected production-inventory cost function $E[TCU(T_1,B)]$ is strictly convex only if Eq. (21) is satisfied for all T_1 and B different from zero. From Eqs. (20) and (21) by computing all the elements of the Hessian matrix equation, one obtains Eq. (22). And it is resulting positive, because all parameters are positive. Hence, $E[TCU(T_1,B)]$ is a strictly convex function.

For locating the optimal production uptime T_1 and optimal backorder quantity B, one can differentiate the expected production-inventory cost function $E[TCU(T_1,B)]$ with respect to T_1 and with respect to B, and then solve linear systems of Eqs.(23) and (24) by setting these partial derivatives equal to zero. The resulting optimal T_1* and B* are shown in Eqs.(25) and (26).

$$\begin{bmatrix} T_1 & B \end{bmatrix} \cdot \begin{pmatrix} \dfrac{\partial^2 E\left[TCU(T_1,B)\right]}{\partial T_1^2} & \dfrac{\partial^2 E\left[TCU(T_1,B)\right]}{\partial T_1 \partial B} \\ \dfrac{\partial^2 E\left[TCU(T_1,B)\right]}{\partial T_1 \partial B} & \dfrac{\partial^2 E\left[TCU(T_1,B)\right]}{\partial B^2} \end{pmatrix} \cdot \begin{bmatrix} T_1 \\ B \end{bmatrix}$$

$$= \frac{1}{1-E[x]\theta} \cdot \frac{\lambda}{T_1 P} \left\{ 2(K+M) + (b+h)\lambda g^2 \cdot E\left(\frac{1-x}{1-x-\dfrac{\lambda}{P}} \right) \right\} > 0 \tag{22}$$

$$\begin{aligned} \frac{\partial E\left[TCU_1(T_1,B)\right]}{\partial T_1} &= \left[\frac{-\lambda(K+M)}{T_1^2 P} + \frac{h(P-\lambda)}{2} \right] E_0 - \left[h \cdot \theta (P-\lambda) \right] E_1 \\ &\quad + \frac{P}{2} \cdot \left[\frac{\lambda(1-\theta)^2}{P_1} (h_1 - h) + h \cdot \theta^2 \right] \cdot E_2 - \frac{1}{2} \frac{Bhg\lambda \cdot E_4}{T_1^2 P} \\ &\quad - \frac{1}{2T_1^2 P} \left\{ g\lambda \left[(b+h)(2B+g\lambda) - Bb \right] + (b+h)B^2 \right\} E_3 \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\partial E\left[TCU_1(T_1,B)\right]}{\partial B} &= -h + \frac{1}{T_1 P} \left\{ g\lambda (b/2 + h) + (b+h)B \right\} E_3 \\ &\quad + \frac{hg\lambda}{2T_1 P} \cdot E_4 \end{aligned} \tag{24}$$

$$\therefore T_1^* = \frac{1}{P} \sqrt{ \frac{ \left[2(K+M)\lambda + (b+h)\lambda^2 g^2 E\left[\dfrac{1-x}{1-x-\lambda/P} \right] - \dfrac{\lambda^2 g^2 \left[1-E[x]\theta\right]}{(b+h)E_3} \left[\left(\dfrac{b}{2} + h \right) E_3 + \dfrac{h}{2} E_4 \right]^2 \right] }{ \left[h\left(1-\dfrac{\lambda}{P}\right) + \left\{ \dfrac{\lambda(1-\theta)^2}{P_1}(h_1 - h) + h \cdot \theta^2 \right\} E\left[x^2\right] - 2h\theta\left(1-\dfrac{\lambda}{P}\right) E[x] - \dfrac{h^2\left[1-E[x]\theta\right]}{(b+h)\cdot E_3} \right] } } \tag{25}$$

$$\therefore B^* = \left(\frac{h}{b+h} \right) \left(\frac{P}{E_3} \right) T_1^* - \frac{\lambda g}{2} \left[1 + \left(\frac{h}{b+h} \right) \cdot \left(1 + \frac{E_4}{E_3} \right) \right] \tag{26}$$

It is noted that practitioner should check both the numerator and the denominator of Eq. (25) are positive before adopting this optimal replenishment run time in real life usage. Proof of positive of denominator can be found in Appendix of [35]. One notes that verification of B* to be a positive number is required when put it in use.

From Eqs. (13), (25), and (26), one can obtain the optimal production lot-size Q^* and the optimal backorder quantity B^* as displayed in Eqs. (27) and (28). The long-run expected production-inventory cost function $E[TCU(T_1,B)]$ can then be obtained by substituting T_1 and B^* from Eqs.(25) and (26) into Eq.(19).

$$\therefore Q^* = \sqrt{ \frac{ \left\{ 2(K+M)\lambda + (b+h)\lambda^2 g^2 E\left[\dfrac{1-x}{1-x-\lambda/P} \right] - \dfrac{\lambda^2 g^2 \left[1-E[x]\theta\right]}{(b+h)E_3} \left[\left(\dfrac{b}{2} + h \right) E_3 + \dfrac{h}{2} \cdot E_4 \right]^2 \right\} }{ \left\{ h\left(1-\dfrac{\lambda}{P}\right) + \left\{ \dfrac{\lambda(1-\theta)^2}{P_1}(h_1 - h) + h\theta^2 \right\} E\left[x^2\right] - 2h\theta\left(1-\dfrac{\lambda}{P}\right) \cdot E[x] - \dfrac{h^2\left[1-E[x]\theta\right]}{(b+h)\cdot E_3} \right\} } } \tag{27}$$

$$\therefore B^* = \left(\frac{h}{b+h} \right) \left(\frac{1}{E_3} \right) Q^* - \frac{\lambda g}{2} \left[1 + \left(\frac{h}{b+h} \right) \cdot \left(1 + \frac{E_4}{E_3} \right) \right] \tag{28}$$

2.2 Result Verification

Suppose one excludes the assumption of machine breakdown factor from the proposed model, then the cost and time for repairing failure machine M=0 and g=0, Eqs. (27) and (28) become the same equations as were given by Chiu [20]:

$$\therefore Q^* = \sqrt{ \frac{ 2K\lambda }{ \left\{ h\left(1-\dfrac{\lambda}{P}\right) + \left\{ \dfrac{\lambda(1-\theta)^2}{P_1}(h_1 - h) + h\theta^2 \right\} E\left[x^2\right] - 2h\theta\left(1-\dfrac{\lambda}{P}\right) \cdot E[x] - \dfrac{h^2\left[1-E[x]\theta\right]}{(b+h)\cdot E_3} \right\} } } \tag{29}$$

$$B^* = \left(\frac{h}{b+h} \right) \left(\frac{1-E[x]\theta}{E\left[\dfrac{1-x}{1-x-\lambda/P} \right]} \right) Q^* \tag{30}$$

Further, if the regular production process produces no defective items, i.e. x=0, then Eqs. (29) and (30) become the same equations as were presented by the classical EPQ model with shortages permitted and backordered [2,36].

3 Numerical Example and Discussion

Consider the production rate of a manufactured item is 9,000 units per year and annual demand of it is 3,600 units. The percentage of defective items produced x, follows a uniform distribution over the interval [0, 0.2]. A θ=0.2 portion of defective items is scrap. The annual rate of rework process is P_1=600 units. Additional parameters are summarized as follows.

- C_R = \$0.5 for each item reworked,
- h_1 = \$0.8 per item per unit time,
- K = \$450 for each production run,
- h = \$0.6 per item per unit time,
- C = \$1 per item,
- C_S = \$0.3 disposal cost for each scrap item,
- b = \$0.2 per item backordered per unit time,
- g = 0.018 years, time needed to repair and restore the machine,

M = \$500 repair cost for each breakdown.

A demonstration of convexity of the long-run average costs $E[TCU(T_1,B)]$ is shown in Figure 4. Applying Eqs. (25), (26), (27), and (20), one can obtain the optimal production run time T_1* =0.8478 years, the optimal backorder quantity B*=3,037, the optimal lot-size Q*=7,630, and the optimal long-run expected cost $E[TCU(T_1*,B*)]$= \$4,754.22.

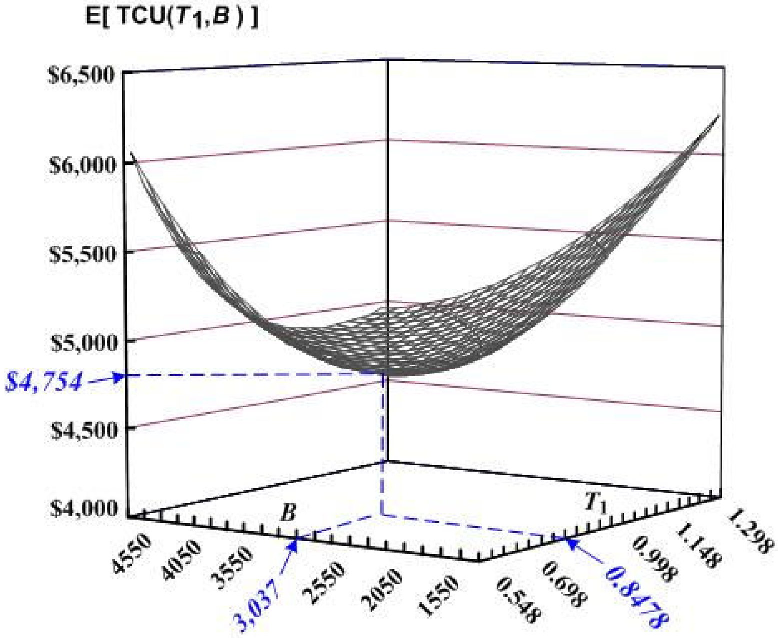

Fig. 4: Convexity of the expected cost function E[TCU(T_1,B)] for the proposed EPQ model with rework and machine failure taking place in backorder replenishing time

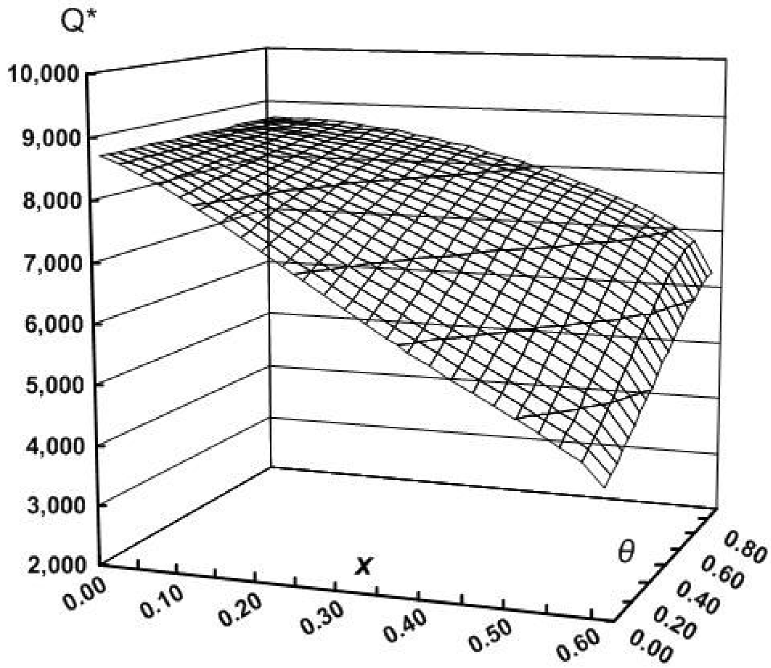

Fig. 5: Variation of the defective rate and the scrap rate effects on the optimal production lot size Q* of the proposed imperfect EPQ model

Figure 5 displays the behavior of the optimal lot size Q^* with respect to random defective rate x and scrap rate θ. It indicates that as the defective rate x increases, the value of optimal lot-size Q^* decreases significantly. It is also noted that for different scrap rates, as θ increases, the optimal production lot-size Q^* increases note-worthily.

Figure 6 illustrates the behavior of the optimal production run time T_1 with respect to the defective rate x and the scrap rate θ. One notes that as the defective rate x increases, the value of production run time T_1^* decreases significantly. It is also noted that for different scrap rates, as θ increases the value of optimal run time T_1^* increases note worthily.

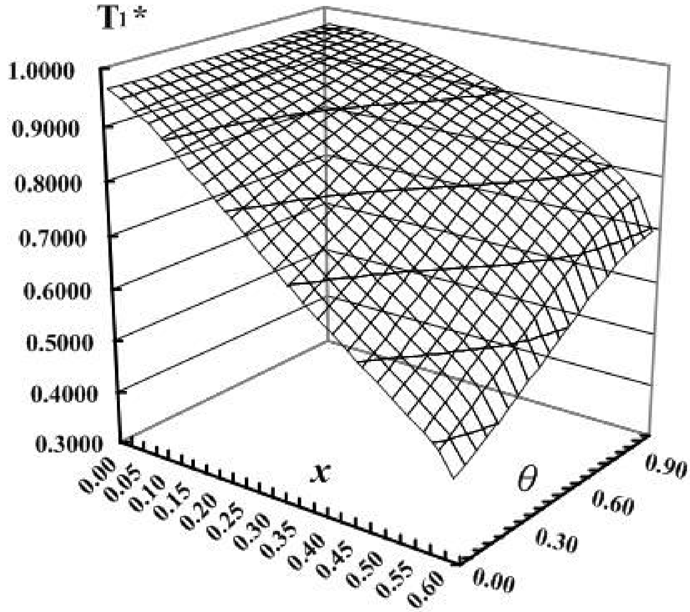

Fig.6: Behavior of the optimal production run time T_1 with respect to the defective rate x and the scrap rate θ of the proposed imperfect EPQ model

Variation of the defective rate x and the scrap rate θ effects on the optimal backorder level B^* is shown in Figure 7. One notes that as the defective rate x increases, the value of optimal backorder quantity B^* decreases significantly. For different scrap rates, as θ increases the value of optimal backorder level B^* has no noteworthy change.

If the result of this investigation is not available, one probably can only use a closely related lot-size solution given by Chiu [20] for solving such an unreliable EPQ model and obtaining a near optimal operating policy in which Q=5,251 (or T_1=0.5834) and B=2,131. Plugging this lot-size solution into Eq. (20), one has the near optimal costs $E[TCU(T_1,B)]$= \$4,819.36. This results a 5.64% more on total setup and holding costs than the optimal production-inventory costs calculated from the present study.

4 Conclusion

This paper studies the optimal replenishment policy for an economic production quantity (EPQ) model with rework and machine failure taking place in backorder-refilling time. In most real-life production settings, random machine breakdown and generation of the defective items are inevitable. The classic EPQ model is not sufficient for solving such a practical system, because neither does it consider the machine failure issue, nor the imperfect quality items matter.

The effects of these reliability situations on EPQ model must be specifically investigated in order to minimize the total production-inventory costs. The mean time between failures is calculated by most production planners to predict the occurrence time for unexpected breakdown. Then robust plan in terms of the optimal replenishment policy that minimizes total production-inventory costs for such an unreliable system can be established accordingly.

This study extends the work of Chiu's [20] and assumes that a random breakdown takes place in the backorder-filling period. Mathematical modeling is employed and convexity of the long-run average cost function is proved. Then an optimal production lot size that minimizes the long-run average costs for such an imperfect quality EPQ model is derived, where shortages are allowed and backordered. Since little attention was paid to the aforementioned area, this paper intends to fill the gap.

For future study, incorporating multiple machine failures during uptime and random repair time for breakdown (these may apply to certain production systems) into the similar imperfect EPQ model may be one of the interesting studies. Also, decisions on maximum repairing time of the machine to be allowed as well as acquisition of the spare machine may also have important effects on the replenishment run time studied.

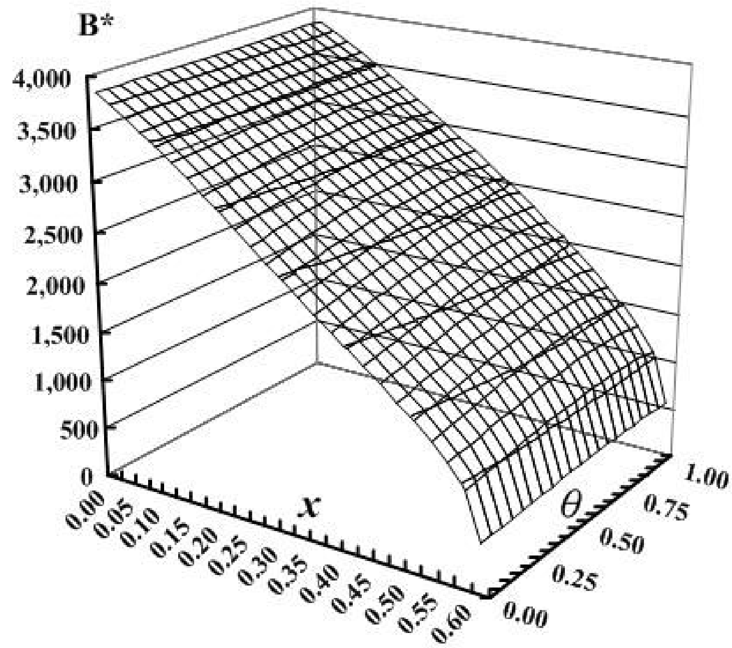

Fig.7: Variation of the defective rate x and the scrap rate θ effects on optimal backordering quantity $B*$ of the proposed imperfect EPQ model

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