Numerical solutions for a class of Nonlinear Systems and Application to Stochastic Resonance

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Abstract: We study a nonlinear system under regime switching and subject to an environmental noise. The stochastic differential equations with Markovian switching (SDEwMSs), one of the important classes of hybrid systems, have been used to model many physical systems that are subject to frequent unpredictable structural changes. The research in this area has been both theoretical and applied. Most of SDEwMSs do not have explicit solutions so it is important to have numerical solutions. It is surprising that there are not any numerical methods established for SDEwMSs yet, although the numerical methods for stochastic differential equations (SDEs) have been well studied. We will considered some more general conditions for the coefficient functions and prove a result on the existence using the Schauder's fixed point theorem extended some similarly results on linear systems. The most important results in this paper is to develop a numerical scheme for SDEwMSs and estimate the error between the numerical and exact solutions. Also, we study the application of these system to control the electronic circuits using the benefit of stochastic resonance

Key–Words:stochastic differential equation with Markovian switching, pathwise uniqueness, fixed point technique, numerical solution, Euler-Maruyama method, stopping times, stochastic resonance

1 Introduction

The hybrid systems driven by continuous-time Markov chains have been used to model many practical systems where they may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. The hybrid systems combine a part of the state that takes values continuously and another part of the state that takes discrete values. The term describing the influence of interest rates was modeled by a finite-state Markov chain to provide a quantitative measure of the effect of interest rate uncertainty on optimal policy (see Bensoussan 2000, Bouks 1993, Ghosh 1993, Hu 2000, etc.) . One of the important classes of the hybrid systems is the stochastic differential equations with Markovian switching (SDEwMSs) (see Ji, 1990, Mao, 1999, Mao, 2006)

In this paper, we shall discuss the existence and uniqueness of the solution on a general nonlinear stochastic differential equations with Markovian switching of the form

$$dx(t) = f(x(t), t, r(t))dt + \\ +g(x(t), t, r(t))dw(t) \tag{1}$$

This equation can be regarded as the result of the following N equations:

$$dx(t) = f(x(t), t, i)dt + g(x(t), t, i)dw(t), \\ 1 \le i \le N \tag{2}$$

switching from one to the others according to the movement of the Markov chain. We consider $r(t),\ t_0 \le t \le T$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \dots, N\}$ with generator $\Gamma = (\gamma_{ij})_{ij=\overline{1,N}}$ given by

$$P[r(t+\Delta) = j | r(t) = i] = \\ = \begin{cases} \gamma_{ij} + o(\Delta), & if\ i \neq j \\ 1 + \gamma_{ij} + o(\Delta), & if\ i = j \end{cases}$$

where $\Delta > 0$ Here γ_{ij} is transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. We assume that the Markov chain $r(t)$ is independent of the Brownian motion $w(t)$. It is well known that almost every sample path of $r(t)$ is a right-continuous step function and $r(t)$ is ergodic. It is known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of R_+ .

Most of SDEwMSsdo not have explicit solutions and hence require numerical solutions. However, there are no numerical methods available for SDEwMSs yet, although the numerical methods for stochastic differential equations (SDEs) have been well studied (see for example [27]). There are several reasons. The main one is that a mathematical difficulty has arisen from the Markovian switching which requires a new technique to be developed. The proof of Theorem 3.1 below demonstrates the new technique to overcome this difficulty. It is there we will see that the proof for SDEwMSs is certainly not a straightforward generalization of that for SDEs. Another reason is that many SDEwMSs do not satisfy the global Lipschitz condition, but even in the case of SDEs (not SDEwMSs, of course), there are only a few papers on the numerical methods without the global Lipschitz condition (see [**?**], Yuan and Mao 2004).

2 Existence and uniqueness

Let $\{w(t)\},\ 0 \le t_0 \le t \le T\ (T \in R_+)$, denote a Wiener process defined on the probability space (Ω, F, P) . Suppose $\{F_t : t_0 \le t \le T\}$ is a nonanticipating family of sub- σ -algebras of F with respect to the Wiener process w_t .

First, in a some similar way as (Athanassov, 1990, Constantin A., 1996, Negrea 2003), we give the following lemma:

Lemma 2.1. *Let $u(t)$ a continuous, positive function on $a < t \le b$ ($a < b$, two real numbers), having nonnegative derivative $u'(t) \in L(a,b)$. Let $v(t)$ a continuous, nonnegative functions for $a \le t \le b$ such that $v(t) = o(u(t))$ as $t \to a^+$ and $v(t) \le \int_{a+}^{b} \frac{u'(s)}{u(s)} v(s)ds,\ \ \forall\, a \le t \le b$.*
Then $v(t) \equiv 0$ on $a \le t \le b$.

The basic idea in proving the existence of the solution in a SDEwMS is to analyze a standard SDE on each interval $[t_k, t_{k+1}]$, k ≥ 0 , where $(t_k)_{k\ge1}$ are the associated stopping times of the Markov process $\{r(t)\}_{t\in[t_0,T]}$. Some results on the existence and uniqueness of solutions and on the convergence of successive approximations for stochastic differential equations, assuming the existence of a function u controlling the growth and the continuity of f and g (as in Constantin A., 1996, Negrea 2003, Constantin I., 2004, etc.) generalizing to the setting of stochastic differential equations driven by Brownian motion a result of Athanassov (see Athanassov 1990).

Now, we recall the SDEwMS (1) and we consider the equivalent integral equation

$$x(t) = x(t_0) + \int_{t_0}^{T} f(x(s), s, r(s))ds + \\ + \int_{t_0}^{T} g(x(s), s, r(s))dW_s \tag{3}$$

with $f, g : \Omega \times R \times [t_0, T] \times R$ and the following hypotheses:

(H1). f, g are $B \otimes P \otimes B$ measurable functions;

(H2). $f(0, \cdot, i) \in M^2([t_0, T], R)$ and $g(0, \cdot, i) \in M^2([t_0, T], R)$;

(H3). there exists $u(t)$ a continuous, positive and derivable function on $t_0 < t \le T$ with $u(t_0) = 0$, having nonnegative derivate $u'(t) \in L(t_0, T)$ such that

$$|f(x,t,i) - f(y,t,i)|^2 \wedge |g(x,t,i) - g(y,t,i)|^2 \le \\ \le \frac{u'(t)}{3u(t)}|x-y|^2, \tag{4}$$

for all $x, y \in R,\ t_0 < t \le T,\ i \in S$;

(H4). with the same function $u(t)$ as above we have

$$|f(x,t,i)|^2 \wedge |g(x,t,i)|^2 \le u'(t)(1+|x|^2); \tag{5}$$

(H5). $x(t_0) = x_0$ is a given F_{t_0} -measurable random variable such that $E[|x_0|^2] < \infty$.

Remark. A result on the existence and uniqueness of the solution for the equation (1) in the above hypotheses was given in [24] (see Negrea et.al. 2009), in a classical way, using an approximation sequence for the existence of the solution. In the following theorem we give an elegant way (and also, a shorter manner) to prove the existence of the solution.

Theorem 2.2. *Let be f and g satisfying the hypotheses (H1)-(H5) and $x_0 \in L^2(\Omega, F_{t_0}, P, R)$, then there exists a unique solution $x \in L^2((t_0, T), R)$ which satisfies the equation (1) for $t_0 \le t \le T$.*

Proof. *Uniqueness.* Let be $x(t)$ nd $y(t)$ two solutions in $L^2([t_0, T], R)$ of the equations (5). We have

$$E|x(t) - y(t)|^2 \le$$

$$\le 3\{E[\int_{t_0}^{t} |f(x(s), s, r(s)) - f(y(s), s, r(s))|^2 ds +$$

$$+\int_{t_0}^{t} |g(x(s),s,r(s)) - g(y(s),s,r(s))|^2 ds]\} \leq$$

$$\leq \int_{t_0}^{t} \frac{u'(s)}{u(s)} E[|x(s)-y(s)|^2]ds$$

and from the Lemma 2.1. with $v(t) = E[|x(t) - y(t)|^2]$ yields that $x(t) \equiv y(t)$ on $[t_0, T]$.

Existence. Recall that almost every simple path of $r(\cdot)$ is a right continuous step function on $[t_0, T]$. Therefore, is a sequence $\{t_k\}_{k\geq 1}$ of stopping times such that for almost every $\omega \in \Omega$ there is a finite $K = K(\omega)$ for $t_0 < t_1 < \ldots < t_k = T$ and $t_k = T$ if $k > K$ and $r(t) = r(t_k) \overset{not}{=} r_k$ on $t_k \leq t < t_{k+1}$ for all $k \geq 1$.

We first consider the equation (1) on $t \in (t_0, t_1)$ which becomes

$$dx(t) = f(x(t),t,r_0)dt + g(x(t),t,r_0)dW_t \qquad (6)$$

with the initial data x_0 and r_0 .

We consider the operator

$$T : \mathcal{C}_a([t_0,t_1]) \to \mathcal{C}_a([t_0,t_1])$$

by

$$Tx(t) = x_0(t) + \int_{t_0}^{t_1} f(s,x(s),r_0)d(s) +$$

$$+\int_{t_0}^{t_1} g(s,x(s),r_0)dw(s)$$

As in [4] we define the set

$$B = \{x \in \mathcal{C}_a([t_0,t_1]) :\ \|x(t)\|_n^2 \leq m(t),\ t_0 \leq t \leq t_1\}$$

where $m(t)$ is the maximal solution of the differential equation

$$m'(t) = 6Ku(t)m(t),\ \ t \in [t_0,t_1]$$

with the initial condition

$$m(0) = Q = 3\sup_{t\in[t_0,t_1]} \|x_0(t)\|^2 + 3KMt_1$$

with

$$M = \max\{\|f(t,0,r_0)\|^2,\ \|g(t,0,r_0)\|^2\}.$$

We deduce that

$$\|Tx(t)\| \leq \|x_0(t)\|^2 + x_0(t) +$$

$$+\int_{t_0}^{t_1} \|f(s,x(s),r_0)\|^2 d(s) +$$

$$+\int_{t_0}^{t_1} \|g(s,x(s),r_0)\|^2 dw(s) \leq$$

$$\leq \sup_{t\in[t_0,t_1]} \|x_0(t)\|$$

$$+[\int_{t_0}^{t_1} (\|f(s,x(s),r_0) - f(s,0,r_0)\| +$$

$$+\|f(s,0,r_0)\|)^2 ds]^{\frac{1}{2}}$$

$$+[\int_{t_0}^{t_1} (\|g(s,x(s),r_0) - g(s,0,r_0)\| +$$

$$+\|g(s,0,r_0)\|)^2 ds]^{\frac{1}{2}}$$

$$\leq \sup_{t\in[t_0,t_1]} \|x_0(t)\| +$$

$$\sqrt{2}K\{\int_{t_0}^{t_1} u'(s)(\|x(s)\|^2)ds + Mt_1\}^{\frac{1}{2}}$$

$$+\sqrt{2}K\{\int_{t_0}^{t_1} u'(s)(\|x(s)\|_n^2)ds + Mt_1\}^{\frac{1}{2}}$$

$$= \sup_{t\in[t_0,t_1]} \|x_0(t)\|_n +$$

$$+\sqrt{2}K\{\int_{t_0}^{t_1} u'(s)(\|x(s)\|^2)ds + Mt_1\}^{\frac{1}{2}}$$

$$+\sqrt{2}K\{\int_{t_0}^{t_1} u'(s)(\|x(s)\|^2)ds + Mt_1\}^{\frac{1}{2}}$$

and for $x \in B$ we obtain that

$$\|Tx(t)\|^2 \leq \sup_{t\in[t_0,t_1]} \|x_0(t)\| + \qquad (7)$$

$$+\sqrt{2}K\{\int_{t_0}^{t_1} u'(s)(m(s))ds + Mt_1\}^{\frac{1}{2}}$$

$$+\sqrt{2}K\{\int_{t_0}^{t_1} u'(s)(m(s))ds + Mt_1\}^{\frac{1}{2}} \leq$$

$$\leq 3\sup_{t\in[t_0,t_2]} \|x_0(t)\|^2 +$$

$$+6K^2\int_{t_0}^{t_1} u'(s)(m(s))ds +$$

$$+6K^2\int_{t_0}^{t_1} u'(s)(m(s))ds + 6K^2Mt_1 =$$

$$= Q + 6K^2\int_{t_0}^{t_1} u'(s)(m(s))ds +$$

$$+6K^2\int_{t_0}^{t_1} u'(s)(m(s))ds = m(t),\ \ t_0 \leq t \leq t_1$$

We proved so that $T(B) \subseteq B$, and it is easy to see that the set B is a closed, bounded and convex subset of the Banach space $(\mathcal{C}_a([t_0,t_1]), |||.|||)$, with

$$|||(x,y)||| = \sqrt{\|x\|^2 + \|y\|^2}$$

where $\|x\|^2 = E[\sup_{0\leq t\leq 1} |x(t)|^2]$.

On the other hand we have that

$$\|Tx(t) - Tx(s)\|^2 \leq \tag{8}$$

$$\leq 6K^2 \int_s^t u(s)(m(s))ds + 6K^2 \int_s^t u(s)(m(s))ds +$$

$$+3 \sup_{t\in[t_0,t_1]} \|x_0(t) - x_0(s)\|^2 + 6K^2 M(t-s),$$

$t_0 \leq s \leq t \leq t_1$ and thus the set $T(B)$ is equicontinuous.

In a similar way we prove that for $x, y \in B$ we have

$$\|Tx(t) - Ty(t)\| \leq \tag{9}$$

$$\{\int_{t_0}^{t_1} \|f(s, x(s), r_0) - f(s, y(s), r_0)\|^2 ds\}^{\frac{1}{2}} +$$

$$+\{\int_{t_0}^{t_1} \|g(s, x(s), r_0) - g(s, y(s), r_0)\|^2 ds\}^{\frac{1}{2}},$$

$t_0 \leq t \leq t_1$.

From (H_3) and the continuity of $f(s, x, r_0)$ and $g(s, x, r_0)$ in x on L_2 we deduce by Lebesgue convergence theorem that T is continuous.

An application of Schauder's fixed point theorem enables us to deduce that T has a fixed point in B, thus equation (1) has a solution on $[t_0, t_1]$.

We repeat this procedure and we can see that the equation has a solution $x(t)$ on $[t_0, T]$. □

Remark. It is easy to see that our results extend more classical results (with lipschitz conditions for example, (see Mao 1999, Mao 2006, etc.). The problems of discontinuities in the stopping times t_i, $(i = 1, 2, \ldots, N)$ appears in more applications when are some changes in the behavior of physical phenomena modeling from the adapted process $x(t)$.

Remark. Some results on the stability properties of the solution of the equation (1), was given by the author (see [25]). Below, we present just two of them.

First, if we consider families of SDEwMS by the form

$$x_\lambda(t) = x(t_0) + \int_{t_0}^T f_\lambda(x_\lambda(s), s, r_\lambda(s))ds +$$

$$\int_{t_0}^T g_\lambda(x_\lambda(s), s, r_\lambda(s))dW_s = X_\lambda,$$

whit $\lambda \in \Lambda$- an open and bounded set $\subset R$.

Theorem 2.3. ([]) *If, for any $\lambda \in \Lambda$, the coefficient functions f_λ and g_λ satisfy the hypothesis (H1)-(H5), then the family equations from above has a unique solution $(x_\lambda) \in M^2([t_0, T], R)$.*

Moreover, if

$$|x_{\lambda,m}(t_0) - x_\lambda(t_0)| \to 0, \quad m \to \infty,$$

then, we have that

$$\lim_{m\to\infty} |x_{\lambda,m} - x_\lambda| = 0.$$

on $[t_0, T]$ and any $\lambda \in \Lambda$.

Another interesting result on the stability of the solution for the equation (1) is the following:

It is known (see A.Constantin A. 1996) that if

$$\varphi_\lambda(x(t), t, r(t)) \xrightarrow{P} \varphi_{\lambda_0}(x(t), t, r(t)), \quad \lambda \to \lambda_0$$

then

$$\lim_{\lambda\to\lambda_0} \int_{t_0}^T |\varphi(x(s), s, r(s)) - \varphi_{\lambda_0}(x(s), s, r(s))|^2 ds = 0$$

Theorem 2.4. *In the hypotheses (H1)-(H5), we have that if*

$$\lim_{\lambda\to\lambda_0} |x_\lambda(t_0) - x_{\lambda_0}(t_0)| = 0,$$

then

$$\lim_{\lambda\to\lambda_0} |x_\lambda - x_{\lambda_0}|^2 = 0,$$

on $[t_0, T]$, where φ is any functions f or g.

Remark. These results assure the stability of the solution for the equation (1) with respect to the initial conditions and with respect to the coefficient functions. Is known that a stability result is very important for to prove the consistency of a numerical method.

3 Numerical method

It known that, except some very few cases, we can't give an analytical solution for an SDEwMS. For this reason we will give some results which assure that a Cauchy-Maruyama (Euler-Maruyama) procedure can be imply in our case.

To analyze the EulerMaruyama method as well as to simulate the approximate solution, we will use the Lemma 2.1 from [27] (see Yuan and Mao 2004).

In a similar way as in the above lemma, for a given Δ >) we consider a discrete Markov chain $r_k^\Delta = r(\Delta k)$, $k - 0, 1, 2, \ldots$ with the one-step transition probability matrix

$$P(\Delta) = \{P_{ij}(\Delta)\}_{i,j\in I} = e^{\Delta\Gamma}$$

(for details see Lemma 2.1 from [27])

Since the γ_{ij} are independent of x, the path of r can be generated independently of x and, in fact, before computing x. In analogous way as in [27] we will describe just few steps for the simulation of the Markov chain (see Yuan and Mao 2004 for more details) :

Let $r_0^\Delta = i_0$ and generate a random number ξ_1 which is uniformly distributed in $[0,1]$. Define

$$r_1^\Delta = \begin{cases} i_1 \text{ if } \xi_1 \in S \setminus \{N\} \text{ such that} \\ \qquad \sum_{j=1}^{i_1-1} P_{i_0,j}(\Delta) \le \xi_1 < \sum_{j=1}^{i_1} P_{i_0,j}(\Delta), \\ N \text{ if } \sum_{j=1}^{N-1} P_{i_0,j}(\Delta) \le \xi_1 \end{cases}$$

where $\sum_{j=1}^{0} P_{i_0,j}(\Delta) = 0$ as usual. Generate independently a new random number ξ_2 which is again uniformly distributed in $[0,1]$ and then define

$$r_2^\Delta = \begin{cases} i_2 \text{ if } \xi_2 \in S \setminus \{N\} \text{ such that} \\ \qquad \sum_{j=1}^{i_2-1} P_{r_1^\Delta,j}(\Delta) \le \xi_1 < \sum_{j=1}^{i_2} P_{r_1^\Delta,j}(\Delta), \\ N \text{ if } \sum_{j=1}^{N-1} P_{r_1^\Delta,j}(\Delta) \le \xi_2 \end{cases}$$

Repeating this procedure a trajectory of (r_k^Δ), $k = 0,1,2,\dots$ can be generated. This procedure can be carried out independently to obtain more trajectories.

Let $\Pi = (t_1,\dots,t_{m+1},\tau_1,\dots,\tau_m)$ be a Cauchy partition in particular of $[a,b)$ and for every function F defined on $[a,b)]$, we define

$$\Delta_k F = F(t_{k+1}) - F(t_k)$$

We denote the Cauchy-Maruyama (Euler-Maruyama) approximation corresponding to the partition Π by x_Π, and we write the successive steps of its definition in the form

$$x_\Pi(a) = x_0$$

$$x_\Pi(t_{k+1}) = x_\Pi(t_k) + f(x_\Pi(t_k), t_j, r(t_k))\Delta_k t +$$
$$+ g(x_\Pi(t_k), t_k, r(t_k))\Delta_k W + \varepsilon(t_k). \tag{10}$$

Theorem 3.1. *Let the hypotheses [H1] - [H5] be satisfied. Assume further that whenever φ is any one of the functions in the set f, g and X is a process $\mathcal{F}_t$-adapted and L_2-continuous on $[a,b]$, the composite function $t \mapsto \varphi(x_t,t,r_t)$ has an integral (Riemman and respectively Ito) from a to b. Assume also that x_Π is $\mathcal{F}_t$-adapted, and that the adjustment term $\varepsilon(t_j)$ defined by (6) is such that there exists a function Ψ on $(0,\infty)$ that has limit 0 at the origin and satisfies for $q = 1,\dots,k$*

$$|\sum_{k=1}^{q} \varepsilon(t_k)| \le$$
$$\le \Psi(\textit{mesh}\,\Pi)[1 + \sup\{|x_\Pi(s)|, \ \ a \le s \le t_q\}]$$

Then as mesh Π tends to 0, x_Π converges uniformly in L_2 to the solution x of (1).

Proof. If x is a solution of the equation (1), from the proof of Theorem 2.2, we have that there exists an upper bound, which we denote by M, for $|x(t)|$. Let ε by positive. There exists a δ_1 such that if $|s-t| < \delta_1$, $|x(s) - x(t)| < \varepsilon$. Define:

$$X(t_q) = x(a) + \sum_{k=1}^{q-1} f(x_\Pi(t_k), t_j, r(t_k))\Delta_k t +$$
$$+ \sum_{k=1}^{q-1} g(x_\Pi(t_k), t_k, r(t_k))\Delta_k W \tag{11}$$

Because $\varphi(x(t),t,r(t))$ has an integral (Riemman and respectively Ito), there exists a positive δ_2 (from Negrea et al 2009) Theorem 1) such that if mesh $(\Pi) < \delta_2$, $|X(t_q) - x(t_q)| < \varepsilon/n$. Also there is a positive δ_3 such that if $0 < u < \delta_3$ then $\psi(s) < \min\{\varepsilon/M, 1/2\}$ (because $\psi(s) \to 0$, $u \to 0$).

Now let Π be a Cauchy partition with mesh Π less than $\min\{\delta_1,\delta_2,\delta_3\}$. Let $t \in [a,b]$ and t_q be the lengthest of the numbers $t_1,\dots,t_{k+1}$ that does not exceed t. Then:

$$x_\Pi(t) - x(t) = \{x(a) + \sum_{k=1}^{q-1} f(x_\Pi(t_k), t_k, r_k)\Delta_k t +$$
$$+ \sum_{k=1}^{q-1} g(x_\Pi(t_k), t_k, r_k)\Delta_k W + \varepsilon_\Pi(t_q)\} +$$
$$+ \{X(t_q) - x(a) + x(a) +$$
$$+ \sum_{k=1}^{q-1} f(x(t_k), t_k, r_k)\Delta_k t -$$
$$- \sum_{k=1}^{q-1} g(x(t_k), t_k, r_k)\Delta_k W - \{x(t_q) - x(t_q)\} =$$

$$= [X(t_q) - x(t_q)] + [x(t_q) - x(t)] + \varepsilon_\Pi(t_q) +$$
$$+ \sum_{k=1}^{q-1} [f(x_\pi(t_k), t_k, r_k) - f(x(t_k), t_k, r_k)]\Delta_k t +$$
$$+ \sum_{k=1}^{q-1} [g(x_\Pi(t_k), t_k, r_k) - g(x(t_k), t_k, r_k)]\Delta_k W$$

From (3), (5) and the Cauchy-Bunyakowsky-Schwarz's inequality we have that

$$\Big|\sum_{k=1}^{q-1}\{[f(x_\pi(t_k),t_k,r_k)-f(x(t_k),t_k,r_k)]\Delta_k t+ [g(x_\Pi(t_k),t_k,r_k)-g(x(t_k),t_k,r_k)]\Delta_k W\Big| \leq$$

$$\leq 2C\Big\{\int_a^{t_q}\frac{u'(s)}{3u(s)}\|x_\Pi(s)-x(s)\|^2ds\Big\}^{1/2} \quad (12)$$

We define $N(t)=\sup\{|x_\Pi(s)-x(s)|: a\leq s\leq t\}$. Since $M-1$ is an upper bound for $x(t)$, we have

$$1+\sup\{\|x_\Pi(t)\|: a\leq s\leq t_q\}\leq M_N(t)$$

so by the hypotheses and the choice of δ ($\delta = \min\{\delta_1,\delta_2,\delta_3\}$) we obtain:

$$\sum_{k=1}^{q-1}\varepsilon_\Pi(t_k)\leq\psi(\text{mesh }\Pi)(M+N(t))\leq\varepsilon/n+\frac{N(t)}{2n}$$

With the help of these estimates, (12) yields

$$|x_\Pi(t)-x(t)|\leq n\Big[\frac{3\varepsilon}{n}+\frac{N(t)}{2}+ +2C\{\int_a^{t_q}\frac{u'(s)}{3u(s)}|x_\Pi(s)-x(s)|^2ds\}^{1/2}\Big]$$

Here we may replace the t in the right member by any larger number in $[a,b]$ or equivalently replace t in the left member by any smaller number in $[a,b]$. That is the estimate holds if we replace the left member by $N(t)$. This implies:

$$2N(t)\leq 6\varepsilon+N(t)+$$

$$+4C\{\int_a^{t_q}\frac{u'(s)}{3u(s)}|x_\Pi(s)-x(s)|^2ds\}^{1/2}\Leftrightarrow$$

$$\Leftrightarrow N(t)\leq 6\varepsilon+$$

$$+4C\{\int_a^{t_q}\frac{u'(s)}{3u(s)}|x_\Pi(s)-x(s)|^2ds\}^{1/2}\Leftrightarrow$$

$$\Leftrightarrow N(t)^2\leq\Big[6\varepsilon+$$

$$+4C\{\int_a^{t_q}\frac{u'(s)}{3u(s)}|x_\Pi(s)-x(s)|^2ds\}^{1/2}\Big]^2\leq$$

$$2\Big[(6\varepsilon)^2+[16C^2\int_a^{t_q}\frac{u'(s)}{3u(s)}|x_\Pi(s)-x(s)|^2ds\leq$$

$$\leq 72\varepsilon^2+32C^2\int_a^{t_q}\frac{u'(s)}{3u(s)}|x_\Pi(s)-x(s)|^2ds$$

We apply the Lemma 2.1 and we obtain that $N(t)^2\to 0$. Therefore, x_Π converges to x uniformly in L^2, as mesh $\Pi\to 0$.

□

Now, in the same way as I .Constantin (2001), we can give an analogous theorem on the Cauchy-Maruyama approximation.

Theorem 3.2. *Let us suppose that the hypotheses [H-1]-[H-5] are satisfied. Let $x(t)$ be a solution of the equation (1) with a.s. continuous functions. Then as mesh$\Pi\to 0$ we have:*
(i) $x_\Pi(t)$ converges to $x(t)$ uniformly in L_2 on $[a,b]$ and $x_\Pi(t,\omega_0)$ converges uniformly to $x(t,\omega_0)$ for each $\omega_0\in\Omega$;
(ii) the random variable

$$\sup_{a\leq t\leq b}|x_\Pi(t,\omega)-x(t,\omega)|$$

converges to zero in probability.

Proof. It easy to see that the process $x_\Pi(t) = X(\tau(t,\Pi))\not= X_\Pi,\ \ t\in[a,b]$, where $\tau(t,\Pi)$ is the greatest number in the set $t_1,\ldots,t_m$ that is $\leq t$, satisfies the hypotheses of the previously lemma with $\varepsilon_\Pi = 0$. Then, by that lemma, $x_\Pi(t)$ converges to $x(t)$ uniformly in L^2 as mesh$\Pi\to 0$.

If $t_k\leq t\leq t_{k+1}$ we have

$$X_\Pi(t)-x_\Pi(t)=f(x_\Pi(t_k),t_k,r_k)[t-t_k]+ +g(x_\Pi(t_k),t_k,r_k)[W(t)-W(t_k)] \quad (13)$$

On the other hand, from the hypotheses [H-4], we have

$$|\varphi(x_\Pi(t_k),t_k,r_k)|^2\leq u'(t_k)(1+|x_\Pi(t_k)|^2)\leq$$

$$\leq\frac{u(t)-u(t_k)}{t-t_k}Q_m\leq M_u\cdot M$$

where $M_u = \sup_{t\in[a,b]}(u(t))$ and M is the upper bound of x. Then, since $t-t_k\leq$ meshΠ, these tend to zero uniformly on $[a,b]$ as mesh$\Pi\to 0$.

The second assertion of the theorem it can prove in a same way as that of the theorem of the paper of I. Constantin (2001). □

Examples. We give the following examples:

$$dx(t)=sin(t)\sqrt{x(t)}r(s)dt+ +sin(s)\sqrt{x(t)}r(t)dW(t) \quad (14)$$

with $x_0=10$, the states of Markov chain $S=(-1\ 1)$, the initial distribution

$$\mu=\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}$$

and the transition probability matrix

$$P = \begin{pmatrix} 0.99 & 0.01 \\ 0.005 & 0.995 \end{pmatrix}.$$

For $n = 100$ and $\Delta = 10^{-4}$ we have the following graph:

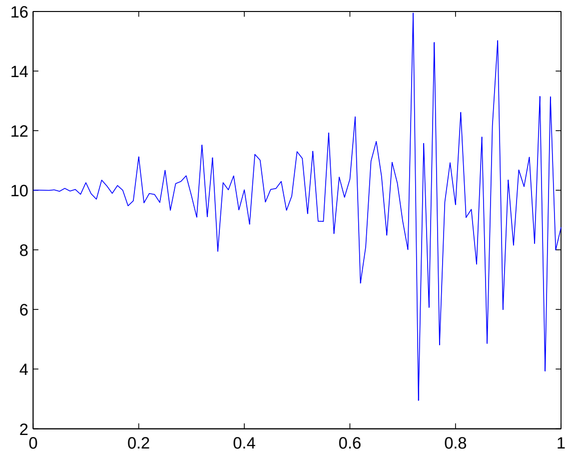

Figure 1: The simple path of the solution for the equation (14).

$$dx(t) = sin(t)\sqrt[3]{x(t)^2}r(t)dt + \\ +sin(t)\sqrt[4]{x(t)^3}r(t)dW(t) \tag{15}$$

with $x_0 = 10$, the states of Markov chain $S = (-1\ 1)$, the initial distribution

$$\mu = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

and the transition probability matrix

$$P = \begin{pmatrix} 0.9 & 0.1 \\ 0.15 & 0.55 \end{pmatrix}.$$

Here, we can give two simple example for the control function control u as $u(t) = sqrt(t)$ or $u(t) = log(t+1)$.

For $n = 100$ and $\Delta = 10^{-4}$ we have the following graph:

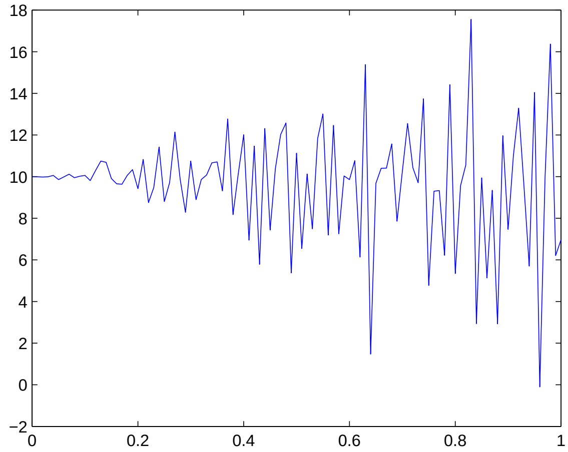

Figure 2: The simple path of the solution for the equation (15).

4 Comments and Applications

The motivation for to study of the SDEwMSs by an interdisciplinary team is based on the more engineering application of these equations which are very good models for to control of some actually devices. It is enough to mention just two current and interesting example, as following:

1. The ABS (Anti-lock Braking System) and the Airbags for a car- work different for different speed levels. For a good and safety control of these special devices we must to use a modeling with a hybrid system.

2. The signal of a mobile phone depend by its link with an external amplifier. There exist a switch moment when we pass from one amplifier area to other.

Remark. About the last example we have an interesting situation: during to a telephone call, we pass form an amplifier to other (i.e. we have the Markovian switching process), and we have a very short delay and this because is changed the link of our mobile phone from the last amplifier to the new amplifier and hence we have a discontinuity point in the behavior of the phone signal. Therefore, is necessary to consider some general coefficient functions and good stability properties of the solutions.

Noise in dynamical system is usually considered a nuisance. However, in certain nonlinear systems, including electronic circuits and biological sensory systems, the presence of noise can enhance the detection of weak signals. The phenomenon is termed stochastic resonance and is of great interest for electronic instrumentation (see Gammaitoni, 1998, McNamara, 1989, Negrea, 2007a, Negrea, 2007b).

The essential ingredient for the stochastic resonance is a nonlinear dynamical system, which typically has a period signal and noise at the input and output that is a function of the input as well as the internal dynamics of the system. The nonlinear component of the dynamical system is sometimes provided by a threshold which must be crossed for the output to be changed or detected. A nonlinear system is essential for stochastic resonance to exists, since in a system that is well characterized by linear response theory, the signal-to-noise ratio at the output must be proportional to the signal-to-noise ratio at the input.

Engineers have normally sought to minimize the effect of noise in electronic circuits and communication systems. Today, however, it is acknowledged that noise or random motion is beneficial in breaking up the quantization pattern in a video signal, in the dithering of analog to digital converters, in the area of Brownian ratchets, etc.

A model of one-dimensional nonlinear system that exhibits stochastic resonance is the damped harmonic oscillators with the Langevin equation of motion (see Gammaitoni, 1998):

$$m\ddot{x}(t) + \gamma\dot{x}(t) = -\frac{dU(x)}{dx} + \sqrt{D}\xi(t)$$

This equation describes the motion of a particle of mass m moving in the presence of friction γ . The restoring force is expressed as the gradient of some bistable or multi-stable potential function $U(x)$. In addition, there is an additive stochastic force $\xi(t)$ with intensity D , and, in generally, it supposed as been a white Gaussian noise.

In the case of symmetrical bistable system, the potential function is a simple symmetric function and, adding a period signal and considering case of time dependent system (see Berglund 2002, Gammaitoni, 1998)

$$U(x,t) = U(x) - Ax\sin(\omega_s t) =$$

$$= -a\frac{x^2}{2} + b\frac{x^4}{4} - Ax\sin(\omega_s t)$$

where A and are the amplitude and the frequency of the periodic signal, respectively.

In the last years, engineers used the asymmetrical bistable system (see Herman 2002, Imkeller 2001), when the potential function has the expression:

$$U(x,t) = \begin{cases} U(x) - Ax\sin(\omega_s t), \text{ if } t \in [0, \frac{1}{2}) \\ U(-x) - Ax\sin(\omega_s t) \text{ if } t \in [\frac{1}{2}, 1] \end{cases},$$

$$U(x) = -a\frac{x^2}{2} + b\frac{x^4}{4}$$

or more general model with k switching times (see Imkeller, 2001)

$$U(x,t) = \sum_{k\geq 0} U(x)1_{[k,k+0.5)} + U(-x)1_{[k+0.5,k+1)}$$

About these models there is a simple observation: we can not say exactly if the external perturbation is present just at the discrete moments. A continuous model appears as more adequate. This approach is possible just using the theory of stochastic differential equations.

A problem, which frequency appear in practice, is the value of initial state x_0 . This value is "proposed" but in some non-standard external conditions, this make an discontinuity of the simple path of the process $\{x(t)\}$ and this phenomena is repeating at any stopping time t_i, $(i = 1, 2, \ldots, N)$ and we will have new discontinuities at these time moments. In applications, we have a right-continuity for the Markov process but just a left-continuity for the process $\{x(t)\}$ (we have a very short delay at the stopping times). Therefore, is necessary to consider some general coefficient functions and good stability properties of the solutions. On the other hand, the stochastic resonance make possible a control of the electronic circuits in some external stochastic perturbations by controlling the adapted process $\{x(t)\}$.

The simulation for this case is given in the following example:

$$dx(t) = (-sin(0.5t) - x(t) + x(t)^3)r(t)dt+$$
$$+0.5r(t)dW(t) \tag{16}$$

with $x_0 = 10$, the states of Markov chain

$$S = (0\ 1\ 2\ 3\ 4\ 5),$$

the initial distribution

$$\mu = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the transition probability matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0.8 & 0 & 0 & 0 \\ 0.01 & 0.01 & 0.1 & 0.88 & 0 & 0 \\ 0.0001 & 0 & 0.01 & 0.1 & 0.8899 & 0 \\ 0.0001 & 0 & 0 & 0.01 & 0.3 & 0.68999 \end{pmatrix}.$$

For $n = 100$ and $\Delta = 10^{-4}$ we have the following graph:

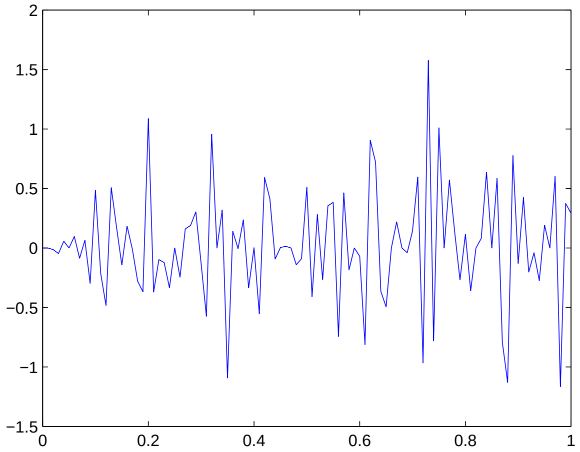

Figure 3: The simple path of the solution for the equation (16).

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