

Traveling Wave Solutions For The Fifth-Order Kdv Equation And The BBM Equation By $(\frac{G'}{G})$ -expansion method

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Abstract: In this paper, we demonstrate the effectiveness of the $(\frac{G'}{G})$ -expansion method by seeking more exact solutions of the fifth-order Kdv equation and the BBM equation. By the method, the two nonlinear evolution equations are separately reduced to non-linear ordinary differential equations (ODE) by using a simple transformation. As a result, the traveling wave solutions are obtained in three arbitrary functions including hyperbolic function solutions, trigonometric function solutions and rational solutions. When the parameters are taken as special values, we also obtain the soliton solutions of the fifth-order Kdv equation. The method appears to be easier and faster by means of a symbolic computation system.

Key-Words: $(\frac{G'}{G})$ -expansion method, Traveling wave solutions, BBM equation, fifth-order Kdv equation, exact solution, evolution equation, nonlinear equation

1 Introduction

The nonlinear phenomena exist in all the fields including either the scientific work or engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on. It is well known that many non-linear evolution equations (NLEEs) are widely used to describe these complex phenomena. Research on solutions of NLEEs is popular. So, the powerful and efficient methods to find analytic solutions and numerical solutions of nonlinear equations have drawn a lot of interest by a diverse group of scientists. Many efficient methods have been presented so far such as in [1-7].

In this paper, we pay attention to the analytical method for getting the exact solution of some NLEEs. Among the possible exact solutions of NLEEs, certain solutions for special form may depend only on a single combination of variables such as traveling wave variables. In the literature, Also there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions. Some of these approaches are the homogeneous balance method [8,9], the hyperbolic tangent expansion method [10,11], the trial function method [12], the tanh-method [13-15], the non-linear transform method [16], the inverse scatter-

ing transform [17], the Backlund transform [18,19], the Hirota's bilinear method [20,21], the generalized Riccati equation [22,23], the Weierstrass elliptic function method [24], the theta function method [25-27], the sineCcosine method [28], the Jacobi elliptic function expansion [29,30], the complex hyperbolic function method [31-33], the truncated Painleve expansion [34], the F-expansion method [35], the rank analysis method [36], the exp-function expansion method [37] and so on.

In [38], Mingliang Wang proposed a new method called $(\frac{G'}{G})$ -expansion method. Recently several authors have studied some nonlinear equations by this method [39-42]. The value of the $(\frac{G'}{G})$ -expansion method is that one can treat nonlinear problems by essentially linear methods. The method is based on the explicit linearization of NLEEs for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Moreover, it transforms a nonlinear equation to a simple algebraic computation. The main merits of the $(\frac{G'}{G})$ -expansion method over the other methods are that it gives more general solutions with some free parameters and it handles NLEEs in a direct manner with no requirement for initial/boundary condition or initial trial function at the outset.

Our aim in this paper is to present an application of the $(\frac{G'}{G})$ -expansion method to some nonlinear problems to be solved by this method for the first time.

The rest of the paper is organized as follows. In Section 2, we describe the $(\frac{G'}{G})$ -expansion method for finding traveling wave solutions of nonlinear evolution equations, and give the main steps of the method. In the subsequent sections, we will apply the method to the fifth-order Kdv equation and the BBM equation. In section 5, the features of the $(\frac{G'}{G})$ -expansion method are briefly summarized.

2 Description of the $(\frac{G'}{G})$ -expansion method

In this section we describe the $(\frac{G'}{G})$ -expansion method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables x, t , is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0, \tag{2.1}$$

or in three independent variables x, y and t , is given by

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{yt}, u_{xx}, u_{yy}, \dots) = 0, \tag{2.2}$$

where $u = u(x, t)$ or $u = u(x, y, t)$ is an unknown function, P is a polynomial in $u = u(x, t)$ or $u = u(x, y, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following, we will give the main steps of the $(\frac{G'}{G})$ -expansion method.

Step 1. Suppose that

$$u(x, t) = u(\xi), \quad \xi = \xi(x, t) \tag{2.3}$$

or

$$u(x, y, t) = u(\xi), \quad \xi = \xi(x, y, t) \tag{2.4}$$

The traveling wave variable (2.3) or (2.4) permits us reducing (2.1) or (2.2) to an ODE for $u = u(\xi)$

$$P(u, u', u'', \dots) = 0. \tag{2.5}$$

Step 2. Suppose that the solution of (2.5) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \alpha_m (\frac{G'}{G})^m + \dots \tag{2.6}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \tag{2.7}$$

α_m, \dots, λ and μ are constants to be determined later, $\alpha_m \neq 0$. The unwritten part in (2.6) is also a polynomial in $(\frac{G'}{G})$, the degree of which is generally equal to or less than $m - 1$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (2.5).

Step 3. Substituting (2.6) into (2.5) and using second order LODE (2.7), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of (2.5) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for α_m, \dots, λ and μ .

Step 4. Assuming that the constants α_m, \dots, λ and μ can be obtained by solving the algebraic equations in Step 3. Since the general solutions of the second order LODE (2.7) have been well known for us, then substituting α_m, \dots and the general solutions of (2.7) into (2.6) we have traveling wave solutions of the nonlinear evolution equation (2.1) or (2.2).

3 Application Of The $(\frac{G'}{G})$ -Expansion Method For The fifth-order Kdv Equation

In this section, we consider the fifth-order Kdv equation [43]:

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0 \tag{3.1}$$

Supposing that

$$u(x, t) = u(\xi), \quad \xi = kx + \omega t \tag{3.2}$$

k, ω are constants that to be determined later.

By (3.2), Eq.(3.1) is converted into an ODE

$$k^5 u^{(5)} + \omega u' + 30ku'u^2 + 20k^3 u'u'' + 10k^3 uu''' = 0 \tag{3.3}$$

Integrating (3.3) once, we obtain

$$k^5 u^{(4)} + \omega u + 10ku^3 + 5k^3 (u')^2 + 10k^3 uu'' = g \tag{3.4}$$

where g is the integration constant that can be determined later.

Suppose that the solution of the ODE (3.4) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^m a_i (\frac{G'}{G})^i \tag{3.5}$$

where a_i are constants, $G = G(\xi)$ satisfies the second order LODE in the form:

$$G'' + \lambda G' + \mu G = 0 \tag{3.6}$$

where λ and μ are constants.

Balancing the order of u^3 and $u^{(4)}$ in Eq.(3.4), we get that $3m = m + 4 \Rightarrow m = 2$. So Eq.(3.5) can be rewritten as

$$u(\xi) = a_2 (\frac{G'}{G})^2 + a_1 (\frac{G'}{G}) + a_0, \quad a_2 \neq 0 \tag{3.7}$$

a_2, a_1, a_0 are constants to be determined later.

Substituting (3.7) into (3.4) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$(\frac{G'}{G})^0 : 10ka_0^3 + 14k^5 a_2 \lambda^2 \mu^2$$

$$+ 10k^3 a_0 a_1 \lambda \mu + 20k^3 a_0 a_2 \mu^2 + 5k^3 a_1^2 \mu^2$$

$$+ \omega a_0 + k^5 a_1 \lambda^3 \mu + 8k^5 a_1 \lambda \mu^2$$

$$- g + 16k^5 a_2 \mu^3 = 0$$

$$(\frac{G'}{G})^1 : \omega a_1 + k^5 a_1 \lambda^4 + 22k^5 a_1 \lambda^2 \mu$$

$$+ 30k^5 a_2 \lambda^3 \mu + 10k^3 a_0 a_1 \lambda^2 + 60k^3 a_0 a_2 \lambda \mu$$

$$+ 20k^3 a_0 a_1 \mu + 16k^5 a_1 \mu^2 + 120k^5 a_2 \mu^2 \lambda$$

$$+ 40k^3 a_1 a_2 \mu^2 + 20k^3 a_1^2 \lambda \mu + 30ka_0^2 a_1 = 0$$

$$(\frac{G'}{G})^2 : 16k^5 a_2 \lambda^4 + 232k^5 a_2 \lambda^2 \mu + 15k^5 a_1 \lambda^3$$

$$+ 30ka_0 a_1^2 + \omega a_2 + 30k^3 a_0 a_1 \lambda$$

$$+ 40k^3 a_2^2 \mu^2 + 15k^3 a_1^2 \lambda^2 + 30ka_0^2 a_2$$

$$+ 136k^5 a_2 \mu^2 + 80k^3 a_0 a_2 \mu + 40k^3 a_0 a_2 \lambda^2$$

$$+ 30k^3 a_1^2 \mu + 60k^5 a_1 \mu \lambda + 110k^3 a_1 a_2 \lambda \mu = 0$$

$$(\frac{G'}{G})^3 : 440k^5 a_2 \mu \lambda + 60ka_0 a_1 a_2 + 70k^3 a_1 a_2 \lambda^2$$

$$+ 140k^3 a_1 a_2 \mu + 40k^3 a_1^2 \lambda + 10ka_1^3$$

$$-80k^3a_0^2\mu - 2k^7\lambda^4\mu - 32k^7\mu^3 \quad (3.8)$$

$$+100k^3a_2^2\lambda\mu + 130k^5a_2\lambda^3 + 100k^3a_0a_2\lambda$$

$$+20k^3a_0a_1 + 40k^5a_1\mu + 50k^5a_1\lambda^2 = 0$$

$$\left(\frac{G'}{G}\right)^4 : 60k^5a_1\lambda + 25k^3a_1^2 + 30ka_0a_2^2$$

$$+120k^3a_2^2\mu + 170k^3a_1a_2\lambda + 240k^5a_2\mu$$

$$+30ka_1^2a_2 + 60k^3a_0a_2 + 330k^5a_2\lambda^2$$

$$+60k^3a_2^2\lambda^2 = 0$$

$$\left(\frac{G'}{G}\right)^5 : 336k^5a_2\lambda + 30ka_1a_2^2 + 100k^3a_1a_2$$

$$+140k^3a_2^2\lambda + 24k^5a_1 = 0$$

$$\left(\frac{G'}{G}\right)^6 : 10ka_2^3 + 80k^3a_2^2 + 120k^5a_2 = 0$$

Solving the algebraic equations above, we can get the results for two cases:

Case 1:

$$a_2 = -2k^2,$$

$$a_1 = -2k^2\lambda,$$

$$a_0 = a_0,$$

$$k = k$$

$$\omega = -k^5\lambda^4 - 12k^5\lambda^2\mu - 10k^3a_0\lambda^2$$

$$-56k^5\mu^2 - 30ka_0^2 - 80k^3a_0\mu$$

$$g = -20ka_0^3 - 24k^7\lambda^2\mu^2 - 32k^5a_0\lambda^2\mu$$

$$-96k^5a_0\mu^2 - a_0k^5\lambda^4 - 10k^3a_0^2\lambda^2$$

Where a_0, k are arbitrary constants.

Substituting (3.8) into (3.7), we get that

$$u(\xi) = -2k^2\left(\frac{G'}{G}\right)^2 - 2k^2\lambda\left(\frac{G'}{G}\right) + a_0$$

$$\xi = kx + (-k^5\lambda^4 - 12k^5\lambda^2\mu - 10k^3a_0\lambda^2 - 56k^5\mu^2 - 30ka_0^2 - 80k^3a_0\mu)t \quad (3.9)$$

Substituting the general solutions of Eq.(3.6) into (3.9), we can obtain three types of travelling wave solutions as follows:

When $\lambda^2 - 4\mu > 0$

$$u_1(\xi) = \frac{k^2\lambda^2}{2} - \frac{k^2}{2}(\lambda^2 - 4\mu)$$

$$\left(\frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi} \right)^2 + a_0$$

where

$$\xi = kx + (-k^5\lambda^4 - 12k^5\lambda^2\mu$$

$$-10k^3a_0\lambda^2 - 56k^5\mu^2 - 30ka_0^2$$

$$-80k^3a_0\mu)t,$$

k, C_1, C_2 are arbitrary constants.

In particular, when $\lambda > 0, \mu = 0, C_1 \neq 0, C_2 = 0$, we can deduce the soliton solutions of the fifth-order Kdv equation as:

$$u_1(\xi) = -\frac{1}{2}k^2\lambda^2 \operatorname{sech}^2\left(\frac{\lambda\xi}{2}\right) + \frac{1}{12}k^2\lambda^2 + a_0$$

When $\lambda^2 - 4\mu < 0$

$$u_2(\xi) = \frac{k^2\lambda^2}{2} - \frac{k^2}{2}(4\mu - \lambda^2)$$

$$\left(\frac{-C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{C_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} \right)^2 + a_0$$

where

$$\xi = kx + (-k^5 \lambda^4 - 12k^5 \lambda^2 \mu - 10k^3 a_0 \lambda^2 - 56k^5 \mu^2 - 30ka_0^2 - 80k^3 a_0 \mu)t,$$

k, C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$

$$u_3(\xi) = \frac{k^2 \lambda^2}{2} - \frac{2k^2 C_2^2}{(C_1 + C_2 \xi)^2} + a_0$$

where

$$\xi = kx + (-k^5 \lambda^4 - 12k^5 \lambda^2 \mu - 10k^3 a_0 \lambda^2 - 56k^5 \mu^2 - 30ka_0^2 - 80k^3 a_0 \mu)t,$$

k, C_1, C_2 are arbitrary constants.

Case 2:

$$\begin{aligned} a_2 &= -6k^2, \\ a_1 &= -6k^2 \lambda, \\ a_0 &= -\frac{1}{2} k^2 (\lambda^2 + 8\mu), \\ k &= k \\ \omega &= -\frac{7}{2} k^5 (\lambda^4 - 8\lambda^2 \mu + 16\mu^2), \\ g &= \frac{1}{2} k^7 \lambda^6 - 6k^7 \lambda^4 \mu + 24k^7 \lambda^2 \mu^2 - 32k^7 \mu^3 \end{aligned} \quad (3.10)$$

where k is an arbitrary constant.

Substituting (3.10) into (3.7), we get that

$$\begin{aligned} u(\xi) &= -6k^2 \left(\frac{G'}{G}\right)^2 - 6k^2 \lambda \left(\frac{G'}{G}\right) - \frac{1}{2} k^2 (\lambda^2 + 8\mu) \\ \xi &= kx - \frac{7}{2} k^5 (\lambda^4 - 8\lambda^2 \mu + 16\mu^2)t \end{aligned} \quad (3.11)$$

where k is an arbitrary constant.

Substituting the general solutions of Eq.(3.6) into (3.11), we can obtain three types of travelling wave solutions as follows:

When $\lambda^2 - 4\mu > 0$

$$\begin{aligned} u_1(\xi) &= \frac{3k^2 \lambda^2}{2} - \frac{3k^2}{2} (\lambda^2 - 4\mu) \\ &\left(\frac{C_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{C_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} \right)^2 \\ &\quad - \frac{1}{2} k^2 (\lambda^2 + 8\mu) \end{aligned}$$

where

$$\xi = kx - \frac{7}{2} k^5 (\lambda^4 - 8\lambda^2 \mu + 16\mu^2)t,$$

k, C_1, C_2 are arbitrary constants.

In particular, when $\lambda > 0, \mu = 0, C_1 \neq 0, C_2 = 0$, we can deduce the soliton solutions of the fifth-order Kdv equation as:

$$u_1(\xi) = -\frac{3}{2} k^2 \lambda^2 \operatorname{sech}^2\left(\frac{\lambda \xi}{2}\right) + \frac{1}{12} k^2 \lambda^2 - \frac{3}{2} k^2 \lambda^2$$

When $\lambda^2 - 4\mu < 0$

$$\begin{aligned} u_2(\xi) &= \frac{3k^2 \lambda^2}{2} - \frac{3k^2}{2} (4\mu - \lambda^2) \\ &\left(\frac{-C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{C_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} \right)^2 \\ &\quad - \frac{1}{2} k^2 (\lambda^2 + 8\mu) \end{aligned}$$

where

$$\xi = kx - \frac{7}{2} k^5 (\lambda^4 - 8\lambda^2 \mu + 16\mu^2)t,$$

k, C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$

$$u_3(\xi) = \frac{3k^2\lambda^2}{2} - \frac{6k^2C_2^2}{(C_1 + C_2\xi)^2} - \frac{1}{2}k^2(\lambda^2 + 8\mu)$$

where

$$\xi = kx - \frac{7}{2}k^5(\lambda^4 - 8\lambda^2\mu + 16\mu^2)t,$$

k, C_1, C_2 are arbitrary constants.

4 Application Of The $(\frac{G'}{G})$ -Expansion Method For The BBM Equation

In this section, we will consider the BBM equation [44]:

$$u_t + uu_x + u_x - \mu u_{xxt} = 0 \tag{4.1}$$

where μ is a constant.

In order to obtain the travelling wave solutions of Eq.(4.1), we suppose that

$$u(x, t) = u(\xi), \xi = x - ct \tag{4.2}$$

c is a constant that to be determined later.

By using the wave variable (4.2), Eq.(4.1) is converted into an ODE

$$-cu' + uu' + u' - c\mu u''' = 0 \tag{4.3}$$

Integrating (4.3) with respect to ξ once, we obtain

$$(-c + 1)u + \frac{1}{2}u^2 - c\mu u'' = g \tag{4.4}$$

where g is the integration constant that can be determined later.

Suppose that the solution of the ODE (4.4) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^m a_i (\frac{G'}{G})^i \tag{4.5}$$

where a_i are constants, $G = G(\xi)$ satisfies the second order LODE in the form:

$$G'' + \lambda G' + \mu G = 0 \tag{4.6}$$

where λ and μ are constants.

Balancing the order of u^2 and u'' in Eq.(4.4), we get that $2m = m + 2 \Rightarrow m = 2$, so Eq.(4.5) can be rewritten as

$$u(\xi) = a_2 (\frac{G'}{G})^2 + a_1 (\frac{G'}{G}) + a_0, a_2 \neq 0 \tag{4.7}$$

a_2, a_1, a_0 are constants to be determined later.

Then we can obtain

$$u'(\xi) = -2a_2 (\frac{G'}{G})^3 + (-a_1 - 2a_2\lambda) (\frac{G'}{G})^2 + (-a_1\lambda - 2a_2\mu) (\frac{G'}{G}) - a_1\mu$$

$$u''(\xi) = 6a_2 (\frac{G'}{G})^4 + (2a_1 + 10a_2\lambda) (\frac{G'}{G})^3 + (8a_2\mu + 3a_1\lambda + 4a_2\lambda^2) (\frac{G'}{G})^2$$

$$+ (6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2) (\frac{G'}{G}) + 2a_2\mu^2 + a_1\lambda\mu$$

On substituting Eq.(4.7) into the ODE (4.4) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$(\frac{G'}{G})^0 : -ca_0 - 2c\mu^3 a_2 + a_0 + \frac{1}{2}a_0^2 - c\mu^2 a_1\lambda - g = 0$$

$$\left(\frac{G'}{G}\right)^1 : -ca_1 + a_1 - c\mu a_1 \lambda^2$$

$$-2ca_1 \mu^2 - 6ca_2 \lambda \mu^2 + a_0 a_1 = 0$$

$$\left(\frac{G'}{G}\right)^2 : -4c\mu a_2 \lambda^2 + \frac{1}{2}a_1^2 + a_2$$

$$-ca_2 - 8ca_2 \mu^2 + a_0 a_2 - 3ca_1 \lambda \mu = 0$$

$$\left(\frac{G'}{G}\right)^3 : -10c\mu a_2 \lambda + a_1 a_2 - 2ca_1 \mu = 0$$

$$\left(\frac{G'}{G}\right)^4 : -6c\mu a_2 + \frac{1}{2}a_2^2 = 0$$

Solving the algebraic equations above yields:

$$a_2 = 12c\mu,$$

$$a_1 = 12c\mu\lambda,$$

$$a_0 = c\mu\lambda^2 + c - 1 + 8c\mu^2,$$

$$g = c - \frac{1}{2}c^2 + 8c^2\mu^4 - \frac{1}{2} + \frac{1}{2}c^2\mu^2\lambda^4 - 4c^2\mu^3\lambda^2,$$

$$c = c \tag{4.8}$$

where c, λ, μ are arbitrary constants.

Substituting (4.8) into (4.7), we get that

$$u(\xi) = 12c\mu\left(\frac{G'}{G}\right)^2 + 12c\mu\lambda\left(\frac{G'}{G}\right) + c\mu\lambda^2$$

$$+ c - 1 + 8c\mu^2$$

$$\xi = x - ct \tag{4.9}$$

where c, λ, μ are arbitrary constants.

Substituting the general solutions of Eq.(4.6) into (4.9), we can obtain three types of travelling wave solutions of the BBM equation (4.1) as follows:

Case (a): when $\lambda^2 - 4\mu > 0$

$$u_1(\xi) =$$

$$c\mu\lambda^2 + c - 1 + 8c\mu^2 - 3c\mu\lambda^2 + 3c\mu(\lambda^2 - 4\mu)$$

$$\cdot \left(\frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2$$

where $\xi = x - ct$, c, C_1, C_2 are arbitrary constants.

Case (b): when $\lambda^2 - 4\mu < 0$

$$u_2(\xi) =$$

$$c\mu\lambda^2 + c - 1 + 8c\mu^2 - 3c\mu\lambda^2 + 3c\mu(4\mu - \lambda^2)$$

$$\cdot \left(\frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right)^2$$

where $\xi = x - ct$, c, C_1, C_2 are arbitrary constants.

Case (c): when $\lambda^2 - 4\mu = 0$

$$u_3(\xi) =$$

$$-3c\mu\lambda^2 + \frac{12c\mu C_2^2}{(C_1 + C_2\xi)^2} + c\mu\lambda^2 + c - 1 + 8c\mu^2$$

where $\xi = x - ct$, c, C_1, C_2 are arbitrary constants.

5 Conclusions

In this paper, a generalized $\left(\frac{G'}{G}\right)$ -expansion method is used to obtain more general exact solutions of the fifth-order Kdv equation and the BBM equation. As a result, exact travelling wave solutions with three arbitrary functions are obtained including hyperbolic function solutions, trigonometric function solutions and rational solutions. The arbitrary functions in the obtained solutions imply that these solutions have rich

local structures. It may be important to explain some physical phenomena.

The value of the $(\frac{G'}{G})$ -expansion method is that one can treat nonlinear problems by essentially linear methods. The method is based on the explicit linearization of NLEEs for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Moreover, it transforms a nonlinear equation to a simple algebraic computation. Compared to the methods used before, one can see that this method is direct, concise and effective. As we can use the MATHEMATICA or MAPLE to find out a useful solution of the algebraic equations resulted, so we can also avoid tedious calculations. This method can also be used to many other nonlinear equations.

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