Generalized Concentration/Inequality Indices of Economic Systems Evolving in Time

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Abstract: In this paper time evolving indices are defined and studied through the introduction of a population structure chancing during the time. These indices are useful to describe the concentration (or inequality) of the wealth in a stochastic model of the wealth evolution. Here some static indices usually used in literature are generalized. In particular the paper shows how it is possible to generalize the Herfindahl-Hirschman and Gini indices and the Theil's entropy. These indices become stochastic processes of which the moments and the asymptotic behavior is provided. A computation of such indices and their asymptotic behavior in different economical scenarios concludes the paper.

Key–Words: Population dynamic, Herfindahl-Hirschman index, Gini index, Theil's entropy, Rewards.

1 Introduction

Among the relevant economic systems problems there is also the measure of changes due to the population dynamics. For this reason, recently, the themes of inequality and wealth concentration in a subclass of a economic system have been extensively studied, see i.e. Quah [18], [19], [20]. Usually such contributions deal with the use of static inequality/concentration indices. In this way, changes of indices values are associated to variations of economic conditions. Anyway it is possible to explain these changes considering a population whose composition changes in time without the occurrence of any relevant economic event. For this reason in this paper a population structure evolving in time is considered and, consequently, dynamic inequality/concentration indices are defined. The relevance of using a dynamical models has been highlighted in [1] and [25]. In order to reach this goal, a discrete time semi-Markov process is used to describe the population dynamic. The choice of the semi-Markov approach is due to the ability to model general real life phenomena, see i.e. [7] and the bibliography therein. In this paper the production of each economic agent is modeled as a reward process. The paper is organized as follows. Section 2 provides a brief introduction to semi-Markov chains. Section 3 contains the definition of the main variables describing the problems and the mathematical assumptions. In Section 4 the dynamic indices are defined and it is shown how to compute them and their asymptotic behavior. Section 5 presents a numerical experience showing the applicability of the model and illustrating some results.

2 Semi-Markov chains

In this section we introduce basic definitions and notation on semi-Markov chains useful to understand next sections in which we propose the stochastic model in order to define and compute the inequality/concentration indices in a dynamic way. The symbology is consistent with [11] and [12].
Let $E = \{1, \ldots, m\}$ be the state space and let (Ω, F, P) be a probability space. Let us also define the following random variables:

$$J_n : \Omega \to E, \quad T_n : \Omega \to \mathbb{N},$$

where J_n represents the state at the *n-th* transition and T_n represents the time of the *n-th* transition.

The $(J_n, T_n)_{n \in \mathbb{N}}$ process is called a *homogeneous Markov renewal process* with kernel Q if

$$P\left[J_{n+1} = j, T_{n+1} - T_n \le t | \,(J_k, T_k), k = 0, ..., n\right]$$

$$= Q_{J_n j}(t) \tag{1}$$

The following probabilities are of interest:

$$p_{ij} = \lim_{t \to \infty} Q_{ij}(t) \quad i, j \in E.$$

which represents the transition probability matrix of the embedded Markov chain in the process.

We set

$$q_{ij}(t) = P\left[J_{n+1} = j, T_{n+1} - T_n = t | \, J_n = i\right]$$

$$= \begin{cases} Q_{ij}(t) = 0 & \text{if } t = 0 \\ Q_{ij}(t) - Q_{ij}(t-1) & \text{if } t > 0. \end{cases}$$

The survival probability in state i is

$$H_i(t) = P\left[T_{n+1} - T_n \leq t \,|\, J_n = i\right] = \sum_{j=1}^{m} Q_{ij}(t)$$

and the waiting time distribution function in state i given that next transition will be to state j is

$$G_{ij}(t) = P\left[T_{n+1} - T_n \leq t \,|\, J_n = i, \; J_{n+1} = j\right]$$

$$= \begin{cases} \frac{Q_{ij}(t)}{p_{ij}} & \text{if } p_{ij} \neq 0, \\ U_1(t) & \text{if } p_{ij} = 0, \end{cases}$$

where $U_1(t) = 1 \;\; \forall t \geq 0$.

The *homogeneous semi-Markov process* Z is defined as $Z(t) = J_{N(t)}$ where

$$N(t) = \max\{n \in \mathbb{N} | T_n \leq t\}.$$

Its transition probabilities are defined and obtained in the following way:

$$\phi_{ij}(t) = P\left[Z(t) = j | Z(0) = i\right],$$

$$\phi_{ij}(t) = (1 - H_i(t))\delta_{ij} + \sum_{l \in E} \sum_{\tau=1}^{t} q_{il}(\tau)\phi_{lj}(\tau, t). \quad (2)$$

Algorithms to solve (2) are well known in literature; see for example [11].

It is well know that an ergodic semi-Markov chain has a limit distribution

$$\mathbf{\Pi^{sm}} = (\pi_1^{sm}, \pi_2^{sm}, ..., \pi_K^{sm})$$

where

$$\pi_i^{sm} = \frac{\pi_i \eta_i}{\sum_k \pi_k \eta_k}$$

being $\eta_i = \sum_{t>0}(1 - H_i(t))$ and $\mathbf{\Pi} = \{\pi_i, i \in E\}$ the limit distribution of the embedded Markov chain.

3 The stochastic model

Let us consider a system of N economic agents which can be countries, regions or individuals. At each time $t \in \mathbb{N}$ each agent $i \in \{1, 2, ..., N\}$ produces a quantity $y_i(t)$ of a good. Suppose that the N time series of productions are observed up to time T. Then data available, in general, are represented by the following N time series:

$$\begin{matrix} y_1(0) & y_1(1) & y_1(2) & \cdots & y_1(T) \\ y_2(0) & y_2(1) & y_2(2) & \cdots & y_2(T) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ y_N(0) & y_N(1) & y_N(2) & \cdots & y_N(T) \end{matrix}$$

The quantity $y_i(s)$ represents the wealth produced (in terms of good production) by agent i at time s.

In order to make things as simple as possible we classify each agent by means of its wealth. To this end each agent is allocated, at each time, in one of K mutually exclusive classes:

$$E = \{C_1, C_2, ..., C_K\}$$

through an allocation map.

The allocation is done using some criterion, for example:

$i)$ consider a partition of the positive real line through a sequence of real numbers $\{l_i\}_{i=1,..,K}$ such that

$$0 = l_0 < l_1 < ... < l_{K-1} < l_k = \infty$$

$ii)$ if the wealth produced by agent i at time t belongs to the $j\!-\!th$ interval, then agent will be allocated in the C_j class. In symbols:

$$y_i(t) \in [l_{j-1}, l_j) \Longrightarrow i \in C_j \;\; at \;\; t.$$

Using the allocation map the N income time series are transformed in N time series of states of E:

$$\begin{matrix} C^1(0) & C^1(1) & C^1(2) & \cdots & C^1(T) \\ C^2(0) & C^2(1) & C^2(2) & \cdots & C^2(T) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ C^N(0) & C^N(1) & C^N(2) & \cdots & C^N(T) \end{matrix}$$

From these sequences we define $\forall h \in \{1, 2, ..., N\}$:

$$T_n^h = \inf\{t \in \mathbb{N} : C^h(t) \neq C^h(T_{n-1}^h)\}, \;\; T_0^h = 0;$$
$$J_n^h = C^h(T_n^h)$$

In this way we obtain N time series of states-times $(J_n^h, T_n^h)_{h=1,...,N}$.

Such time series can be considered as realizations of a random process. Therefore the following hypothesis is formulated:

Assumption 1 *The (J_n^h, T_n^h) are N independent trajectories of a discrete time Markov renewal process.*

As stated in Section 1 the process behavior is described by the kernel $\mathbf{Q}$ which is a matrix of functions. The probabilistic meaning of the element $Q_{ij}(t)$ has been described in relation (1).

In this model $Q_{ij}(t)$ denotes the probability that an agent now allocated in wealth class C_i will receive with next allocation wealth class C_j within a time t.

The kernel can be estimated using techniques developed by [15]:

$$\hat{Q}_{ij}(t) = \frac{1}{N_i} \sum_{h=1}^{N} \sum_{l=1}^{N^h} 1_{\{J_{l-1}^h = i, J_l^h = j, T_l^h - T_{l-1}^h \leq t\}} \quad (3)$$

and $N^h = N^h(T) = \sup\{n \in \mathbb{N} : T_n^h \leq T\}$ is the total number of transitions held by the h-*th* agent.

The (3) consists of computing the number of times a transition from state i towards state j occurred in a time no longer than t in the whole dataset, divided by the total number of transitions in state i.

For each $C_j \in E$ set

$$y_{C_j}^{(1)} = \frac{\sum_{h=1}^{N} \sum_{t=0}^{T} 1_{\{Z^h(t) = C_j\}} y_h(t)}{\sum_{h=1}^{N} \sum_{t=0}^{T} 1_{\{Z^h(t) = C_j\}}} \quad (4)$$

It represents the average wealth of agent in class C_j estimated from data. Therefore the following hypothesis is formulated:

Assumption 2 *Each time an agent is in state C_j it produces a wealth equal to $y_{C_j}^{(1)}$ which can be considered as a permanence reward in the class C_j.*

In order to define dynamic inequality/concentration indices we introduce a population structure at some starting time, denoted by zero, . At starting time $t = 0$ we have the population configuration

$$\{n_{C_1}(0), n_{C_2}(0), ..., n_{C_K}(0)\}. \quad (5)$$

We would analyze the time evolution of some indices (stochastic processes) which describe the concentration/inequality of the total wealth in the K classes. To this end, for each class $C_j \in E$, let $a_{C_j}(0, n)$ be the initial share of production (wealth) due to class C_j.

$$a_{C_j}(0) = \frac{n_{C_j}(0) y_{C_j}^{(1)}}{\sum_{h=1}^{K} n_{C_h}(0) y_{C_h}^{(1)}} = \frac{n_{C_j}(0) y_{C_j}^{(1)}}{< \underline{n}(0), \underline{y}^{(1)} >}. \quad (6)$$

We define the multivariate stochastic process

$$\underline{a}(t, n) = (a_{C_1}(t, n), a_{C_2}(t, n), ..., a_{C_K}(t, n))$$

which describes the time evolution of the shares of production among the classes of population:

$$a_{C_j}(t, n) = \frac{n_{C_j}(t) y_{C_j}^{(1)}}{< \underline{n}(t), \underline{y}^{(1)} >}. \quad (7)$$

It is a function of the multivariate counting process

$$\underline{n}(t) = \{n_{C_1}(t), n_{C_2}(t), ..., n_{C_K}(t)\}, \quad (8)$$

and of the vector of rewards

$$\underline{y}^{(1)}(t) = \{y_{C_1}^{(1)}, y_{C_2}^{(1)}, ..., y_{C_K}^{(1)}\}. \quad (9)$$

4 Dynamic inequality/concentration indices

Economists showed that it is important to have measures of how much concetrated is the production of the wealth inside an economic system.

Indeed for economic growth considerations and for social welfare needs, it is necessary to execute economic policies aimed to an increase of the average gross domestic product but also lead to lower income distribution inequality in the society, see i.e. Campano and Salvatore [4], Dasgupta et al. [8].

In literature are available many indices useful to the measurement of concentration and inequality of wealth.

In this section we define and analyze some dynamic inequality/concentration indices usually computed in the economic analyses in a static way, see i.e [10], [9], [23] and [26].

Concentration indices have been mailnly used in the industrial organization literature to provide information on the degree of competition inside an industry.

The book by Tirole [24] still constitutes a key reference.

Concentration indices considers as relevant factors the inequality with respect to the size of the firms and the number of the firms. So it is evident that some connections should exist between concentration and inequality iondices.

Marfels [13] was one of the first researcher who investigated the relationship between concentration and inequality measures.

A more recent contribution to this subject is given by Bajo and Salas [3] were a connection between concentration and inequality indices is provided.

For this reason, henceforth, we will refer to concentration or inequality indices indifferently.

4.1 The Herfindahl-Hirschman index

This index is widely used as a measure of inequality/concentration. It is defined by $C_{HH} = \sum_{i=1}^{K} a_i^2$, see [10]. In order to measure the time evolution of the inequality/concentration of wealth in an economic system like that described in Section 3, it is necessary to replace the static Herfindahl-Hirschman index with a dynamic one. It is possible to reach this goal because the shares of production among the classes of population are here considered as a multivariate stochastic process through (7).

Definition 3 *Given the population configuration (5) at time $t = 0$ and the vector of average productions (9), the dynamic Herfindahl-Hirschman index is the stochastic process:*

$$C_{HH}(t) := \sum_{i=1}^{K} a_{C_i}^2(t,n) = \sum_{i=1}^{K} \left(\frac{n_{C_j}(t) y_{C_j}^{(1)}}{< \underline{n}(t), \underline{y}^{(1)} >}\right)^2. \quad (10)$$

Because it is difficult to characterize the evolution of the process $C_{HH}(t)$, we concentrate on its moments:

$$E[C_{HH}(t)] = \sum_{i=1}^{K} E\left[\left(\frac{n_{C_i}(t) y_{C_i}^{(1)}}{< \underline{n}(t), \underline{y}^{(1)} >}\right)^2\right] =$$

$$\sum_{i=1}^{K} \sum_{\underline{n}' \in p.c.} P[\underline{n}(t) = \underline{n}' | \underline{n}(0) = \underline{n}] (a_{C_i}(t, n'))^2, \quad (11)$$

where p.c. is the set of all possible population configuration. It is easy to determine the cardinality of the population configuration set:

$$|p.c| = \binom{N+K-1}{K-1}. \quad (12)$$

To compute $P[\underline{n}(t) = \underline{n}' | \underline{n}(0) = \underline{n}]$ we use the following Theorem, see [14]:

Theorem 4 *For each $t > 0$, the joint distribution of the number of particles present in each compartments follows multinomial distribution with parameters*

$$\{N, P_1(t), P_2(t), ..., P_K(t)\} \quad (13)$$

where $\forall i = 1, ..., K$ $P_i(t) = \sum_{h=1}^{K} \frac{n_h(0)}{N} \phi_{hi}(t)$ and $\phi(t)$ is the transition probability matrix at time t of the homogeneous semi-Markov process.

The $\phi_{hi}(t)$ express the transition probabilities of the semi-Markov chain associated with the Markov renewal process obtained by solving equation (2).

First moment can be evaluated by:

$$E[C_{HH}(t)] =$$

$$\sum_{i=1}^{K} \sum_{\underline{n}' \in p.c.} \frac{N!}{\prod_{h=1}^{K} n'_{C_h}!} \prod_{h=1}^{K} (P_h(t))^{n'_{C_h}} (a_{C_i}(t, n'))^2.$$

Second moment can be evaluated by:

$$E[C_{HH}^2(t)] =$$

$$\sum_{i=1}^{K} \sum_{\underline{n}' \in p.c.} \frac{N!}{\prod_{h=1}^{K} n'_{C_h}!} \prod_{h=1}^{K} (P_h(t))^{n'_{C_h}} (a_{C_i}(t, n'))^4 +$$

$$2 \sum_{i<j} \sum_{\underline{n}' \in p.c.} \frac{N!}{\prod_{h=1}^{K} n'_{C_h}!} \prod_{h=1}^{K} (P_h(t))^{n'_{C_h}} \left(\frac{a_{C_i}(t, n')}{a_{C_j}^{-1}(t, n')}\right)^2.$$

It is no difficult to obtain with the same techniques moments of higher order.
If the semi-Markov process is ergodic with limit distribution

$$\mathbf{\Pi^{sm}} = (\pi_1^{sm}, \pi_2^{sm}, ..., \pi_K^{sm}),$$

it is possible to determine the limit behavior as follows:

$$\lim_{t \to \infty} E[C_{HH}(t)] =$$

$$\sum_{i=1}^{K} \sum_{\underline{n}' \in p.c.} \frac{N!}{\prod_{h=1}^{K} n'_{C_h}!} \prod_{h=1}^{K} (\pi_h^{sm})^{n'_{C_h}} (a_{C_i}(t, n'))^2.$$

4.2 The Gini index

The Gini index is one of the most used inequality/concentration index since it satisfies common criteria of inequality measures, see [21] and [2]:

a) Invariance to unit of measurement of incomes.

If the incomes of all the economic agents increase (decrease) of the same percentage, then the inequality measure should not change.

b) Invariance to replication of population.

It means that if i.e. population size is doubled by adding an exact replica of every individual to the population, then the inequality measure does not change.

c) Compliance with the Pigou-Dalton principle of transfers, see i.e. Dalton [6], Quah [17].

This principle requires the inequality measure to not increase any time that income is redistributed from a richer agent to a poorer agent and viceversa.

A computationally efficient representation of the Gini index is the following, see [9]:

$$G = 1 + \frac{1}{K} - \left(\frac{2}{K^2 \overline{a}}\right) \sum_{i=1}^{K} i a_{(i)}, \quad (14)$$

where $a_{(i)}$ is the i-*th* highest share and $\overline{a}$ is the average population share. A sorting of shares is required first of computing the index.

The Gini index has values between zero and one. If the index value is equal to zero then the wealth is equidistributed in the population.

On the contrary a value equal to one denotes the fact that only one economic agent possesses all the wealth.

Definition 5 *Given the population configuration (5) at time $t = 0$ and the vector of average productions (9), the dynamic Gini index is the stochastic process:*

$$G(t) := 1 + \frac{1}{K} - \Big(\frac{2\sum_{i=1}^{K} i a_{(i)}(t,n)}{K}\Big).$$

First moment can be evaluated by:

$$E\Big[G(t)\Big] = 1 + \frac{1}{K} -$$

$$\frac{2}{K}\sum_{i=1}^{K}\sum_{\underline{n}'\in p.c.}\frac{i\cdot N!}{\prod_{h=1}^{K} n'_{C_h}!}\prod_{h=1}^{K}(P_h(t))^{n'_{C_h}}(a_{C_i}(t,n')).$$

The asymptotic behavior of the expected value can be computed as follows:

$$\lim_{t\to\infty} E[G(t)] = 1 + \frac{1}{K}$$

$$-\frac{2}{K}\sum_{i=1}^{K}\sum_{\underline{n}'\in p.c.}\frac{i\cdot N!}{\prod_{h=1}^{K} n'_{C_h}!}\prod_{h=1}^{K}(\pi_h^{sm})^{n'_{C_h}}(a_{C_i}(t,n')).$$

It is no difficult to obtain with the same techniques moments of higher order.

4.3 The Theil's Entropy

This measure has been proposed by Theil and is derived from the mathematical theory of communication founded by Shannon, see [22].

In its static form the Theil's entropy is defined in [23] by

$$T_e = \sum_{i=1}^{K} a_i(\log K a_i). \tag{15}$$

This index satisfies the following criteria:

a) Strong principle of transfers.

It is more sensitive to transfers of wealth in the lower tail of the wealth distribution than it is to the transfers in the upper tail, see [5].

b) It is additively decomposable.

This means that for any significant grouping of the population (for example by age, region, occupation) the Theil measure in the entire population can be additively decomposed to inequality between subgroups and an appropriately weighted average of inequality within each group.

Definition 6 *Given the population configuration (5) at time $t = 0$ and the vector of average productions (9), the dynamic Theil's Entropy is the stochastic process:*

$$T_e(t) := \sum_{i=1}^{K} a_{C_i}(t,n)(\log K\, a_{C_i}(t,n)). \tag{16}$$

First moment can be evaluated by:

$$E[T_e(t)] =$$

$$\sum_{i=1}^{K}\sum_{\underline{n}'\in p.c.}\frac{N!}{\prod_{h=1}^{K} n'_{C_h}!}\prod_{h=1}^{K}(P_h(t))^{n'_{C_h}}\Big(\frac{logKa_{C_i}(t,n')}{a_{C_i}^{-1}(t,n')}\Big).$$

The asymptotic behavior of the expected value can be computed as follows:

$$\lim_{t\to\infty} E[T_e(t)] =$$

$$\sum_{i=1}^{K}\sum_{\underline{n}'\in p.c.}\frac{N!}{\prod_{h=1}^{K} n'_{C_h}!}\prod_{h=1}^{K}(\pi_h^{sm})^{n'_{C_h}}\Big(\frac{logKa_{C_i}(t,n')}{a_{C_i}^{-1}(t,n')}\Big).$$

It is no difficult to obtain with the same techniques moments of higher order.

5 Concentration/inequality indices in different economical scenarios

Now we provide numerical examples performed by *MATHEMATICA* software. In our *Mathematica* session we loaded the adds-on packages *MultivariateStatistics* and *Combinatorica* that are able to compute in a direct way the binomial coefficient (12) and the multinomial distribution values as stated by Theorem 4 .

We consider different economical scenarios through the consideration of different initial economical conditions expressed by changes in the initial configuration population and/or by the vector of agent's productions.

In all scenarios we assume a number N of economic agents that are classified in the set $E = \{1/4, 1/2, 1, 2, \infty\}$, according with the paper by Quah [16]. In other words in the first class there will be all the agents with a wealth less than a quarter of the world average gross domestic product (GDP) per capita income. In the second class there will be all the agents with a wealth within a quarter and one half of the world average GDP per capita and so on. In the last class there will be all the agents with a wealth equal to more than 2 time the world average GDP per capita.

Moreover Quah [16], working with a simple Markov chain, estimated the following one step transition probability matrix.

$$\mathbf{P} = \begin{array}{c} \\ 1/4 \\ 1/2 \\ 1 \\ 2 \\ \infty \end{array} \begin{array}{c} \begin{array}{ccccc} 1/4 & 1/2 & 1 & 2 & \infty \end{array} \\ \begin{pmatrix} 0.97 & 0.03 & 0.00 & 0.00 & 0.00 \\ 0.04 & 0.92 & 0.04 & 0.00 & 0.00 \\ 0.00 & 0.04 & 0.92 & 0.04 & 0.00 \\ 0.00 & 0.00 & 0.04 & 0.94 & 0.02 \\ 0.00 & 0.00 & 0.00 & 0.01 & 0.99 \end{pmatrix} \end{array}$$

For all cases that follow we consider $\mathbf{P}$ as the matrix of the embedded Markov chain in the semi-Markov process.

In order to get the semi-Markov kernel we simulated the conditional waiting time cumulative distribution functions. In this way we obtained a semi-Markov kernel and we computed the corresponding transition probabilities each time for the different economical scenarios.

The limit distributions of the Markov chain and of the semi-Markov chain are the following for all the scenarios:

$$\begin{array}{c} \\ MC \\ SMC \end{array} \begin{array}{c} \begin{array}{ccccc} 1/4 & 1/2 & 1 & 2 & \infty \end{array} \\ \begin{pmatrix} 0.24 & 0.18 & 0.16 & 0.16 & 0.27 \\ 0.214 & 0.167 & 0.194 & 0.147 & 0.278 \end{pmatrix} \end{array}$$

5.1 Highly dispersed economy

In this Subsection we suppose that the economical population consists in $N = 15$ agents which are distributed according to the starting population configuration $\underline{n}(0)$ and the wealth production $\underline{y}^{(1)}(0)$ given in the Table 1. Then i.e. in the third class we have 4 agents each of them producing ten units of good.

	1/4	1/2	1	2	∞
$\underline{n}(0)$	2	4	4	2	3
$\underline{y}^{(1)}(0)$	2	4	10	18	50

Table 1 : Economic Configuration 1 (EC 1)

Table 1 shows that the wealth production is highly dispersed. Indeed the average and the variance of the production are respectively:

$$m_1 = 16.4, \quad v_1 = 306.$$

Table 2 shows the dynamic indices values proposed in Section 4 corresponding to times indicated in the first column. In the last row it is possible to read the corresponding asymptotic values.

	$C_{HH}(t)$	$G(t)$	$T_e(t)$
$t = 1$	0.4466	0.5265	1.1054
$t = 2$	0.4476	0.5275	1.1088
$t = 3$	0.4484	0.5281	1.1105
$t = 4$	0.4495	0.5290	1.1239
$t = 5$	0.4505	0.5298	1.1167
$t = 6$	0.4515	0.5307	1.1199
$t = 7$	0.4526	0.5315	1.1232
$t = 8$	0.4537	0.5324	1.1267
$t = 9$	0.4548	0.5332	1.1303
$t = 10$	0.4559	0.5341	1.1339
$t = 11$	0.4571	0.5350	1.1377
$t = 12$	0.4582	0.5359	1.1415
$t = 13$	0.4594	0.5368	1.1453
$t = 14$	0.4606	0.5377	1.1492
$t = 15$	0.4618	0.5386	1.1532
$t = 16$	0.4630	0.5395	1.1571
$t = 17$	0.4642	0.5404	1.1611
$t = 18$	0.4654	0.5413	1.1651
$t = 19$	0.4666	0.5422	1.1691
$t = 20$	0.4678	0.5431	1.1732
$t = \infty$	0.5383	0.5904	1.4094

Table2 : Dynamic Indices EC 1

5.2 Lower dispersed economy

Now we consider some variations to the economy represented by changes in the vector of wealth production and in the initial distribution of economic agents in the five classes.

In particular we assume that the $N = 15$ economic agents are distributed according to the starting population configuration $\underline{n}(0)$ and the wealth production $\underline{y}^{(1)}(0)$ given in Table 3:

	1/4	1/2	1	2	∞
$\underline{n}(0)$	2	4	4	2	3
$\underline{y}^{(1)}(0)$	2	4	7	11	16

Table 3 : Economic Configuration 2

Table 3 shows that the wealth production is characterized by a lower dispersion of the production. Indeed the average and the variance of the production are respectively:

$$m_2 = 7.9, \quad v_2 = 23.$$

Table 4 shows the time evolution of the indices values corresponding to the Economic Configuration 2.

	$C_{HH}(t)$	$G(t)$	$T_e(t)$
$t=1$	0.3272	0.4034	0.6535
$t=2$	0.3280	0.4043	0.6549
$t=3$	0.3283	0.4043	0.6537
$t=4$	0.3289	0.4049	0.6546
$t=5$	0.3294	0.4053	0.6548
$t=6$	0.3300	0.4058	0.6556
$t=7$	0.3305	0.4062	0.6565
$t=8$	0.3311	0.4068	0.6576
$t=9$	0.3317	0.4073	0.6589
$t=10$	0.3323	0.4079	0.6603
$t=11$	0.3329	0.4085	0.6619
$t=12$	0.3335	0.4091	0.6635
$t=13$	0.3341	0.4097	0.6652
$t=14$	0.3347	0.4104	0.6670
$t=15$	0.3354	0.4110	0.6689
$t=16$	0.3360	0.4117	0.6708
$t=17$	0.3366	0.4123	0.6727
$t=18$	0.3373	0.4130	0.6747
$t=19$	0.3379	0.4136	0.6767
$t=20$	0.3385	0.4143	0.6788
$t=\infty$	0.3808	0.4566	0.8995

Table 4 : Dynamic Indices EC 2

In the last row it is possible to read the corresponding asymptotic values.

As it is possible to see, the Economic Configuration 2 is carachterized by less concentration of the wealth production and less inequality among the classes in which agents are allocated. Indeed, for each time and for each of the three indices, the values read in Tab. 2 are greater than the corresponding in Tab. 4.

5.3 Average dispersed economy

In this case we suppose that the $N = 15$ economic agents are distributed according to the starting population configuration $\underline{n}(0)$ and the wealth production $\underline{y}^{(1)}(0)$ given in Table 5:

	1/4	1/2	1	2	∞
$\underline{n}(0)$	2	4	4	2	3
$\underline{y}^{(1)}(0)$	2	4	8	15	30

Table 5 : Economic Configuration 3

Table 5 shows that the wealth production is characterized by an average dispersion with respect to the previous cases. Indeed the average and the variance of the production are respectively:

$$m_3 = 11, \quad v_3 = 100.$$

Table 6 shows the time evolution of the indices values corresponding to the Economic Configuration 3. As usual, in the last row it is possible to read the corresponding asymptotic values.

	$C_{HH}(t)$	$G(t)$	$T_e(t)$
$t=1$	0.3868	0.4707	0.8819
$t=2$	0.3878	0.4718	0.8847
$t=3$	0.3885	0.4724	0.8857
$t=4$	0.3894	0.4733	0.8883
$t=5$	0.3903	0.4741	0.8904
$t=6$	0.3913	0.4750	0.8930
$t=7$	0.3922	0.4758	0.8956
$t=8$	0.3931	0.4767	0.8984
$t=9$	0.3941	0.4775	0.9013
$t=10$	0.3951	0.4784	0.9044
$t=11$	0.3961	0.4793	0.9075
$t=12$	0.3971	0.4802	0.9106
$t=13$	0.3981	0.4811	0.9138
$t=14$	0.3991	0.4820	0.9171
$t=15$	0.4001	0.4829	0.9204
$t=16$	0.4011	0.4838	0.9237
$t=17$	0.4021	0.4846	0.9271
$t=18$	0.4031	0.4855	0.9304
$t=19$	0.4041	0.4864	0.9338
$t=20$	0.4051	0.4873	0.9372
$t=\infty$	0.4657	0.5356	1.1264

Table 6 : Dynamic Indices EC 3

The Economic Configuration 3 is carachterized by an average concentration of the wealth production and by an average inequality among the classes in which agents are allocated.

Indeed, for each time and for each of the three indices, the values read in Table 6 are between the corresponding values in Table 4 and Table 2.

5.4 Immigration effect with one entry in the poorest class

Now we would like to quantify the effects of the immigration in economy. The first case we analyze is the Economic Configuration 1 perturbed by the entrance of one agent from outside in the poorest class (the agents with a wealth less than a quarter of the world average GDP per capita income).

Consequently the new economic configuration we are analyzing, that is EC 4, is formed by $N = 16$ economic agents which are distributed according to the starting population configuration $\underline{n}(0)$ and the wealth production $\underline{y}^{(1)}(0)$ given in the Table 7:

	1/4	1/2	1	2	∞
$\underline{n}(0)$	3	4	4	2	3
$\underline{y}^{(1)}(0)$	2	4	10	18	50

Table 7 : Economic Configuration 4

Table 7 shows that the wealth production is characterized by an average dispersion with respect to the previous cases. Indeed the average and the variance of the production are respectively:

$$m_4 = 15.5, \quad v_4 = 299.$$

The comparison between Table 7 and Table 1 shows that the new entry led to a decrease of both average and variance. Then the average income of this economy is less than the corresponding in absence of immigration. Moreover the dispersion of the wealth production is decreased.

Table 8 shows the time evolution of the indices values corresponding to the Economic Configuration 4.

As usual, in the last row it is possible to read the corresponding asymptotic values.

	$C_{HH}(t)$	$G(t)$	$T_e(t)$
$t=1$	0.4396	0.5159	1.0586
$t=2$	0.4404	0.5169	1.0623
$t=3$	0.4411	0.5175	1.0643
$t=4$	0.4421	0.5184	1.0677
$t=5$	0.4429	0.5192	1.0706
$t=6$	0.4439	0.5200	1.0739
$t=7$	0.4448	0.5208	1.0771
$t=8$	0.4458	0.5217	1.0805
$t=9$	0.4467	0.5225	1.0839
$t=10$	0.4477	0.5234	1.0875
$t=11$	0.4488	0.5242	1.0910
$t=12$	0.4498	0.5251	1.0946
$t=13$	0.4508	0.5260	1.0983
$t=14$	0.4519	0.5268	1.1020
$t=15$	0.4529	0.5277	1.1057
$t=16$	0.4540	0.5286	1.1094
$t=17$	0.4551	0.5295	1.1132
$t=18$	0.4562	0.5303	1.1170
$t=19$	0.4573	0.5312	1.3508
$t=\infty$	0.5375	0.5899	.

Table 8 : Dynamic Indices EC 4

5.5 Immigration effect with one entry in the richest class

In order to make a thorough investigation on the effects of the immigration in economy, in this subsection we consider the Economic Configuration 1 perturbed by the entrance of one agent from outside in the richest class (the agents with a wealth more than 2 time the world average GDP per capita income).

Consequently in this case, the Economic Configuration 5 is formed by $N = 16$ agents initially distributed as in first row of the Table 9 which have an initial production as in the second row of the Table 9.

	1/4	1/2	1	2	∞
$\underline{n}(0)$	2	4	4	2	4
$\underline{y}^{(1)}(0)$	2	4	10	18	50

Table9 : Economic Configuration 5

Table 9 shows that the wealth production is characterized by a dispersion lightly grater than the previous cases. Indeed the average and the variance of the production are respectively:

$$m_5 = 18.5, \quad v_5 = 353.$$

	$C_{HH}(t)$	$G(t)$	$T_e(t)$
$t=1$	0.4983	0.5649	1.2771
$t=2$	0.4984	0.5653	1.2779
$t=3$	0.4985	0.5653	1.2775
$t=4$	0.4989	0.5658	1.2788
$t=5$	0.4992	0.5661	1.2797
$t=6$	0.4997	0.5664	1.2810
$t=7$	0.5001	0.5668	1.2824
$t=8$	0.5006	0.5672	1.2841
$t=9$	0.5012	0.5677	1.2859
$t=10$	0.5018	0.5681	1.2879
$t=11$	0.5023	0.5685	1.2899
$t=12$	0.5030	0.5690	1.2921
$t=13$	0.5036	0.5695	1.2944
$t=14$	0.5043	0.5700	1.2967
$t=15$	0.5050	0.5706	1.2992
$t=16$	0.5058	0.5711	1.3017
$t=17$	0.5065	0.5716	1.3042
$t=18$	0.5073	0.5722	1.3068
$t=19$	0.5081	0.5727	1.3094
$t=\infty$	0.5375	0.5898	1.5641

Table 10 : Dynamic Indices EC 5

The comparison between Table 9 and Table 1 shows that the new entry led to an increase of both mean and variance. Then the mean income of this economy is greater than the corresponding in absence of immigration. Moreover the dispersion of the wealth production is increased.

Table 10 shows the time evolution of the indices values corresponding to the Economic Configuration 5.

In the last row it is possible to read the corresponding asymptotic values.

6 Conclusions

In this paper we provided a stochastic model in order to study the wealth evolution.

In particular we generalized three concentration/inequality indices used in economic literature in a static way by means of indices evolving in time.

The mean idea was that to introduce population dynamics.

We shown how to compute the moments of these indices represented through stocastic processes and their asymptotic values.

Finally we performed a numerical experience aimed to analyze the variations of the economy depending on changes on distribution of the agents and on their wealth production.

Further research on the subject could be addressed to:

- the consideration of a interdependence relations in the population dynamics;

- the construction of a geographical model;

- the stochastic orders of the semi-Markovian processes involved;

- the research of numerical bounds aimed to explain the differences among the indices values.

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