Traveling Wave Solution For The BBM Equation With Any Order By $(\frac{G'}{G})$-expansion method

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Abstract: In this paper, we will try to construct the travelling wave solutions of the BBM equation with any order by using a generalized $(\frac{G'}{G})$-expansion method. The BBM equation is reduced to a nonlinear ordinary differential equations (ODE) by using a simple transformation. As a result, the traveling wave solutions of the BBM equation are expressed in three forms: hyperbolic function solutions, trigonometric function solutions and rational solutions.

Key–Words: $(\frac{G'}{G})$-expansion method, Traveling wave solutions, BBM equation, exact solution, evolution equation, nonlinear equation

1 Introduction

Many non-linear evolution equations(NLEEs) are widely used to describe complex physical phenomena. Research on NLEEs is a hot topic. Many efficient methods for finding analytic solutions and numerical solutions of nonlinear equations have been presented so far such as in [1-7].

In this paper, we will construct the travelling wave solutions of a non-linear equation. Also many approaches have been presented for constructing traveling wave solutions of NLEEs. Some of these approaches are the homogeneous balance method [8,9], the hyperbolic tangent expansion method [10,11], the trial function method [12], the tanh-method [13-15], the non-linear transform method [16], the inverse scattering transform [17], the Backlund transform [18,19], the Hirotas bilinear method [20,21], the generalized Riccati equation [22,23], the Weierstrass elliptic function method [24], the theta function method [25-27], the sineCcosine method [28], the Jacobi elliptic function expansion [29,30], the complex hyperbolic function method [31-33], the truncated Painleve expansion [34], the F-expansion method [35], the rank analysis method [36], the exp-function expansion method [37] and so on.

In [38], Mingliang Wang proposed a new method called $(\frac{G'}{G})$-expansion method. Recently several authors have studied some nonlinear equations by this method [39-42]. The main merits of the $(\frac{G'}{G})$-expansion method over the other methods are that it gives more general solutions with some free parameters and it handles NLEEs in a direct manner with no requirement for initial/boundary condition or initial trial function at the outset.

The rest of the paper is organized as follows. In Section 2, we describe the $(\frac{G'}{G})$-expansion method and give the main steps of the method. In the subsequent sections, we will apply the method to the BBM equation with Any Order. Some conclusions will be given in the last section.

2 Description of the $(\frac{G'}{G})$-expansion method

In this section we describe the $(\frac{G'}{G})$-expansion method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables x, t, is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{2.1}$$

or in three independent variables x, y and t, is given by

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{yt}, u_{xx}, u_{yy}, ...) = 0, \tag{2.2}$$

where $u = u(x,t)$ or $u = u(x,y,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ or $u = u(x,y,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following, we will give the main steps of the $(\frac{G'}{G})$-expansion method.

Step 1. Suppose that

$$u(x,\ t) = u(\xi), \quad \xi = \xi(x,t) \tag{2.3}$$

or

$$u(x,\ y,\ t) = u(\xi), \quad \xi = \xi(x,y,t) \tag{2.4}$$

The traveling wave variable (2.3) or (2.4) permits us reducing (2.1) or (2.2) to an ODE for $u = u(\xi)$

$$P(u,\ u', u'',\ ...) = 0. \tag{2.5}$$

Step 2. Suppose that the solution of (2.5) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \alpha_m (\frac{G'}{G})^m +\ ... \tag{2.6}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \tag{2.7}$$

$\alpha_m, ..., \lambda$ and μ are constants to be determined later, $\alpha_m \neq 0$. The unwritten part in (2.6) is also a polynomial in $(\frac{G'}{G})$, the degree of which is generally equal to or less than $m - 1$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (2.5).

Step 3. Substituting (2.6) into (2.5) and using second order LODE (2.7), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of (2.5) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_m, ..., \lambda$ and μ.

Step 4. Assuming that the constants $\alpha_m, ..., \lambda$ and μ can be obtained by solving the algebraic equations in Step 3. Since the general solutions of the second order LODE (2.7) have been well known for us, then substituting $\alpha_m, ...$ and the general solutions of (2.7) into (2.6) we have traveling wave solutions of the nonlinear evolution equation (2.1) or (2.2).

3 Application Of The $(\frac{G'}{G})$-Expansion Method For The BBM Equations With Any Order

We consider the BBM equation with any order [43]:

$$u_t + au_x + bu^n u_x - ru_{xxt} = 0, \ \ n > 0 \tag{3.1}$$

where $a,\ b$ and r are known constants.

In order to obtain the travelling wave solutions of Eq.(3.1), we suppose that

$$u(x,\ t) = u(\xi),\ \xi = x - ct \tag{3.2}$$

c is a constant that to be determined later.

By using the wave variable (3.2), (3.1) is converted into an ODE

$$-cu' + au' + bu^n u' + cru''' = 0 \tag{3.3}$$

Suppose that the solution of (3.3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i (\frac{G'}{G})^i \tag{3.4}$$

where a_i are constants.

Balancing the order of $u^n u'$ and u''' in Eq.(3.3), we have $mn + m + 1 = m + 3 \ \Rightarrow\ m = \frac{2}{n}$. So we make a variable $u = v^{\frac{2}{n}}$, then (3.3) is converted into

$$2(a - c + bv)n^2 v^2 v' + 2cr(2 - n)(2 - 2n)(v')^3$$
$$+bcrnvv'v'' + 2n^2 crv^2 v''' = 0 \tag{3.5}$$

Suppose that the solution of (3.5) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$v(\xi) = \sum_{i=0}^{l} b_i (\frac{G'}{G})^i \tag{3.6}$$

where b_i are constants, $G = G(\xi)$ satisfies the second order LODE in the form:

$$G'' + \lambda G' + \mu G = 0 \quad (3.7)$$

where λ and μ are constants.

Balancing the order of $v^3 v'$ and $(v')^3$ in Eq.(3.5), we have $4l + 1 = 3l + 3 \Rightarrow l = 2$. So Eq.(3.6) can be rewritten as

$$v(\xi) = b_2(\frac{G'}{G})^2 + b_1(\frac{G'}{G}) + b_0,\ b_2 \neq 0 \quad (3.8)$$

$b_2,\ b_1,\ b_0$ are constants to be determined later. Then we can obtain

$$v'(\xi) = b_1[-\lambda(\frac{G'}{G}) - \mu - (\frac{G'}{G})^2]$$

$$v''(\xi) = 2b_1(\frac{G'}{G})^3 + 3\lambda b_1(\frac{G'}{G})^2$$
$$+(\lambda^2 b_1 + 2b_1\mu)(\frac{G'}{G}) + \lambda\mu b_1$$

$$v'''(\xi) = -24b_2(\frac{G'}{G})^5 + (-54b_2\lambda - 6b_1)(\frac{G'}{G})^4$$
$$+(-12b_1\lambda - 38b_2\lambda^2 - 40b_2\mu)(\frac{G'}{G})^3$$
$$+(-52b_2\lambda\mu - 7b_1\lambda^2 - 8b_2\lambda^3 - 8b_1\mu)(\frac{G'}{G})^2$$
$$+(-14b_2\lambda^2\mu - b_1\lambda^3 - 16b_2\mu^2 - 8b_1\lambda\mu)(\frac{G'}{G})$$
$$-b_1\lambda^2\mu - 2b_1\mu^2 - 6b_2\lambda\mu^2$$

Substituting (3.8) into (3.5) and collecting all the terms with the same power of $(\frac{G'}{G})$ together and equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$(\frac{G'}{G})^0:\ -2n^2bb_0^3b_1\mu + 12crnb_1^3\mu^3 - 2n^2crb_0^2b_1\lambda^2\mu$$

$$-4crn^2b_1^3\mu^3 - 2n^2ab_0^2b_1\mu - 4n^2crb_0^2b_1\mu^2$$

$$-2bcnrb_0b_1\mu^3b_2 - bcrnb_0b_1^2\mu^2\lambda - 8crb_1^3\mu^3$$

$$+2n^2cb_0^2b_1\mu - 12n^2crb_0^2b_2\lambda\mu^2 = 0$$

$$(\frac{G'}{G})^1:\ -12n^2crb_1^3\lambda\mu^2 - 24crn^2b_1^2b_2\mu^3 - bcrnb_1^3\lambda\mu^2$$

$$-10bcrnb_0b_1b_2\lambda\mu^2 - 2bcnrb_0b_1^2\mu^2 - 2bcnrb_1^2b_2\mu^3$$

$$-2bcnrb_0b_1^2\lambda^2\mu - 6bn^2b_0^2b_1^2\mu - 4bcnrb_0b_2^2\mu^3$$

$$-2n^2ab_0^2b_1\lambda + 2n^2cb_0^2b_1\lambda - 2n^2bb_0^3b_1\lambda$$

$$-4n^2ab_0^2b_2\mu - 4n^2ab_0b_1^2\mu + 4n^2cb_0b_1^2\mu$$

$$+4n^2cb_0^2b_2\mu - 4n^2bb_0^3b_2\mu + 36crnb_1^3\lambda\mu^2$$

$$-24crb_1^3\lambda\mu^2 - 48crb_1^2b_2\mu^3 + 72crnb_1^2b_2\mu^3$$

$$-24crn^2b_0b_1b_2\lambda\mu^2 - 16crn^2b_0^2b_1\lambda\mu - 4n^2crb_0b_1^2\lambda^2\mu$$

$$-2n^2crb_0^2b_1\lambda^3 - 32n^2crb_0^2b_2\mu^2 - 8n^2crb_0b_1^2\mu^2$$

$$-28n^2crb_0^2b_2\lambda^2\mu = 0$$

$$(\frac{G'}{G})^2:\ -14crn^2b_1^3\lambda^2\mu + 144crnb_1b_2^2\mu^3 - 84crn^2b_1^2b_2\lambda\mu^2$$

$$-48crn^2b_1b_2^2\mu^3 - 2bcrnb_1^3\lambda^2\mu - 2bcrnb_1^3\mu^2$$

$$-14bcrnb_0b_1b_2\mu^2 - 16bcrnb_0b_2^2\lambda\mu^2 - 6bcrnb_1b_2^2\mu^3$$

$$-11bcrnb_1^2b_2\lambda\mu^2 - 14bcrnb_0b_1b_2\lambda^2\mu - 6bcrnb_0b_1^2\lambda\mu$$

$$-4an^2b_0^2b_2\lambda - 4an^2b_0b_1^2\lambda + 4cn^2b_0b_1^2\lambda$$

$$+4cn^2b_0^2b_2\lambda - 4bn^2b_0^3b_2\lambda - 18bn^2b_0^2b_1b_2\mu$$

$$-12an^2b_0b_1b_2\mu + 12cn^2b_0b_1b_2\mu - bcrnb_0b_1^2\lambda^3$$

$$-6bn^2b_0^2b_1^2\lambda - 6bn^2b_0b_1^3\mu - 144crb_1^2b_2\lambda\mu^2$$

$$+36crnb_1^3\lambda^2\mu + 216crnb_1^2b_2\lambda\mu^2 - 24crb_1^3\lambda^2\mu$$

$$-96crb_1b_2^2\mu^3 + 36crnb_1^3\mu^2 - 16crn^2b_1^3\mu^2$$

$$-72crn^2b_0b_1b_2\mu^2 - 60crn^2b_0b_1b_2\lambda^2\mu - 32crn^2b_0b_1^2\lambda\mu$$
$$-24crn^2b_0b_2^2\lambda\mu^2 - 14crn^2b_0^2b_1\lambda^2 - 16crn^2b_0^2b_2\lambda^3$$

$$-104crn^2b_0^2b_2\lambda\mu - 4crn^2b_0b_1^2\lambda^3 - 16crn^2b_0^2b_1\mu$$

$$+2cn^2b_0^2b_1 - 2bn^2b_0^3b_1 - 2an^2b_0^2b_1$$

$$-2an^2b_1^3\mu + 2cn^2b_1^3\mu - 24crb_1^3\mu^2 = 0$$

$$(\frac{G'}{G})^3 :\ -8an^2b_1^2b_2\mu + 432crnb_1b_2^2\lambda\mu^2 - 112crn^3b_1^2b_2\mu^2$$

$$-40crn^2b_1^3\lambda\mu - 104crn^2b_1^2b_2\lambda^2\mu - 168crn^2b_1b_2^2\lambda\mu^2$$

$$-6bcrnb_1^3\lambda\mu - 4bcrnb_2^3\mu^3 - bcrnb_1^3\lambda^3$$

$$-20bcrnb_0b_2^2\lambda^2\mu - 6bcrnb_0b_1b_2\lambda^3 - 16bcrnb_1^2b_2\lambda^2\mu$$

$$-26bcrnb_1b_2^2\lambda\mu^2 - 16bcrnb_1^2b_2\mu^2 - 36bcrnb_0b_1b_2\lambda\mu$$

$$-6bn^2b_0b_1^3\lambda - 24bn^2b_0b_1^2b_2\mu + 8cn^2b_1^2b_2\mu$$

$$-4bcrnb_0b_1^2\lambda^2 - 4bcrnb_0b_1^2\mu - 20bcrnb_0b_2^2\mu^2$$

$$-18bn^2b_0^2b_1b_2\lambda - 12an^2b_0b_1b_2\lambda + 12cn^2b_0b_1b_2\lambda$$

$$-12bn^2b_0^2b_2^2\mu - 8an^2b_0b_2^2\mu + 8cn^2b_0b_2^2\mu$$

$$-32crn^2b_2^2\mu^3 - 144crb_1^2b_2\lambda^2\mu - 288crb_1b_2^2\lambda\mu^2$$

$$+72crnb_1^3\lambda\mu + 216crnb_1^2b_2\lambda^2\mu - 48crb_1^3\lambda\mu$$

$$-144crb_1^2b_2\mu^2 + 12crnb_1^3\lambda^3 + 96crnb_2^3\mu^3$$

$$-6crn^2b_1^3\lambda^3 + 216crnb_1^2b_2\mu^2 - 240crn^2b_0b_1b_2\lambda\mu$$

$$-56crn^2b_0b_2^2\lambda^2\mu - 36crn^2b_0b_1b_2\lambda^3 - 24crn^2b_0^2b_1\lambda$$

$$-76crn^2b_0^2b_2\lambda^2 - 28crn^2b_0b_1^2\lambda^2 - 32crn^2b_0b_1^2\mu$$

$$-64crn^2b_0b_2^2\mu^2 - 80crn^2b_0^2b_2\mu - 2an^2b_1^3\lambda$$

$$+2cn^2b_1^3\lambda - 6bn^2b_0^2b_1^2 - 4an^2b_0^2b_2$$

$$-4an^2b_0b_1^2 + 4cn^2b_0b_1^2 + 4cn^2b_0^2b_2$$

$$-4bn^2b_0^3b_2 - 2bn^2b_1^4\mu - 64crb_2^3\mu^3$$

$$-8crb_1^3\lambda^3 = 0$$

$$(\frac{G'}{G})^4 :\ 2cn^2b_1^3 - 2an^2b_1^3 + 432crnb_1b_2^2\lambda^2\mu$$

$$+432crnb_1b_2^2\mu^2+288crnb_2^3\lambda\mu^2-44crn^2b_1^2b_2\lambda^3$$

$$-280crn^2b_1^2b_2\lambda\mu-108crn^2b_2^3\lambda\mu^2-4bcrnb_1^3\lambda^2$$

$$-202crn^2b_1b_2^2\lambda^2\mu-212crn^2b_1b_2^2\mu^2-34bcrnb_1b_2^2\lambda^2\mu$$

$$-7bcrnb_1^2b_2\lambda^3-4bcrnb_1^3\mu-42bcrnb_1^2b_2\lambda\mu$$

$$-48bcrnb_0b_2^2\lambda\mu bcrn-22b_0b_1b_2\lambda^2-34bcrnb_1b_2^2\mu^2$$

$$-16bcrnb_2^3\lambda\mu^2-22bcrnb_0b_1b_2\mu-8an^2b_0b_2^2\lambda$$

$$+8cn^2b_0b_2^2\lambda-30bn^2b_0b_1b_2^2\mu-10bn^2b_1^3b_2\mu$$

$$-5bcrnb_0b_1^2\lambda-8bcrnb_0b_2^2\lambda^3-18bn^2b_0^2b_1b_2$$

$$-24bn^2b_0b_1^2b_2\lambda-12an^2b_0b_1b_2+12cn^2b_0b_1b_2$$

$$-8an^2b_1^2b_2\lambda+8cn^2b_1^2b_2\lambda-12bn^2b_0^2b_2^2\lambda$$

$$-288crb_1^2b_2\lambda\mu-288crb_1b_2^2\lambda^2\mu-10an^2b_1b_2^2\mu$$

$$+10cn^2b_1b_2^2\mu+72crnb_1^2b_2\lambda^3+432crnb_1^2b_2\lambda\mu$$

$$-48crb_1^2b_2\lambda^3-288crb_1b_2^2\mu^2-192crb_2^3\lambda\mu^2$$

$$+36crnb_1^3\lambda^2+36crnb_1^3\mu-26crn^2b_1^3\lambda^2$$

$$-28crn^2b_1^3\mu-12crn^2b_0^2b_1-192crn^2b_0b_1b_2\mu$$

$$-208crn^2b_0\lambda\mu-180crn^2b_0b_1b_2\lambda^2-108crn^2b_0^2b_2\lambda$$

$$-48crn^2b_0b_1^2\lambda-32crn^2b_0b_2^2\lambda^3-24crb_1^3\lambda^2$$

$$-2n^2bb_1^4\lambda-6n^2bb_0b_1^3-24crb_1^3\mu=0$$

$$(\frac{G'}{G})^5:\ -2bn^2b_1^4+216crnb_1^2b_2\mu+144crnb_1b_2^2\lambda^3$$

$$+864crnb_1b_2^2\lambda\mu+288crnb_2^3\lambda^2\mu-184crn^2b_1^2b_2\mu$$

$$-82crn^2b_1b_2^2\lambda^3-176crn^2b_1^2b_2\lambda^2-20bcrnb_2^3\mu^2$$

$$-2bcrnb_0b_1^2-512crn^2b_1b_2^2\lambda\mu-124crn^2b_2^3\lambda^2\mu$$

$$-26bcrnb_1^2b_2\lambda^2-5bcrnb_1^3\lambda-84bcrnb_1b_2^2\lambda\mu$$

$$-26bcrnb_1^2b_2\mu-14bcrnb_1b_2^2\lambda^3-20bcrnb_2^3\lambda^2\mu$$

$$-26bcrnb_0b_1b_2\lambda-10an^2b_1b_2^2\lambda+10cn^2b_1b_2^2\lambda$$

$$-28bcrnb_0b_2^2\mu-28bcrnb_0b_2^2\lambda^2-24crn^2b_0b_1^2$$

$$-30bn^2b_0b_1b_2^2\lambda-24bn^2b_0b_1^2b_2-10bn^2b_1^3b_2\lambda$$

$$-12bn^2b_0b_2^3\mu-18bn^2b_1^2b_2^2\mu-128crn^2b_2^3\mu^2$$

$$-576crb_1b_2^2\lambda\mu+216crnb_1^2b_2\lambda^2-144crb_1^2b_2\lambda^2$$

$$-144crb_1^2b_2\mu-96crb_1b_2^2\lambda^3-192crb_2^3\lambda^2\mu$$

$$+36crnb_1^3\lambda + 288crnb_2^3\mu^2 - 36crn^2b_1^3\lambda$$

$$-48crn^2b_0^2b_2 - 264crn^2b_0b_1b_2\lambda - 160crn^2b_0b_2^2\mu$$

$$-152crn^2b_0b_2^2\lambda^2 - 192crb_2^3\mu^2 - 8an^2b_1^2b_2$$

$$+8cn^2b_1^2b_2 - 12bn^2b_0^2b_2^2 - 8an^2b_0b_2^2$$

$$+8cn^2b_0b_2^2 - 4an^2b_2^3\mu + 4cn^2b_2^3\mu$$

$$-24crb_1^3\lambda = 0$$

$$(\frac{G'}{G})^6:\ -8crb_1^3 + 432crnb_1b_2^2\lambda^2 + 576crnb_2^3\lambda\mu$$

$$+432crnb_1b_2^2\mu - 228crn^2b_1^2b_2\lambda - 296crn^2b_2^3\lambda\mu$$

$$-8bcrnb_2^3\lambda^3 - 310crn^2b_1b_2^2\lambda^2 - 320crn^2b_1b_2^2\mu$$

$$-2bcrnb_1^3 - 10bcrnb_0b_1b_2 - 31bcrnb_1^2b_2\lambda$$

$$-50bcrnb_1b_2^2\lambda^2 - 50bcrnb_1b_2^2\mu - 48bcrnb_2^3\lambda\mu$$

$$-12bn^2b_0b_2^3\lambda - 18bn^2b_1^2b_2^2\lambda - 32bcrnb_0b_2^2\lambda$$

$$-30bn^2b_0b_1b_2^2 - 14bn^2b_1b_2^3\mu + 216crnb_1^2b_2\lambda$$

$$-144crb_1^2b_2\lambda - 288crb_1b_2^2\lambda^2 - 384crb_2^3\lambda\mu$$

$$+96crnb_2^3\lambda^3 - 48crn^2b_2^3\lambda^3 - 120crn^2b_0b_1b_2$$

$$-216crn^2b_0b_2^2\lambda - 64crb_2^3\lambda^3 - 4an^2b_2^3\lambda$$

$$+4cn^2b_2^3\lambda - 10bn^2b_1^3b_2 - 10an^2b_1b_2^2$$

$$+10cn^2b_1b_2^2 - 16crn^2b_1^3 + 12crnb_1^3 = 0$$

$$(\frac{G'}{G})^7:\ -28bcrnb_2^3\mu - 12bcrnb_1^2b_2 - 4an^2b_2^3$$

$$+72crnb_1^2b_2 + 288crnb_2^3\lambda^2 - 12bcrnb_0b_2^2$$

$$-28bcrnb_2^3\lambda^2 - 172crn^2b_2^3\lambda^2 - 176crn^2b_2^3\mu$$

$$-14bn^2b_2^3b_1\lambda - 58bcrnb_1b_2^2\lambda - 192crb_2^3\mu$$

$$+432crnb_1b_2^2\lambda - 96crn^2b_1^2b_2 - 48crb_1^2b_2$$

$$+4cn^2b_2^3 - 288crb_1b_2^2\lambda - 96crn^2b_0b_2^2$$

$$-4bn^2b_2^4\mu + 288crnb_2^3\mu - 18bn^2b_1^2b_2^2$$

$$-384crn^2b_1b_2^2\lambda - 12bn^2b_0b_2^3 = 0$$

$$(\frac{G'}{G})^8:\ 144crnb_1b_2^2 - 156crn^2b_1b_2^2 - 192crb_2^3\lambda$$

$$+288crnb_2^3\lambda - 32bcrnb_2^3\lambda - 4bn^2b_2^4\lambda$$

$$-22bcrnb_1b_2^2 - 204crn^2b_2^3\lambda - 14bn^2b_2^3b_1$$

$$-96crb_1b_2^2 = 0$$

$$(\frac{G'}{G})^9:\ -4bn^2b_2^4 - 64crb_2^3 - 12bcrnb_2^3$$

$$+96crnb_2^3 - 80crn^2b_2^3 = 0$$

Solving the algebraic equations above, yields:

$$b_2 = [2ar(16+3bn-24n+20n^2)]/$$

$$[b(-12rn\lambda^2-24rn^2\mu-4brn\mu-2n^2$$

$$+6rn^2\lambda^2+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]$$

$$b_1 = [2\lambda ar(16+3bn-24n+20n^2)]/$$

$$[b(-12rn\lambda^2-24rn^2\mu-4brn\mu-2n^2$$

$$+6rn^2\lambda^2+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]$$

$$b_0 = [2\mu ar(16+3bn-24n+20n^2)]/$$

$$[b(-12rn\lambda^2-24rn^2\mu-4brn\mu-2n^2$$

$$+6rn^2\lambda^2+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]$$

$$c = (-2an^2)/(-12rn\lambda^2-24rn^2\mu-4brn\mu-2n^2$$

$$+6rn^2\lambda^2+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)$$

$$\xi = x+y-[(-2an^2)/(-12rn\lambda^2-24rn^2\mu-4brn\mu$$

$$-2n^2+6rn^2\lambda^2+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]t \tag{3.9}$$

Substituting the general solutions of Eq.(3.7) into (3.9), we have three types of travelling wave solutions of the BBM equation (3.1) as follows:

Case 1: when $\lambda^2-4\mu>0$

$$v_1(\xi) = \{[2ar(16+3bn-24n+20n^2)(\lambda^2-4\mu)]/$$

$$[4b(-12rn\lambda^2-24rn^2\mu-4brn\mu-2n^2+6rn^2\lambda^2$$

$$+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]\}$$

$$\times\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2$$

$$-\{[2ar(16+3bn-24n+20n^2)\lambda^2]/$$

$$[4b(-12rn\lambda^2-24rn^2\mu-4brn\mu$$

$$-2n^2+6rn^2\lambda^2+8r\lambda^2$$

$$+48rn\mu+brn\lambda^2-32r\mu)]\}$$

$$+\{[2\mu ar(16+3bn-24n+20n^2)]/$$

$$[b(-12rn\lambda^2-24rn^2\mu-4brn\mu$$

$$-2n^2+6rn^2\lambda^2+8r\lambda^2$$

$$+48rn\mu+brn\lambda^2-32r\mu)]\}$$

Then

$$u_1(\xi) = (v_1(\xi))^{\frac{2}{n}},$$

where

$$\xi = x+y-[(-2an^2)/$$

$$(-12rn\lambda^2-24rn^2\mu-4brn\mu-2n^2$$

$$+6rn^2\lambda^2+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]t,$$

C_1 and C_2 are two arbitrary constants.

Case 2: when $\lambda^2-4\mu<0$

$$v_2(\xi) = \{[2ar(16+3bn-24n+20n^2)(4\mu-\lambda^2)]/$$

$$[4b(-12rn\lambda^2-24rn^2\mu-4brn\mu-2n^2+6rn^2\lambda^2$$

$$+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]\}$$

$$\times\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2$$

$$-\{[2ar(16+3bn-24n+20n^2)\lambda^2]/$$

$$[4b(-12rn\lambda^2-24rn^2\mu-4brn\mu-2n^2+6rn^2\lambda^2$$

$$+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]\}$$

$$+\{[2\mu ar(16+3bn-24n+20n^2)]/$$

$$[b(-12rn\lambda^2-24rn^2\mu-4brn\mu$$

$$-2n^2+6rn^2\lambda^2+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]\}$$

Then

$$u_2(\xi) = (v_2(\xi))^{\frac{2}{n}},$$

where

$$\xi = x+y-[(-2an^2)/$$

$$(-12rn\lambda^2-24rn^2\mu-4brn\mu-2n^2+6rn^2\lambda^2$$

$$+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]t,$$

C_1 and C_2 are two arbitrary constants.

Case 3: when $\lambda^2-4\mu=0$

$$v_3(\xi) = \{[2arc_2^2(16+3bn-24n+20n^2)]/$$

$$[b(-12rn\lambda^2-24rn^2\mu-4brn\mu-2n^2+6rn^2\lambda^2+8r\lambda^2$$

$$+48rn\mu+brn\lambda^2-32r\mu)(c_1+c_2\xi)^2]\}$$

$$-\{[2ar\lambda^2(16+3bn-24n+20n^2)]/$$

$$[4b(-12rn\lambda^2-24rn^2\mu-4brn\mu-2n^2+6rn^2\lambda^2$$

$$+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]\}$$

$$+\{[2\mu ar(16+3bn-24n+20n^2)]/$$

$$[b(-12rn\lambda^2-24rn^2\mu-4brn\mu$$

$$-2n^2+6rn^2\lambda^2+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]\}$$

Then

$$u_3(\xi) = (v_3(\xi))^{\frac{2}{n}},$$

where

$$\xi = x+y-[(-2an^2)/$$

$$(-12rn\lambda^2-24rn^2\mu-4brn\mu-2n^2+6rn^2\lambda^2$$

$$+8r\lambda^2+48rn\mu+brn\lambda^2-32r\mu)]t,$$

C_1 and C_2 are two arbitrary constants.

4 Conclusions

On comparing between the $(\frac{G'}{G})$-expansion method and the other methods, we come to the conclusion that the $(\frac{G'}{G})$- expansion method is more powerful, effective and convenient. It is a standard and computerizable method which allows us to solve complicated non-linear evolution equations in mathematical physics. We have noted that the $(\frac{G'}{G})$-expansion method changes the given difficult problems into simple problems which can be solved easily.

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