Traveling Wave Solutions For The Variant Boussinseq Equation And The (2+1)-dimensional Nizhnik-Novikov-Veselov (NNV) System By $(\frac{G'}{G})$-expansion method

Qinghua Feng
Shandong University of Technology
School of Science
Zhangzhou Road 12, Zibo, 255049
China
fqhua@sina.com

Bin Zheng
Shandong University of Technology
School of Science
Zhangzhou Road 12, Zibo, 255049
China
zhengbin2601@126.com

Abstract: In this paper, using the generalized $(\frac{G'}{G})$-expansion method, new explicit travelling wave solutions for the variant Boussinseq equation and the (2+1)-dimensional Nizhnik-Novikov-Veselov (NNV) system are obtained with the aid of Mathematica.

Key–Words: $(\frac{G'}{G})$-expansion method, Traveling wave solutions, (2+1)-dimensional Nizhnik-Novikov-Veselov (NNV) system, variant Boussinseq equation, exact solution, evolution equation, nonlinear equation

1 Introduction

Research on non-linear equations is a hot topic. The powerful and efficient methods to find analytic solutions and numerical solutions of nonlinear equations have drawn a lot of interest by a diverse group of scientists. Many efficient methods have been presented so far such as in [1-7]. In recent years, exact solutions of non-linear PDEs have been investigated by many authors. Many powerful methods have been presented by those authors such as the homogeneous balance method [8,9], the hyperbolic tangent expansion method [10,11], the trial function method [12], the tanh-method [13-15], the non-linear transform method [16], the inverse scattering transform [17], the Backlund transform [18,19], the Hirotas bilinear method [20,21], the generalized Riccati equation [22,23], the Weierstrass elliptic function method [24], the theta function method [25-27], the sineCcosine method [28], the Jacobi elliptic function expansion [29,30], the complex hyperbolic function method [31-33], the truncated Painleve expansion [34], the F-expansion method [35], the rank analysis method [36], the exp-function expansion method [37] and so on.

The objective of this paper is to use a new method which is called the $(\frac{G'}{G})$-expansion method [38-42]. The value of the $(\frac{G'}{G})$-expansion method is that one can treat nonlinear problems by essentially linear methods. Moreover, it transforms a nonlinear equation to a simple algebraic computation.

We organize the rest of the paper as follows. In Section 2, we give the main steps of the $(\frac{G'}{G})$-expansion method. In the subsequent sections, we will apply the method to obtain the travelling wave solutions of the variant Boussinseq equation and the (2+1)-dimensional Nizhnik-Novikov-Veselov (NNV) system. Some conclusions are presented in section 5.

2 Description of the $(\frac{G'}{G})$-expansion method

In this section we describe the $(\frac{G'}{G})$-expansion method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables $x,\ t$, is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{2.1}$$

or in three independent variables $x,\ y$ and t, is given by

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{yt}, u_{xx}, u_{yy}, ...) = 0, \tag{2.2}$$

where $u = u(x,t)$ or $u = u(x,y,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ or $u = u(x,y,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following, we will give the main steps of the $(\frac{G'}{G})$-expansion method.

Step 1. Suppose that

$$u(x,\ t) = u(\xi), \quad \xi = \xi(x,t) \tag{2.3}$$

or

$$u(x,\ y,\ t) = u(\xi), \quad \xi = \xi(x,y,t) \tag{2.4}$$

The traveling wave variable (2.3) or (2.4) permits us reducing (2.1) or (2.2) to an ODE for $u = u(\xi)$

$$P(u,\ u', u'',\ ...) = 0. \tag{2.5}$$

Step 2. Suppose that the solution of (2.5) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \alpha_m(\frac{G'}{G})^m + \ ... \tag{2.6}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \tag{2.7}$$

$\alpha_m, ..., \lambda$ and μ are constants to be determined later, $\alpha_m \neq 0$. The unwritten part in (2.6) is also a polynomial in $(\frac{G'}{G})$, the degree of which is generally equal to or less than $m-1$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (2.5).

Step 3. Substituting (2.6) into (2.5) and using second order LODE (2.7), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of (2.5) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_m, ..., \lambda$ and μ.

Step 4. Assuming that the constants $\alpha_m, ..., \lambda$ and μ can be obtained by solving the algebraic equations in Step 3. Since the general solutions of the second order LODE (2.7) have been well known for us, then substituting $\alpha_m, ...$ and the general solutions of (2.7) into (2.6) we have traveling wave solutions of the nonlinear evolution equation (2.1) or (2.2).

3 Application Of The $(\frac{G'}{G})$-Expansion Method For The Variant Boussinseq Equation

We consider the variant Boussinseq equation [43]:

$$u_t + uu_x + v_x + \alpha u_{xxt} = 0 \tag{3.1}$$

$$v_t + (uv)_x + \beta u_{xxx} = 0 \tag{3.2}$$

where α and β are arbitrary constants, $\beta > 0$.

Supposing that

$$\xi = k(x - ct) \tag{3.3}$$

By (3.3), (3.1) and (3.2) are converted into ODEs

$$-cu' + uu' + v' - \alpha k^2 cu''' = 0 \tag{3.4}$$

$$-cv' + (uv)' + \beta k^2 u''' = 0 \tag{3.5}$$

Integrating (3.4) and (3.5) once, we have

$$-cu + \frac{1}{2}u^2 + v - \alpha k^2 cu'' = g_1 \tag{3.6}$$

$$-cv + uv + \beta k^2 u'' = g_2 \tag{3.7}$$

where g_1 and g_2 are the integration constants.

Suppose that the solution of (3.6) and (3.7) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i(\frac{G'}{G})^i \tag{3.8}$$

$$v(\xi) = \sum_{i=0}^{n} b_i(\frac{G'}{G})^i \tag{3.9}$$

where a_i, b_i are constants, $G = G(\xi)$ satisfies the second order LODE in the form:

$$G'' + \lambda G' + \mu G = 0 \tag{3.10}$$

where λ and μ are constants.

Balancing the order of u^2 and v in Eq.(3.6), the order of u'' and uv in Eq.(3.7), then we can obtain

$2m = n,\ n+2 = m+n \Rightarrow\ m = 1,\ n = 2$, so Eq.(3.8) and (3.9) can be rewritten as

$$u(\xi) = a_1(\frac{G'}{G}) + a_0,\ a_1 \neq 0 \quad (3.11)$$

$$v(\xi) = b_2(\frac{G'}{G})^2 + b_1(\frac{G'}{G}) + b_0,\ b_2 \neq 0 \quad (3.24)$$

$a_1,\ a_0,\ b_2,\ b_1,\ b_0$ are constants to be determined later.

Substituting (3.11) and (3.12) into (3.6) and (3.7) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

For Eq.(3.6):

$(\frac{G'}{G})^0:\ -ca_0 + b_0 + \frac{1}{2}a_0^2 - g_1 - \alpha k^2 ca_1\lambda\mu = 0$

$(\frac{G'}{G})^1:\ a_0a_1 - \alpha k^2 ca_1\lambda^2 + b_1 - 2\alpha k^2 ca_1\mu - ca_1 = 0$

$(\frac{G'}{G})^2:\ \frac{1}{2}a_1^2 - 3\alpha k^2 ca_1\lambda + b_2 = 0$

$(\frac{G'}{G})^3:\ -2\alpha k^2 ca_1 = 0$

For Eq.(3.7):

$(\frac{G'}{G})^0:\ -cb_0 - g_2 + \beta k^2 a_1\lambda\mu + a_0b_0 = 0$

$(\frac{G'}{G})^1:\ b_1a_0 + \beta k^2 a_1\lambda^2 + a_1b_0 - cb_1 + 2\beta k^2 a_1\mu = 0$

$(\frac{G'}{G})^2:\ a_1b_1 + b_2a_0 + 3\beta k^2 a_1\lambda - cb_2 = 0$

$(\frac{G'}{G})^3:\ 2\beta k^2 a_1 + b_2a_1 = 0$

Solving the algebraic equations above yields:

$$a_1 = a_1,\ \ a_0 = \frac{1}{2}a_1\lambda,\ \ b_2 = -\frac{1}{2}a_1^2$$

$$b_1 = -\frac{1}{2}a_1^2\lambda,\ b_0 = -\frac{1}{2}a_1^2\mu$$

$$k = \pm\frac{1}{2}\sqrt{\frac{1}{\beta}},\ \ c = 0,\ \ g_1 = \frac{1}{8}(\lambda^2 - 4\mu),\ \ g_2 = 0 \quad (3.13)$$

where a_1 is an arbitrary constant.

Substituting (3.13) into (3.11) and (3.12), yields:

$$u(\xi) = a_1(\frac{G'}{G}) + \frac{1}{2}a_1\lambda \quad (3.14)$$

$$v(\xi) = -\frac{1}{2}a_1^2(\frac{G'}{G})^2 - \frac{1}{2}a_1^2\lambda(\frac{G'}{G}) - \frac{1}{2}a_1^2\mu \quad (3.15)$$

where $\xi = kx$.

Substituting the general solutions of (3.10) into (3.14) and (3.15), we have:

When $\lambda^2 - 4\mu > 0$

$$u_1(\xi) = \frac{a_1\sqrt{\lambda^2 - 4\mu}}{2} .(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi})$$

$$v_1(\xi) = \frac{a_1^2\lambda^2}{8} - \frac{1}{2}a_1^2\mu - \frac{a_1^2}{8}(\lambda^2 - 4\mu) .(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi})^2$$

Where $\xi = kx, a_1, C_1, C_2$ are arbitrary constants.

When $\lambda^2 - 4\mu < 0$

$$u_2(\xi) = \frac{a_1\sqrt{4\mu - \lambda^2}}{2} .(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi})$$

$$v_2(\xi) = \frac{a_1^2\lambda^2}{8} - \frac{1}{2}a_1^2\mu - \frac{a_1^2}{8}(4\mu - \lambda^2) .(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi})^2$$

Where $\xi = kx, a_1, C_1, C_2$ are arbitrary constants.

When $\lambda^2 - 4\mu = 0$

$$u_3(\xi) = \frac{a_1(2C_2 - C_1\lambda - C_2\lambda\xi)}{2(C_1 + C_2\xi)} + \frac{1}{2}a_1\lambda$$

$$v_3(\xi) = \frac{a_1^2\lambda^2}{8} - \frac{a_1^2C_2^2}{2(C_1 + C_2\xi)^2} - \frac{1}{2}a_1^2\mu$$

Where $\xi = kx, a_1, C_1, C_2$ are arbitrary constants.

4 Application Of The $(\frac{G'}{G})$-Expansion Method For The (2+1)-dimensional Nizhnik-Novikov-Veselov System

In this section we will consider the (2+1)-dimensional Nizhnik-Novikov-Veselov (NNV) system [44-45]:

$$u_t+au_{xxx}+bu_{yyy}+cu_x+du_y=3a(uv)_x+3b(uw)_y \quad (4.1)$$

$$u_x=v_y \quad (4.2)$$

$$u_y=w_x \quad (4.3)$$

Supposing that

$$\xi=kx+ly+\omega t \quad (4.4)$$

By (4.4), (4.1), (4.2) and (4.3) are converted into ODEs

$$\omega u'+ak^3u'''+bl^3u'''+cku'+dlu'=3ak(uv)'+3bl(uw)' \quad (4.5)$$

$$ku'=lv' \quad (4.6)$$

$$lu'=kw' \quad (4.7)$$

Integrating (4.5), (4.6) and (4.7) once, we have

$$\omega u+ak^3u''+bl^3u''+cku+dlu=3akuv+3bluw+g_1 \quad (4.8)$$

$$ku=lv+g_2 \quad (4.9)$$

$$lu=kw+g_3 \quad (4.10)$$

where g_1, g_2, g_3 are the integration constants.

Suppose that the solution of (4.8), (4.9) and (4.10) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi)=\sum_{i=0}^{m}a_i(\frac{G'}{G})^i \quad (4.11)$$

$$v(\xi)=\sum_{i=0}^{n}b_i(\frac{G'}{G})^i \quad (4.12)$$

$$w(\xi)=\sum_{i=0}^{s}c_i(\frac{G'}{G})^i \quad (4.13)$$

where a_i, b_i, c_i are constants, $G=G(\xi)$ satisfies the second order LODE in the form:

$$G''+\lambda G'+\mu G=0 \quad (4.14)$$

where λ and μ are constants.

Balancing the order of u'' and uv in Eq.(4.8), the order of u and v in Eq.(4.9), the order of u and w in Eq.(4.10), then we can obtain $m+2=m+n,\ m=n,\ m=s\Rightarrow\ m=n=s=2$, so Eq.(4.11), (4.12) and (4.13) can be rewritten as

$$u(\xi)=a_2(\frac{G'}{G})^2+a_1(\frac{G'}{G})+a_0,\ a_2\neq 0 \quad (4.15)$$

$$v(\xi)=b_2(\frac{G'}{G})^2+b_1(\frac{G'}{G})+b_0,\ b_2\neq 0 \quad (4.16)$$

$$w(\xi)=c_2(\frac{G'}{G})^2+c_1(\frac{G'}{G})+c_0,\ c_2\neq 0 \quad (4.17)$$

$a_2,\ a_1,\ a_0,\ b_2,\ b_1,\ b_0,\ c_2,\ c_1,\ c_0$ are constants to be determined later.

Substituting (4.15), (4.16) and (4.17) into (4.8), (4.9) and (4.10) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

For Eq.(4.8):

$$(\frac{G'}{G})^0:\ 2ak^3a_2\mu^2+bl^3a_1\lambda\mu+2bl^3a_2\mu^2$$
$$+dla_0+cka_0-3aka_0b_0-3bla_0c_0$$
$$+ak^3a_1\lambda\mu+\omega a_0-g_1=0$$

$$(\frac{G'}{G})^1:\ 6ak^3a_2\lambda\mu+cka_1+bl^3a_1\lambda^2$$
$$-3bla_1c_0+dla_1+6bl^3a_2\lambda\mu$$
$$+\omega a_1+ak^3a_1\lambda^2-3aka_0b_1+2ak^3a_1\mu$$
$$+2bl^3a_1\mu-3bla_0c_1-3aka_1b_0=0$$

$$(\frac{G'}{G})^2:\ -3aka_0b_2+4ak^3a_2\lambda^2+3bl^3a_1\lambda$$
$$+dla_2-3bla_1c_1+4bl^3a_2\lambda^2$$
$$-3bla_2c_0+8ak^3a_2\mu-3aka_2b_0$$
$$+8bl^3a_2\mu-3aka_1b_1-3bla_0c_2$$
$$+\omega a_2+3ak^3a_1\lambda+cka_2=0$$

$$(\frac{G'}{G})^3:\ 2bl^3a_1-3aka_2b_1+10ak^3a_2\lambda$$
$$+10bl^3a_2\lambda-3bla_1c_2+2ak^3a_1$$
$$-3aka_1b_2-3bla_2c_1=0$$

$$(\frac{G'}{G})^4: \ -3aka_2b_2+6ak^3a_2+6bl^3a_2-3bla_2c_2=0$$

For Eq.(4.9):

$$(\frac{G'}{G})^0: \ ka_0-lb_0-g_2=0$$

$$(\frac{G'}{G})^1: \ ka_1-lb_1=0$$

$$(\frac{G'}{G})^2: \ ka_2-lb_2=0$$

For Eq.(4.10):

$$(\frac{G'}{G})^0: \ la_0-kc_0-g_3=0$$

$$(\frac{G'}{G})^1: \ la_1-kc_1=0$$

$$(\frac{G'}{G})^2: \ la_2-kc_2=0$$

Solving the algebraic equations above yields:

Case 1:

$$a_2=2kl,$$
$$a_1=2kl\lambda,$$
$$a_0=2kl\mu$$
$$b_2=2k^2,$$
$$b_1=2k^2\lambda,$$
$$b_0=2k^2\mu$$
$$c_2=2l^2,$$
$$c_1=2l^2\lambda,$$
$$c_0=2l^2\mu,$$
$$k=k, \ l=l$$
$$\omega=4ak^3\mu-ak^3\lambda^2-bl^3\lambda^2-dl+4bl^3\mu-ck$$
$$g_1=g_2=g_3=0 \qquad (4.18)$$

where k,l are arbitrary constants.

Substituting (4.18) into (4.15), (4.16) and (4.17), yields:

$$u(\xi)=2kl(\frac{G'}{G})^2+2kl\lambda(\frac{G'}{G})+2kl\mu \qquad (4.19)$$

$$v(\xi)=2k^2(\frac{G'}{G})^2+2k^2\lambda(\frac{G'}{G})+2k^2\mu \qquad (4.20)$$

$$w(\xi)=2l^2(\frac{G'}{G})^2+2l^2\lambda(\frac{G'}{G})+2l^2\mu \qquad (4.21)$$

where

$$\xi=kx+ly+(4ak^3\mu-ak^3\lambda^2-bl^3\lambda^2-dl+4bl^3\mu-ck)t.$$

Substituting the general solutions of (4.14) into (4.19), (4.20) and (4.21), we have:

When $\lambda^2-4\mu>0$

$$u_1(\xi)=2kl\mu-\frac{kl\lambda^2}{2}+\frac{kl(\lambda^2-4\mu)}{2}$$

$$.(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2$$

$$v_1(\xi)=2k^2\mu-\frac{k^2\lambda^2}{2}+\frac{k^2}{2}(\lambda^2-4\mu)$$

$$.(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2$$

$$w_1(\xi)=2l^2\mu-\frac{l^2\lambda^2}{2}+\frac{l^2}{2}(\lambda^2-4\mu)$$

$$.(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2$$

Where

$$\xi=kx+ly+(4ak^3\mu-ak^3\lambda^2-bl^3\lambda^2-dl+4bl^3\mu-ck)t,$$

k,l,C_1,C_2 are arbitrary constants.

When $\lambda^2-4\mu<0$

$$u_2(\xi)=2kl\mu-\frac{kl\lambda^2}{2}+\frac{kl(4\mu-\lambda^2)}{2}$$

$$.(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^2$$

$$v_2(\xi)=2k^2\mu-\frac{k^2\lambda^2}{2}+\frac{k^2}{2}(4\mu-\lambda^2)$$

$$.(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^2$$

$$w_2(\xi)=2l^2\mu-\frac{l^2\lambda^2}{2}+\frac{l^2}{2}(4\mu-\lambda^2)$$

$$.(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^2$$

Where

$$\xi=kx+ly+(4ak^3\mu-ak^3\lambda^2-bl^3\lambda^2-dl+4bl^3\mu-ck)t,$$

k,l,C_1,C_2 are arbitrary constants.

When $\lambda^2-4\mu=0$

$$u_3(\xi)=-\frac{kl\lambda^2}{2}+\frac{2klC_2^2}{(C_1+C_2\xi)^2}+2kl\mu$$

$$v_3(\xi)=-\frac{k^2\lambda^2}{2}+\frac{2k^2C_2^2}{(C_1+C_2\xi)^2}+2k^2\mu$$

$$w_3(\xi)=-\frac{l^2\lambda^2}{2}+\frac{2l^2C_2^2}{(C_1+C_2\xi)^2}+2l^2\mu$$

Where

$$\xi=kx+ly+(4ak^3\mu-ak^3\lambda^2-bl^3\lambda^2-dl+4bl^3\mu-ck)t,$$

k,l,C_1,C_2 are arbitrary constants.

Case 2:

$$a_2=2kl,$$
$$a_1=2kl\lambda,$$
$$a_0=\frac{1}{3}kl(\lambda^2+2\mu)$$
$$b_2=2k^2,$$
$$b_1=2k^2\lambda,$$
$$b_0=\frac{1}{3}k^2(\lambda^2+2\mu)$$
$$c_2=2l^2,$$
$$c_1=2l^2\lambda,$$
$$c_0=\frac{1}{3}l^2(\lambda^2+2\mu),$$
$$k=k,\ \ l=l$$
$$\omega=-4ak^3\mu+ak^3\lambda^2+bl^3\lambda^2-dl-4bl^3\mu-ck$$
$$g_1=g_2=g_3=0 \qquad (4.22)$$

where k,l are arbitrary constants.

Substituting (4.22) into (4.15), (4.16) and (4.17), yields:

$$u(\xi)=2kl(\frac{G'}{G})^2+2kl\lambda(\frac{G'}{G})+\frac{1}{3}kl(\lambda^2+2\mu) \quad (4.23)$$

$$v(\xi)=2k^2(\frac{G'}{G})^2+2k^2\lambda(\frac{G'}{G})+\frac{1}{3}k^2(\lambda^2+2\mu) \quad (4.24)$$

$$w(\xi)=2l^2(\frac{G'}{G})^2+2l^2\lambda(\frac{G'}{G})+\frac{1}{3}l^2(\lambda^2+2\mu) \quad (4.25)$$

where

$$\xi=kx+ly+(-4ak^3\mu+ak^3\lambda^2+bl^3\lambda^2-dl-4bl^3\mu-ck)t.$$

Substituting the general solutions of (4.14) into (4.23), (4.24) and (4.25), we have:

When $\lambda^2-4\mu>0$

$$u_1(\xi)=\frac{1}{3}kl(\lambda^2+2\mu)-\frac{kl\lambda^2}{2}+\frac{kl(\lambda^2-4\mu)}{2}$$

$$\cdot(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2$$

$$v_1(\xi) = \frac{1}{3}k^2(\lambda^2+2\mu) - \frac{k^2\lambda^2}{2} + \frac{k^2}{2}(\lambda^2-4\mu)$$

$$\cdot(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2$$

$$w_1(\xi) = \frac{1}{3}l^2(\lambda^2+2\mu) - \frac{l^2\lambda^2}{2} + \frac{l^2}{2}(\lambda^2-4\mu)$$

$$\cdot(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2$$

Where

$\xi = kx+ly+(-4ak^3\mu+ak^3\lambda^2+bl^3\lambda^2-dl-4bl^3\mu-ck)t$,

k, l, C_1, C_2 are arbitrary constants.

When $\lambda^2-4\mu<0$

$$u_2(\xi) = \frac{1}{3}kl(\lambda^2+2\mu) - \frac{kl\lambda^2}{2} + \frac{kl(4\mu-\lambda^2)}{2}$$

$$\cdot(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^2$$

$$v_2(\xi) = \frac{1}{3}k^2(\lambda^2+2\mu) - \frac{k^2\lambda^2}{2} + \frac{k^2}{2}(4\mu-\lambda^2)$$

$$\cdot(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^2$$

$$w_2(\xi) = \frac{1}{3}l^2(\lambda^2+2\mu) - \frac{l^2\lambda^2}{2} + \frac{l^2}{2}(4\mu-\lambda^2)$$

$$\cdot(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^2$$

Where

$\xi = kx+ly+(-4ak^3\mu+ak^3\lambda^2+bl^3\lambda^2-dl-4bl^3\mu-ck)t$,

k, l, C_1, C_2 are arbitrary constants.

When $\lambda^2-4\mu=0$

$$u_3(\xi) = -\frac{kl\lambda^2}{2} + \frac{2klC_2^2}{(C_1+C_2\xi)^2} + \frac{1}{3}kl(\lambda^2+2\mu)$$

$$v_3(\xi) = -\frac{k^2\lambda^2}{2} + \frac{2k^2C_2^2}{(C_1+C_2\xi)^2} + \frac{1}{3}k^2(\lambda^2+2\mu)$$

$$w_3(\xi) = -\frac{l^2\lambda^2}{2} + \frac{2l^2C_2^2}{(C_1+C_2\xi)^2} + \frac{1}{3}l^2(\lambda^2+2\mu)$$

Where

$\xi = kx+ly+(-4ak^3\mu+ak^3\lambda^2+bl^3\lambda^2-dl-4bl^3\mu-ck)t$,

k, l, C_1, C_2 are arbitrary constants.

5 Conclusions

From above we have seen that the traveling wave solutions of the variant Boussinseq equation and the (2+1)-dimensional Nizhnik-Novikov-Veselov (NNV) system are successfully found by using the $(\frac{G'}{G})$-expansion method.

Compared to the methods used before, one can see that this method is direct, concise and effective. As we can use the MATHEMATICA or MAPLE to find out a useful solution of the algebraic equations resulted, so we can also avoids tedious calculations. This method can also be used to many other nonlinear equations.

6 Acknowledgements

I would like to thank the anonymous referees for their useful and valuable suggestions.

References:

[1] Damelys Zabala, Aura L. Lopez De Ramos, Effect of the Finite Difference Solution Scheme in a Free Boundary Convective Mass Transfer Model, WSEAS Transactions on Mathematics, Vol. 6, No. 6, 2007, pp. 693-701

[2] Raimonds Vilums, Andris Buikis, Conservative Averaging and Finite Difference Methods for Transient Heat Conduction in 3D Fuse, WSEAS Transactions on Heat and Mass Transfer, Vol 3, No. 1, 2008

[3] Mastorakis N E., An Extended Crank-Nicholson Method and its Applications in the Solution of Partial Differential Equations: 1-D and 3-D Conduction Equations, WSEAS Transactions on Mathematics, Vol. 6, No. 1, 2007, pp 215-225

[4] Nikos E. Mastorakis, Numerical Solution of Non-Linear Ordinary Differential Equations via Collocation Method (Finite Elements) and Genetic Algorithm, WSEAS Transactions on Information Science and Applications, Vol. 2, No. 5, 2005, pp. 467-473

[5] Inc M, New exact solutions for the ZK-MEW equation by using symbolic computation. Appl. Math. Comput. 189 (2007) 508.

[6] Zhang Huiqun, Extended Jacobi elliptic function expansion method and its applications. Communications in Nonlinear Science and Numerical Simulation 12 (2007) 627-635.

[7] A.M. Wazwaz, Multiple-front solutions for the burgers equation and the coupled burgers equations, Appl. Math. Comput. 190 (2007) 1198-1206.

[8] M. Wang, Solitary wave solutions for variant Boussinesq equations, Phys. Lett. A 199 (1995) 169-172.

[9] E.M.E. Zayed, H.A. Zedan, K.A. Gepreel, On the solitary wave solutions for nonlinear Hirota-Satsuma coupled KdV equations, Chaos, Solitons and Fractals 22 (2004) 285-303.

[10] L. Yang, J. Liu, K. Yang, Exact solutions of nonlinear PDE nonlinear transformations and reduction of nonlinear PDE to a quadrature, Phys. Lett. A 278 (2001) 267-270.

[11] E.M.E. Zayed, H.A. Zedan, K.A. Gepreel, Group analysis and modified tanh-function to find the invariant solutions and soliton solution for nonlinear Euler equations, Int. J. Nonlinear Sci. Numer. Simul. 5 (2004) 221-234.

[12] M. Inc, D.J. Evans, On traveling wave solutions of some nonlinear evolution equations, Int. J. Comput. Math. 81 (2004) 191-202.

[13] M.A. Abdou, The extended tanh-method and its applications for solving nonlinear physical models, Appl. Math. Comput. 190 (2007) 988-996

[14] E.G. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277 (2000) 212-218.

[15] W. Malfliet, Solitary wave solutions of nonlinear wave equations, Am. J. Phys. 60 (1992) 650-654.

[16] J.L. Hu, A new method of exact traveling wave solution for coupled nonlinear differential equations, Phys. Lett. A 322 (2004) 211-216.

[17] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge University Press, Cambridge, 1991.

[18] M.R. Miura, Backlund Transformation, Springer-Verlag, Berlin, 1978.

[19] C. Rogers, W.F. Shadwick, Backlund Transformations, Academic Press, New York, 1982.

[20] R. Hirota, Exact envelope soliton solutions of a nonlinear wave equation, J. Math. Phys. 14 (1973) 805-810.

[21] R. Hirota, J. Satsuma, Soliton solution of a coupled KdV equation, Phys. Lett. A 85 (1981) 407-408.

[22] Z.Y. Yan, H.Q. Zhang, New explicit solitary wave solutions and periodic wave solutions for WhithamCBroerCKaup equation in shallow water, Phys. Lett. A 285 (2001) 355-362.

[23] A.V. Porubov, Periodical solution to the nonlinear dissipative equation for surface waves in a convecting liquid layer, Phys. Lett. A 221 (1996) 391-394.

[24] K.W. Chow, A class of exact periodic solutions of nonlinear envelope equation, J. Math. Phys. 36 (1995) 4125-4137.

[25] E.G. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277 (2000) 212-218.

[26] Engui Fan, Multiple traveling wave solutions of nonlinear evolution equations using a unifiex algebraic method, J. Phys. A, Math. Gen. 35 (2002) 6853-6872.

[27] Z.Y. Yan, H.Q. Zhang, New explicit and exact traveling wave solutions for a system of variant Boussinesq equations in mathematical physics, Phys. Lett. A 252 (1999) 291-296.

[28] S.K. Liu, Z.T. Fu, S.D. Liu, Q. Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A 289 (2001) 69-74.

[29] Z. Yan, Abundant families of Jacobi elliptic functions of the (2 + 1)-dimensional integrable DaveyCStawartson-type equation via a new method, Chaos, Solitons and Fractals 18 (2003) 299-309.

[30] C. Bai, H. Zhao, Complex hyperbolic-function method and its applications to nonlinear equations, Phys. Lett. A 355 (2006) 22-30.

[31] E.M.E. Zayed, A.M. Abourabia, K.A. Gepreel, M.M. Horbaty, On the rational solitary wave solutions for the nonlinear HirotaCSatsuma coupled KdV system, Appl. Anal. 85 (2006) 751-768.

[32] K.W. Chow, A class of exact periodic solutions of nonlinear envelope equation, J. Math. Phys. 36 (1995) 4125-4137.

[33] M.L. Wang, Y.B. Zhou, The periodic wave equations for the KleinCGordonCSchordinger equations, Phys. Lett. A 318 (2003) 84-92.

[34] M.L. Wang, X.Z. Li, Extended F-expansion and periodic wave solutions for the generalized Zakharov equations, Phys. Lett. A 343 (2005) 48-54.

[35] M.L. Wang, X.Z. Li, Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation, Chaos, Solitons and Fractals 24 (2005) 1257-1268.

[36] X. Feng, Exploratory approach to explicit solution of nonlinear evolutions equations, Int. J. Theo. Phys. 39 (2000) 207-222.

[37] J.H. He, X.H. Wu, Exp-function method for nonlinear wave equations, Chaos, Solitons and Fractals 30 (2006) 700-708.

[38] Mingliang Wang, Xiangzheng Li, Jinliang Zhang, The $(\frac{G'}{G})$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Physics Letters A, 372 (2008) 417-423.

[39] Mingliang Wang, Jinliang Zhang, Xiangzheng Li, Application of the $(\frac{G'}{G})$-expansion to travelling wave solutions of the Broer-Kaup and the approximate long water wave equations. Appl. Math. Comput. , 206 (2008) 321-326.

[40] Ismail Aslan, Exact and explicit solutions to some nonlinear evolution equations by utilizing the $(\frac{G'}{G})$-expansion method. Appl. Math. Comput. In press, (2009).

[41] Xun Liu, Lixin Tian, Yuhai Wu, Application of $(\frac{G'}{G})$-expansion method to two nonlinear evolution equations. Appl. Math. Comput. , in press, (2009).

[42] Ismail Aslan, Turgut $\ddot{O}$zis, Analytic study on two nonlinear evolution equations by using the $(\frac{G'}{G})$-expansion method. Appl. Math. Comput. 209 (2009) 425-429.

[43] Z. Huiqun, Commun. Nonlinear Sci. Numer. Simul. 12 (5) (2007) 627-635.

[44] Wazwa Abdul-Majid. New solitary wave and periodic wave solutions to the (2+1)-dimensional Nizhnik-Nivikov-veselov system. Appl. Math. Comput. 187 (2007) 1584-1591.

[45] Senthil kumar C, Radha R, lakshmanan M. Trilinearization and localized coherent structures and periodic solutions for the (2+1) dimensional K-dv and NNV equations. Chaos, Solitons and Fractals. 39 (2009) 942-955.