Traveling Wave Solutions For The Fifth-Order Sawada-Kotera Equation And The General Gardner Equation By $(\frac{G'}{G})$-expansion method

Qinghua Feng
Shandong University of Technology
School of Science
Zhangzhou Road 12, Zibo, 255049
China
fqhua@sina.com

Bin Zheng
Shandong University of Technology
School of Science
Zhangzhou Road 12, Zibo, 255049
China
zhengbin2601@126.com

Abstract: In this paper, a generalized $(\frac{G'}{G})$-expansion method is used to seek more general exact solutions of the fifth-order Sawada-Kotera equation and the general Gardner equation. As a result, the traveling wave solutions with three arbitrary functions are obtained including hyperbolic function solutions, trigonometric function solutions and rational solutions. The method appears to be easier and faster by means of some mathematical software.

Key–Words: $(\frac{G'}{G})$-expansion method, Traveling wave solutions, fifth-order Sawada-Kotera equation, general Gardner equation, exact solution, evolution equation, nonlinear equation

1 Introduction

During the past four decades or so searching for explicit solutions of nonlinear evolution equations (NLEEs) by using various different methods has been the main goal for many researchers, and many powerful methods for constructing exact solutions of nonlinear evolution equations have been established and developed. Some of these approaches are the homogeneous balance method [8,9], the hyperbolic tangent expansion method [10,11], the trial function method [12], the tanh-method [13-15], the non-linear transform method [16], the inverse scattering transform [17], the Backlund transform [18,19], the Hirotas bilinear method [20,21], the generalized Riccati equation [22,23], the Weierstrass elliptic function method [24], the theta function method [25-27], the sineCcosine method [28], the Jacobi elliptic function expansion [29,30], the complex hyperbolic function method [31-33], the truncated Painleve expansion [34], the F-expansion method [35], the rank analysis method [36], the exp-function expansion method [37] and so on. Yet there is no unified method that can be used to deal with all types of nonlinear evolution equations.

Recently a so-called $(\frac{G'}{G})$-expansion method has drawn a lot of attention. The method was presented by Mingliang Wang in [38] at first. The main merits of the $(\frac{G'}{G})$-expansion method over the other methods are that it gives more general solutions with some free parameters and it handles NLEEs in a direct manner with no requirement for initial/boundary condition or initial trial function at the outset. The method was soon been applied to other non-linear problems by several authors [39-42].

In this paper we will apply the $(\frac{G'}{G})$-expansion method to some nonlinear problems. In Section 2, we describe the universe process of the $(\frac{G'}{G})$-expansion method. In section 3 and 4, we will obtain the travelling wave solutions of the fifth-order Sawada-Kotera equation and the general Gardner equation by the method respectively. In section 5, we will give some conclusions on the $(\frac{G'}{G})$-expansion method.

2 Description of the $(\frac{G'}{G})$-expansion method

In this section we describe the $(\frac{G'}{G})$-expansion method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables $x,\ t$, is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{2.1}$$

or in three independent variables $x,\ y$ and t, is given by

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{yt}, u_{xx}, u_{yy}, ...) = 0, \quad (2.2)$$

where $u = u(x,t)$ or $u = u(x,y,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ or $u = u(x,y,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following, we will give the main steps of the $(\frac{G'}{G})$-expansion method.

Step 1. Suppose that

$$u(x,\ t) = u(\xi), \quad \xi = \xi(x,t) \quad (2.3)$$

or

$$u(x,\ y,\ t) = u(\xi), \quad \xi = \xi(x,y,t) \quad (2.4)$$

The traveling wave variable (2.3) or (2.4) permits us reducing (2.1) or (2.2) to an ODE for $u = u(\xi)$

$$P(u,\ u', u'',\ ...) = 0. \quad (2.5)$$

Step 2. Suppose that the solution of (2.5) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \alpha_m(\frac{G'}{G})^m +\ ... \quad (2.6)$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \quad (2.7)$$

$\alpha_m, ..., \lambda$ and μ are constants to be determined later, $\alpha_m \neq 0$. The unwritten part in (2.6) is also a polynomial in $(\frac{G'}{G})$, the degree of which is generally equal to or less than $m-1$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (2.5).

Step 3. Substituting (2.6) into (2.5) and using second order LODE (2.7), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of (2.5) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_m, ..., \lambda$ and μ.

Step 4. Assuming that the constants $\alpha_m, ..., \lambda$ and μ can be obtained by solving the algebraic equations in Step 3. Since the general solutions of the second order LODE (2.7) have been well known for us, then substituting $\alpha_m, ...$ and the general solutions of (2.7) into (2.6) we have traveling wave solutions of the nonlinear evolution equation (2.1) or (2.2).

3 Application Of The $(\frac{G'}{G})$-Expansion Method For The Fifth-Order Sawada-Kotera Equation

We begin with the fifth-order Sawada-Kotera equation [43]:

$$u_{xxxxx} + u_t + 45u_x u^2 + 15(u_x u_{xx} + u u_{xxx}) = 0 \quad (3.1)$$

In order to obtain the traveling wave solutions of Eq.(3.1), we suppose that

$$u(x,\ t) = u(\xi),\ \xi = x - ct \quad (3.2)$$

c is a constant that to be determined later.
By using the wave variable (3.2), Eq.(3.1) is converted into an ODE

$$u^{(5)} - cu' + 45u'u^2 + 15u'u'' + 15uu''' = 0 \quad (3.3)$$

Integrating (3.3) with respect to ξ once, we obtain

$$u^{(4)} - cu + 15u^3 + 15uu'' = g \quad (3.4)$$

where g is the integration constant that can be determined later.

Suppose that the solution of the ODE (3.4) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i(\frac{G'}{G})^i \quad (3.5)$$

where a_i are constants, $G = G(\xi)$ satisfies the second order LODE in the form:

$$G'' + \lambda G' + \mu G = 0 \quad (3.6)$$

where λ and μ are constants.
Balancing the order of u^3 and $u^{(4)}$ in Eq.(3.4), we get that $3m = m + 4 \ \Rightarrow\ m = 2$, so Eq.(3.5) can be

rewritten as

$$u(\xi)=a_2(\frac{G'}{G})^2+a_1(\frac{G'}{G})+a_0,\ a_2\neq 0 \qquad (3.7)$$

$a_2,\ a_1,\ a_0$ are constants to be determined later. Then we can obtain

$$u'(\xi)=-2a_2(\frac{G'}{G})^3+(-a_1-2a_2\lambda)(\frac{G'}{G})^2$$
$$+(-a_1\lambda-2a_2\mu)(\frac{G'}{G})-a_1\mu$$

$$u''(\xi)=6a_2(\frac{G'}{G})^4+(2a_1+10a_2\lambda)(\frac{G'}{G})^3$$
$$+(8a_2\mu+3a_1\lambda+4a_2\lambda^2)(\frac{G'}{G})^2$$
$$+(6a_2\lambda\mu+2a_1\mu+a_1\lambda^2)(\frac{G'}{G})+2a_2\mu^2+a_1\lambda\mu$$

$$u'''(\xi)=-24a_2(\frac{G'}{G})^5+(-54a_2\lambda-6a_1)(\frac{G'}{G})^4$$
$$+(-12a_1\lambda-38a_2\lambda^2-40a_2\mu)(\frac{G'}{G})^3$$
$$+(-52a_2\lambda\mu-7a_1\lambda^2-8a_2\lambda^3-8a_1\mu)(\frac{G'}{G})^2$$
$$+(-14a_2\lambda^2\mu-a_1\lambda^3-16a_2\mu^2-8a_1\lambda\mu)(\frac{G'}{G})$$
$$-a_1\lambda^2\mu-2a_1\mu^2-6a_2\lambda\mu^2$$

$$u^{(4)}(\xi)=120a_2(\frac{G'}{G})^6+(24a_1+336a_2\lambda)(\frac{G'}{G})^5$$
$$+(330a_2\lambda^2+240a_2\mu+60a_1\lambda)(\frac{G'}{G})^4$$
$$+(50a_1\lambda^2+130a_2\lambda^3+40a_1\mu+440a_2\lambda\mu)(\frac{G'}{G})^3$$
$$+(16a_2\lambda^4+15a_1\lambda^3+136a_2\mu^2+60a_1\lambda\mu+232a_2\lambda^2\mu)(\frac{G'}{G})^2$$
$$+(22a_1\lambda^2\mu+16a_1\mu^2+120a_2\lambda\mu^2+30a_2\lambda^3\mu+a_1\lambda^4)(\frac{G'}{G})$$
$$+14a_2\lambda^2\mu^2+16a_2\mu^3+a_1\lambda^3\mu+8a_1\lambda\mu^2$$

On substituting Eq.(3.7) into the ODE (3.4) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$(\frac{G'}{G})^0:\ 16a_2\mu^3+15a_0a_1\lambda\mu+14a_2\lambda^2\mu^2$$
$$+8a_1\lambda\mu^2-ca_0+15a_0^3+a_1\lambda^3\mu$$
$$-g+30a_0a_2\mu^2=0$$

$$(\frac{G'}{G})^1:\ 30a_2\lambda^3\mu+30a_0a_1\mu+30a_1a_2\mu^2$$
$$+45a_0^2a_1+90a_0a_2\lambda\mu+15a_0a_1\lambda^2$$
$$+22a_1\lambda^2\mu-ca_1+16a_1\mu^2+a_1\lambda^4$$
$$+15a_1^2\lambda\mu+120a_2\lambda\mu^2=0$$

$$(\frac{G'}{G})^2:\ 120a_0a_2\mu+15a_1^2\lambda^2+16a_2\lambda^4$$
$$+30a_1^2\mu+45a_0^2a_2+136a_2\mu^2$$
$$+105a_1a_2\lambda\mu+232a_2\lambda^2\mu+15a_1\lambda^3$$
$$+60a_1\lambda\mu+60a_0a_2\lambda^2+45a_0a_1^2+45a_0a_1\lambda$$
$$-ca_2+30a_2^2\mu^2=0$$

$$(\frac{G'}{G})^3:\ 75a_1a_2\lambda^2+130a_2\lambda^3+90a_2^2\lambda\mu$$
$$+90a_0a_1a_2+30a_0a_1+150a_1a_2\mu$$
$$+15a_1^3+440a_2\lambda\mu+40a_1\mu$$
$$+45a_1^2\lambda+50a_1\lambda^2+150a_0a_2\lambda=0$$

$$(\frac{G'}{G})^4:\ 330a_2\lambda^2+30a_1^2+90a_0a_2$$
$$+120a_2^2\mu+60a_2^2\lambda^2+60a_1\lambda$$
$$+45a_1^2a_2+195a_1a_2\lambda$$
$$+240a_2\mu+45a_0a_2^2=0$$

$$(\frac{G'}{G})^5:\ 45a_1a_2^2+150a_2^2\lambda+336a_2\lambda$$
$$+24a_1+120a_1a_2=0$$

$$(\frac{G'}{G})^6:\ 15a_2^3 + 90a_2^2 + 120a_2 = 0$$

Solving the algebraic equations above, we can get the results for two cases:

Case 1:

$$a_2 = -4,$$
$$a_1 = -4\lambda,$$
$$a_0 = -\frac{1}{3}(\lambda^2 + 8\mu)$$
$$c = -8\lambda^2\mu + 16\mu^2 + \lambda^4,$$
$$g = \frac{128}{9}\mu^3 + \frac{8}{3}\lambda^4\mu - \frac{32}{3}\lambda^2\mu^2 - \frac{2}{9}\lambda^6 \tag{3.8}$$

where λ, μ are arbitrary constants.

Substituting (3.8) into (3.7), we get that

$$u(\xi) = -4(\frac{G'}{G})^2 - 4\lambda(\frac{G'}{G}) - \frac{1}{3}(\lambda^2 + 8\mu)$$

$$\xi = x + (-8\lambda^2\mu + 16\mu^2 + \lambda^4)t \tag{3.9}$$

where λ, μ are arbitrary constants.

Substituting the general solutions of Eq.(3.6) into (3.9), we can obtain three types of traveling wave solutions of the fifth-order SawadaCKotera equation (3.1) as follows:

When $\lambda^2 - 4\mu > 0$

$$u_1(\xi) = \lambda^2 - (\lambda^2 - 4\mu).$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2$$
$$-\frac{1}{3}(\lambda^2 + 8\mu),$$

where

$$\xi = x + (-8\lambda^2\mu + 16\mu^2 + \lambda^4)t.$$

C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$

$$u_2(\xi) = \lambda^2 - (4\mu - \lambda^2).$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right)^2$$
$$-\frac{1}{3}(\lambda^2 + 8\mu),$$

where

$$\xi = x + (-8\lambda^2\mu + 16\mu^2 + \lambda^4)t.$$

C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$

$$u_3(\xi) = \lambda^2 - \frac{4C_2^2}{(C_1 + C_2\xi)^2} - \frac{1}{3}(\lambda^2 + 8\mu),$$

where

$$\xi = x + (-8\lambda^2\mu + 16\mu^2 + \lambda^4)t.$$

C_1, C_2 are arbitrary constants.

Case 2:

$$a_2 = -2,$$
$$a_1 = -2\lambda,$$
$$a_0 = a_0,$$
$$c = 120a_0\mu + 45a_0^2 + 22\lambda^2\mu$$
$$+15a_0\lambda^2 + \lambda^4 + 76\mu^2$$
$$g = -32\mu^3 - 2\lambda^4\mu - 44\lambda^2\mu^2$$
$$-120a_0^2\mu - 30a_0^3 - 52a_0\lambda^2\mu$$
$$-15a_0^2\lambda^2 - a_0\lambda^4 - 136a_0\mu^2 \tag{3.10}$$

where a_0, λ, μ are arbitrary constants.

Substituting (3.10) into (3.7), we get that

$$u(\xi) = -2(\frac{G'}{G})^2 - 2\lambda(\frac{G'}{G}) + a_0$$
$$\xi = x + (120a_0\mu + 45a_0^2 + 22\lambda^2\mu + 15a_0\lambda^2 + \lambda^4 + 76\mu^2)t \tag{3.11}$$

where a_0, λ, μ are arbitrary constants.

Substituting the general solutions of Eq.(3.6) into (3.11), we can obtain three types of traveling wave solutions of the fifth-order SawadaCKotera equation (3.1) as follows:

When $\lambda^2 - 4\mu > 0$

$$u_1(\xi) = \frac{1}{2}\lambda^2 - \frac{1}{2}(\lambda^2 - 4\mu).$$

$$\left(\frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2$$

$$+a_0,$$

where

$$\xi = x + (120a_0\mu + 45a_0^2 + 22\lambda^2\mu + 15a_0\lambda^2 + \lambda^4 + 76\mu^2)t.$$

a_0, C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$

$$u_2(\xi) = \frac{1}{2}\lambda^2 - \frac{1}{2}(4\mu - \lambda^2).$$

$$\left(\frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right)^2$$

$$+a_0,$$

where

$$\xi = x + (120a_0\mu + 45a_0^2 + 22\lambda^2\mu + 15a_0\lambda^2 + \lambda^4 + 76\mu^2)t.$$

a_0, C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$

$$u_3(\xi) = \frac{1}{2}\lambda^2 - \frac{2C_2^2}{(C_1 + C_2\xi)^2} + a_0,$$

where

$$\xi = x + (120a_0\mu + 45a_0^2 + 22\lambda^2\mu + 15a_0\lambda^2 + \lambda^4 + 76\mu^2)t.$$

a_0, C_1, C_2 are arbitrary constants.

4 Application Of The $(\frac{G'}{G})$-Expansion Method For The General Gardner Equation

In this section, we will consider the general Gardner equation[44]:

$$u_t + (p + qu^n + ru^{2n})u_x + u_{xxx} = 0, \quad n \geq 0,\ r < 0 \tag{4.1}$$

when $n = 1, q \neq 0, r \neq 0$, Eq.(4.1) becomes the KdV-mKdV equation

$$u_t + (p + qu + ru^2)u_x + u_{xxx} = 0,$$

when $n = 1, q \neq 0, r = 0$, Eq.(4.1) becomes the KdV equation

$$u_t + (p + qu)u_x + u_{xxx} = 0,$$

when $n = 1, q = 0, r \neq 0$, Eq.(4.1) becomes the mKdV equation

$$u_t + (p + ru^2)u_x + u_{xxx} = 0$$

In the following, we shall construct exact traveling wave solutions of Eq.(4.1).

In order to obtain the traveling wave solutions of Eq.(4.1), we suppose that

$$u(x,\ t) = u(\xi),\ \xi = k(x - \omega t) \tag{4.2}$$

k, ω are constants that to be determined later.

By using (4.2), (4.1) is converted into an ODE

$$-k\omega u' + k(p + qu^n + ru^{2n})u' + k^3u''' = 0 \tag{4.3}$$

Suppose that the solution of (4.3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i(\frac{G'}{G})^i \tag{4.4}$$

where a_i are constants.

Balancing the order of $u^{2n}u'$ and u''' in Eq.(4.3), we have $2mn + m + 1 = m + 3 \ \Rightarrow\ m = \frac{1}{n}$. So we make a variable $u = v^{\frac{1}{n}}$, then (4.3) is converted into

$$-k(\omega - p - qv - rv^2)n^2v^2v' + k^3(1-n)(1-2n)(v')^3$$
$$+3k^3n(1-n)vv'v'' + k^3n^2v^2v''' = 0 \tag{4.5}$$

Suppose that the solution of (4.5) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$v(\xi) = \sum_{i=0}^{l} b_i(\frac{G'}{G})^i \tag{4.6}$$

where b_i are constants, $G = G(\xi)$ satisfies the second order LODE in the form:

$$G'' + \lambda G' + \mu G = 0 \tag{4.7}$$

where λ and μ are constants.

Balancing the order of v^4v' and $(v')^3$ in Eq.(4.5), we have $4l+l+1=3l+3 \Rightarrow l=1$. So Eq.(4.6) can be rewritten as

$$v(\xi)=b_1(\frac{G'}{G})+b_0,\ b_1\neq 0 \qquad (4.8)$$

$b_1,\ b_0$ are constants to be determined later.

Substituting (4.8) into (4.5) and collecting all the terms with the same power of $(\frac{G'}{G})$ together and equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$(\frac{G'}{G})^0:\ -k^3a_1^3\mu^3+3k^3a_1^3n\mu^3+kn^2a_1\omega a_0^2\mu$$
$$-k^3n^2a_0^2a_1\lambda^2\mu+3k^3n^2a_1^2a_0\mu^2\lambda-kn^2a_1qa_0^3\mu$$
$$-2k^3a_1^3n^2\mu^3-2k^3n^2a_0^2a_1\mu^2-3k^3na_1^2a_0\mu^2\lambda$$
$$-kn^2a_1ra_0^4\mu-kn^2a_1pa_0^2\mu=0$$

$$(\frac{G'}{G})^1:\ -3kn^2a_1^2qa_0^2\mu+6k^3a_1^3n\lambda\mu^2$$
$$+4k^3n^2a_1^2a_0\lambda^2\mu+2k^3n^2a_1^2a_0\mu^2-4kn^2a_1^2ra_0^3\mu$$
$$-kn^2a_1qa_0^3\lambda-3k^3a_1^{n3}\lambda\mu^2-kn^2a_1pa_0^2\lambda$$
$$-8k^3n^2a_0^2a_1\lambda\mu-kn^2a_1ra_0^4\lambda-6k^3na_1^2a_0\lambda^2\mu$$
$$-2kn^2a_1^2pa_0\mu+2kn^2a_1^2\omega a_0\mu+kn^2a_1\omega a_0^2\lambda$$
$$-3k^3a_1^3n^2\lambda\mu^2-k^3n^2a_0^2a_1\lambda^3-6k^3na_1^2a_0\mu^2=0$$

$$(\frac{G'}{G})^2:\ -kn^2a_1pa_0^2+kn^2a_1\omega a_0^2-k^3a_1^3n^2\lambda^2\mu$$
$$-3k^3a_1^3\lambda^2\mu-3kn^2a_1^3qa_0\mu-2kn^2a_1^2pa_0\lambda+kn^2a_1^3\omega\mu$$
$$-kn^2a_1^3p\mu-kn^2a_1qa_0^3-8k^3n^2a_0^2a_1\mu-3k^3na_1^2a_0\lambda^3$$
$$+k^3n^2a_1^2a_0\lambda^3-4kn^2a_1^2ra_0^3\lambda+2kn^2a_1^2\omega a_0\lambda$$
$$+2k^3n^2a_1^2a_0\lambda\mu-3k^3a_1^3\mu^2-kn^2a_1ra_0^4+3k^3a_1^3n\mu^2$$
$$+3k^3a_1^3n\lambda^2\mu-2k^3a_1^3n^2\mu^2-3kn^2a_1^2qa_0^2\lambda$$
$$-6kn^2a_1^3ra_0^2\mu-18k^3na_1^2a_0\lambda\mu-7k^3n^2a_0^2a_1\lambda^2=0$$

$$(\frac{G'}{G})^3:\ -6kn^2a_1^3ra_0^2\lambda+2kn^2a_1^2\omega a_0$$
$$-3kn^2a_1^{n3}qa_0\lambda-2k^{nn3}n^2a_1^2a_0\lambda^2-4k^{n3}n^2a_1^2a_0\mu$$
$$-k^3a_1^3\lambda^3-12k^3na_1^2a_0\mu-4kn^2a_1^4ra_0\mu-kn^2a_1^3p\lambda$$
$$-6k^3a_1^3\lambda\mu-4kn^2a_1^2ra_0^3-2kn^2a_1^2pa_0-kn^2a_1^4q\mu$$
$$-12k^3na_1^2a_0\lambda^2+kn^2a_1^3\omega\lambda-2k^3a_1^3n^2\lambda\mu$$
$$-3kn^2a_1^2qa_0^2-12k^3n^2a_0^2a_1\lambda=0$$

$$(\frac{G'}{G})^4:\ kn^2a_1^3\omega-3k^3a_1^3n\mu-kn^2a_1^4q\lambda-kn^2a_1^5r\mu$$
$$-3k^3a_1^3\mu-3kn^2a_1^3qa_0-3k^3a_1^3\lambda^2-15k^3na_1^2a_0\lambda$$
$$-kn^2a_1^3p-2k^3a_1^3n^2\mu-3k^3a_1^3n\lambda^2-k^3a_1^3n^2\lambda^2$$
$$-9k^3n^2a_1^2a_0\lambda-4kn^2a_1^4ra_0\lambda-6kn^2a_1^3ra_0^2$$
$$-6k^3n^2a_0^2a_1=0$$

$$(\frac{G'}{G})^5:\ -kn^2a_1^5r\lambda-6k^3n^2a_1^2a_0-kn^2a_1^4q$$
$$-3k^3a_1^3n^2\lambda-3k^3a_1^3\lambda-6k^3na_1^2a_0-4kn^2a_1^4ra0$$
$$-6k^3a_1^3n\lambda=0$$

$$(\frac{G'}{G})^6:\ -2k^3a_1^3n^2-k^3a_1^3-3k^3a_1^3n$$
$$-kn^2a_1^5r=0$$

Solving the algebraic equations above, yields:
Case 1: when $\lambda^2-4\mu>0$

$$b_1=\pm\frac{(2n+1)q}{(n+2)r}\sqrt{\frac{1}{\lambda^2-4\mu}}$$

$$b_0=\frac{q}{2(n+2)r}\left[-2n\pm(2n+1)\lambda\sqrt{\frac{1}{\lambda^2-4\mu}}\mp 1\right]$$

$$k=\pm\frac{qn}{n+2}\sqrt{\frac{-(2n+1)}{(\lambda^2-4\mu)(rn+r)}}$$

$$\omega=\frac{prn^3+5prn^2+8npr-2nq^2+4pr-q^2}{r(n+2)^2(n+1)} \qquad (4.9)$$

Substituting (4.9) into (4.8), we have

$$v(\xi)=\pm\frac{(2n+1)q}{(n+2)r}\sqrt{\frac{1}{\lambda^2-4\mu}}(\frac{G'}{G})+$$
$$\frac{q}{2(n+2)r}\left[-2n\pm(2n+1)\lambda\sqrt{\frac{1}{\lambda^2-4\mu}}\mp 1\right],$$

$$\xi=\pm\frac{qn}{n+2}\sqrt{\frac{-(2n+1)}{(\lambda^2-4\mu)(rn+r)}}$$

$$[x-\frac{prn^3+5prn^2+8npr-2nq^2+4pr-q^2}{r(n+2)^2(n+1)}t] \tag{4.10}$$

Substituting the general solutions of (4.7) into (4.10), we have:

$$v_1(\xi)=\mp\frac{(2n+1)q\lambda}{2(n+2)r}\sqrt{\frac{1}{\lambda^2-4\mu}}\pm$$

$$\frac{(2n+1)q}{2(n+2)r}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)$$

$$+\frac{q}{2(n+2)r}\left[-2n\pm(2n+1)\lambda\sqrt{\frac{1}{\lambda^2-4\mu}}\mp 1\right],$$

then

$$u_1(\xi)=(v_1(\xi))^{\frac{1}{n}}.$$

Where

$$\xi=\pm\frac{qn}{n+2}\sqrt{\frac{-(2n+1)}{(\lambda^2-4\mu)(rn+r)}}$$

$$[x-\frac{prn^3+5prn^2+8npr-2nq^2+4pr-q^2}{r(n+2)^2(n+1)}t],$$

C_1 and C_2 are two arbitrary constants.

Case 2: when $\lambda^2-4\mu<0$

$$b_1=\pm\frac{(2n+1)q}{(n+2)r}\sqrt{\frac{1}{4\mu-\lambda^2}}i$$

$$b_0=\frac{q}{2(n+2)r}\left[-2n\pm(2n+1)\lambda\sqrt{\frac{1}{4\mu-\lambda^2}}i\mp 1\right]$$

$$k=\pm\frac{qn}{n+2}\sqrt{\frac{-(2n+1)}{(4\mu-\lambda^2)(rn+r)}}i$$

$$\omega=\frac{prn^3+5prn^2+8npr-2nq^2+4pr-q^2}{r(n+2)^2(n+1)} \tag{4.11}$$

Substituting (4.11) into (4.8), we have

$$v(\xi)=\pm\frac{(2n+1)q}{(n+2)r}\sqrt{\frac{1}{4\mu-\lambda^2}}i(\frac{G'}{G})+\frac{q}{2(n+2)r}$$

$$[-2n\pm(2n+1)\lambda\sqrt{\frac{1}{4\mu-\lambda^2}}i\mp 1],$$

$$\xi=\pm\frac{qn}{n+2}\sqrt{\frac{-(2n+1)}{(4\mu-\lambda^2)(rn+r)}}i$$

$$[x-\frac{prn^3+5prn^2+8npr-2nq^2+4pr-q^2}{r(n+2)^2(n+1)}t] \tag{4.12}$$

Substituting the general solutions of (4.7) into (4.12), we have:

$$v_2(\xi)=\mp\frac{(2n+1)q\lambda}{2(n+2)r}\sqrt{\frac{1}{4\mu-\lambda^2}}i\pm$$

$$\frac{(2n+1)q}{2(n+2)r}i(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})$$

$$+\frac{q}{2(n+2)r}\left[-2n\pm(2n+1)\lambda\sqrt{\frac{1}{4\mu-\lambda^2}}i\mp 1\right],$$

then

$$u_2(\xi)=(v_2(\xi))^{\frac{1}{n}}.$$

Where

$$\xi=\pm\frac{qn}{n+2}\sqrt{\frac{-(2n+1)}{(4\mu-\lambda^2)(rn+r)}}i$$

$$[x-\frac{prn^3+5prn^2+8npr-2nq^2+4pr-q^2}{r(n+2)^2(n+1)}t],$$

C_1 and C_2 are two arbitrary constants.

5 Conclusions

The main points of the method are that assuming the solution of the ODE reduced by using the traveling wave variable as well as integrating can be expressed by an m-th degree polynomial in $(\frac{G'}{G})$, where $G = G(\xi)$ is the general solutions of a second order LODE. The positive integer m is determined by the homogeneous balance between the highest order derivatives and nonlinear terms appearing in the reduced ODE, and the coefficients of the polynomial can be obtained by solving a set of simultaneous algebraic equations resulted from the process of using the method. Compared to the methods used before, one can see that this method is direct, concise and effective. Moreover, the method can also be used to many other nonlinear equations.

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